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Research article

Approximate solutions of Atangana-Baleanu variable order fractional problems

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Abstract: The main aim of this paper is to propose a new approach for Atangana-Baleanu variable order fractional problems. We introduce a new reproducing kernel function with polynomial form. The advantage is that its fractional derivatives can be calculated explicitly. Based on this kernel function, a new collocation technique is developed for variable order fractional problems in the Atangana-Baleanu fractional sense. To show the accuracy and effectiveness of our approach, we provide three numerical experiments.

Keywords: reproducing kernel method; Atangana-Baleanu derivative; variable fractional order; collocation method
Mathematics Subject Classification: 65L60, 65R20

1. Introduction

We are concerned with Atangana-Baleanu variable order fractional problems:

$$\begin{cases} Lu(x) = {}^{ABC} D^{\alpha(x)} u(x) + a(x)u(x) = f(x,u), \ x \in [0,1], \\ B(u) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha(x) < 1$, ${}^{ABC}D^{\alpha(x)}(x)$ denotes the $\alpha(x)$ order Atangana-Baleanu Caputo derivatives, $B(u)$ is the linear boundary condition, which includes initial value condition, periodic condition, final value condition and so on.

The $\alpha(x)(0 < \alpha(x) < 1)$ order Atangana-Baleanu Caputo derivatives of a function $u(x)$ is firstly defined by Atangana and Baleanu [1]

$$ {}^{ABC}D^{\alpha(x)}u(x) = \frac{M(\alpha(x))}{1-\alpha(x)} \int_0^x E_{\alpha(x)}(-\frac{\alpha(x)}{1-\alpha(x)}(x-t)^{\alpha(x)})u'(t)dt, \tag{1.2} $$

where $E_{\alpha(x)}(x)$ is the Mittag-Leffler function.

Fractional order differential equations (FDEs) have important applications in several fields such as materials, chemistry transmission dynamics, optimal control and engineering [2–6]. In fact, the classical fractional derivatives are defined with weak singular kernels and the solutions of FDEs inherit the weak singularity. The Mittag-Leffler (ML) function was firstly introduced by Magnus Gösta Mittag-Leffler. Recently, it is found that this function has close relation to FDEs arising in real applications.

Atangana and Baleanu [1] introduced a new fractional derivative by using the ML function, which is nonlocal and nonsingular. The new fractional derivatives is very important and have been applied to several different fields (see e.g. [7–9]). Up to now, several numerical algorithms have been developed for solving Atangana-Baleanu FDEs. Akgül et al. [10–12] proposed effective difference techniques and kernels based approaches for Atangana-Baleanu FDEs. On the basis of the Sobolev kernel functions, Arqub et al. [13–17] proposed the numerical techniques for Atangana-Baleanu fractional Riccati and Bernoulli equations, Bagley-Torvik and Painlev equations, Volterra and Fredholm integro-differential equations. Yadav et al. [18] introduced the numerical algorithms and application of Atangana-Baleanu FDEs. El-Ajou, Hadid, Al-Smadi et al. [19] developed approximated technique for solutions of population dynamics of Atangana-Baleanu fractional order.

Reproducing kernel Hilbert space (RKHS) is ideal for function approximation and estimate of fractional derivatives. In recent years, reproducing kernel functions (RKF) theory have been employed to solve linear and nonlinear fractional order problems, singularly perturbed problems, singular integral equations, fuzzy differential equations, and so on (see, e.g. [10–17, 19–35]). However, there exists little discussion on numerical schemes for solving variable order Atangana-Baleanu FDEs.

In this paper, by using polynomials RKF, we will present a new collocation method for solving variable order Atangana-Baleanu FDEs.

This work is organized as follows. We summarize fractional derivatives and RKHS theory in Section 2. In Section 3, we develop RKF based collocation technique for Atangana-Baleanu variable order FDEs. Numerical experiments are provided in Section 4. Concluding remarks are included in the last section.

2. Preliminaries to fractional derivatives and RKHS theory

Definition 2.1. *Let H be a Hilbert function space defined on E. The function $K : E \times E \to R$ is known as an RKF of space H if*

$$ (1)\ K(\cdot,\ t) \in H for\ all\ t \in E, $$
$$ (2)\ w(t) = (w(\cdot),\ K(\cdot,\ t)), for\ all\ t \in E\ and\ all\ w \in H. $$

If there exists a RKF in a Hilbert space, then the space is a RKHS.

Definition 2.2. *Symmetric function $K : E \times E \to R$ is known as a positive definite kernel (PDK) if $\sum_{i,j=1}^{n} c_i c_j K(x_i, x_j) \geq 0$ for any $n \in N$, $x_1, x_2, \ldots, x_n \in E, c_1, c_2, \ldots, c_n \in R$.*

Theorem 2.1. *[36] The RKF of an RKHS is positive definite. Besides, every PDK can define a unique RKHS, of which it is the RKF.*

Definition 2.3. *Let $q > 0$. The one parameter Mittag-Leffler function of order q is defined by*

$$E_q(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jq+1)}. \tag{2.1}$$

Definition 2.4. *Let $q_1, q_2 > 0$. The two-parameter Mittag-Leffler function is defined by*

$$E_{q_1,q_2}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jq_1+q_2)}. \tag{2.2}$$

For the domains of convergence of the Mittag-Leffler functions, please refer to the following theorem.

Theorem 2.2. *[37] For $q_1, q_2 > 0$, the two-parameter Mittag-Leffler function $E_{q_1,q_2}(z)$ is convergent for all $z \in \mathbb{C}$.*

Definition 2.5. *The Sobolev space $H^1(0,T)$ is defined as follows*

$$H^1(0,T) = \{u | \, u \in L^2(0,T), \; u' \in L^2(0,T)\}.$$

Definition 2.6. *The $\alpha \in (0,1)$ order Atangana- Baleanu fractional derivative of a function $u \in H^1(a,b)$ is defined*

$$^{ABC}D^{\alpha}u(x) = \frac{M(\alpha)}{1-\alpha} \int_0^x E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(x-t)^{\alpha}\right)u'(t)dt, \tag{2.3}$$

where $M(\alpha)$ is the normalization term satisfying $M(0) = M(1) = 1$.

Theorem 2.3. *[38] The function $k(x,y) = (xy+c)^m$ for $c > 0, m \in N$ is a PDK.*

According to Theorem 2.1, there exists an associated RKHS Q_m with k as an RKF.

3. Collocation method

To solve (1.1), we will construct the RKF which satisfies the homogenous boundary condition.

Definition 3.1.

$$Q_{m,0} = \{w(t) \mid w(t) \in Q_m, B(w) = 0\}.$$

Theorem 3.1. *The space $Q_{m,0}$ is an RKHS and its RKF is expressed by*

$$K(x,y) = k(x,y) - \frac{B_x k(x,y) B_y k(x,y)}{B_x B_y k(x,y)}.$$

Proof. If $B_y k(x,y) = 0$ or $B_x k(x,y) = 0$, then

$$K(x,y) = k(x,y).$$

If $B_y k(x,y) \neq 0$, then

$$\begin{aligned} B_x K(x,y) &= B_x k(x,y) - \tfrac{B_x k(x,y) B_x B_y k(x,y)}{B_x B_y k(x,y)}, \\ &= 0, \end{aligned}$$

and naturally $K(x,y) \in Q_{m,0}$.

For all $u(y) \in Q_{m,0}$, we have $u(y) \in Q_m$ and $B_y u(y) = 0$.

We have

$$\begin{aligned} (u(y), K(x,y)) &= (u(y), k(x,y)) - (u(y), \tfrac{B_x k(x,y) B_y k(x,y)}{B_x B_y k(x,y)} \\ &= u(x) - \tfrac{B_y k(x,y)}{B_x B_y k(x,y)}(u(y), B_x k(x,y)) \\ &= u(x) - \tfrac{B_y k(x,y)}{B_x B_y k(x,y)} B_x(u(y), k(x,y)) \\ &= u(x) - \tfrac{B_y k(x,y)}{B_x B_y k(x,y)} B_x u(x) \\ &= u(x) - 0 \\ &= 0. \end{aligned}$$

Thus, $K(x,y)$ is the RKF of space $Q_{m,0}$ and the proof is complete. □

Suppose that $L : Q_{m,0} \to H^1$ is a bounded linear operator. It is easy to proved that its inverse operator L^{-1} is also bounded since both $Q_{m,0}$ and H^1 are Banach spaces.

Choose N distinct scattered points in $[0,1]$, such as $\{x_1, x_2, \ldots, x_N\}$. Put $\psi_i(x) = K(x, x_i), i = 1, 2, \ldots, N$. By using RKF basis, the RKF collocation solution $u_N(x)$ for (1.1) can be written as follows

$$u_N(x) = \sum_{i=1}^{N} c_i \psi_i(x), \tag{3.1}$$

where $\{c_i\}_{i=1}^N$ are undetermined constants.

Collocating (1.1) at N nodes $x_1, x_2, \ldots, x_N$ provides N equations:

$$Lu_N(x_k) = \sum_{i=1}^{N} c_i L\psi_i(x_k) = f(x_k, u_N(x_k)),\ k = 1, 2, \ldots, N. \tag{3.2}$$

System (3.3) of equations is simplified to the matrix form:

$$A\mathbf{c} = \mathbf{f}, \tag{3.3}$$

where $A_{ik} = L_x \psi_k(x)|_{x=x_i}, i, k = 1, 2, \ldots, N$, $\mathbf{f} = (f(x_1, u_N(x_1)), f(x_2, u_N(x_2)), \ldots, f(x_N, u_N(x_N))$.

Theorem 3.2. *If $\gamma > 0$, then*

$${}^{ABC}D^{\alpha(x)} x^\gamma = \frac{M(\alpha(x))}{1-\alpha(x)} \Gamma(\gamma+1) x^\gamma E_{\alpha(x),\gamma+1}(-\frac{\alpha(x)}{1-\alpha(x)} x^{\alpha(x)}),$$

and therefore matrix A can be computed exactly.

Proof. It is noticed that

$$\begin{aligned}
{}^{ABC}D^{\alpha(x)}x^{\gamma} &= \frac{M(\alpha(x))}{1-\alpha(x)}\int_0^x E_{\alpha(x)}(-\frac{\alpha(x)}{1-\alpha(x)}(x-t)^{\alpha(x)})\gamma t^{\gamma-1}dt \\
&= \frac{M(\alpha(x))}{1-\alpha(x)}\int_0^x \sum_{j=0}^{\infty}\frac{(-\frac{\alpha(x)}{1-\alpha(x)}(x-t)^{\alpha(x)})^j}{\Gamma(j\alpha(x)+1)}\gamma t^{\gamma-1}dt \\
&= \frac{M(\alpha(x))}{1-\alpha(x)}\gamma\sum_{j=0}^{\infty}\frac{(-\frac{\alpha(x)}{1-\alpha(x)})^j}{\Gamma(j\alpha(x)+1)}\int_0^x (x-t)^{\alpha(x)}t^{\gamma-1}dt \\
&= \frac{M(\alpha(x))}{1-\alpha(x)}\gamma\sum_{j=0}^{\infty}\frac{(-\frac{\alpha(x)}{1-\alpha(x)})^j}{\Gamma(j\alpha(x)+1)}\frac{\Gamma(j\alpha(x)+1)\Gamma(\gamma)}{\Gamma(j\alpha(x)+\gamma+1)}x^{j\alpha(x)+\gamma} \\
&= \frac{M(\alpha(x))}{1-\alpha(x)}\Gamma(\gamma+1)x^{\gamma}\sum_{j=0}^{\infty}\frac{(-\frac{\alpha(x)}{1-\alpha(x)}x^{\alpha(x)})^j}{\Gamma(j\alpha(x)+\gamma+1)} \\
&= \frac{M(\alpha(x))}{1-\alpha(x)}\Gamma(\gamma+1)x^{\gamma}E_{\alpha(x),\gamma+1}(-\frac{\alpha(x)}{1-\alpha(x)}x^{\alpha(x)}).
\end{aligned}$$

Since RKF $K(x, y)$ is a polynomials, matrix A in (3.3) can be calculated exactly. The proof is complete. □

If $f(x, u)$ is linear, then (3.3) is a system of linear equations and it is convenient to determine the value of the unknowns $\{c_i\}_{i=1}^N$. If $f(x, u)$ is nonlinear, then (3.3) is a system of nonlinear equations, we solve it by using the tool "FindRoot" in soft Mathematica 11.0.

The residual function is defined as

$$R_N(x) = Lu_N(x) - f(x, u_N(x)).$$

Theorem 3.3. *If* $a(x)$ *and* $f(x, u) \in C^4[0, 1]$*, then*

$$\| R_N(x) \|_\infty \triangleq \max_{x\in[x_1,x_N]} | R_N(x) | \le c\, h^4,$$

where $c > 0$ *is a real number,* $h = \max_{1\le i\le N} | x_{i+1} - x_i |$.

Proof. For the proof, please refer to [22]. □

4. Numerical experiments

Three experiments are illustrated in this section to show the applicability and effectiveness of the mentioned approach. We take $M(\alpha) = 1$ in the following experiments.

Problem 4.1

Solve fractional linear initial value problems (IVPs) as follows:

$$\begin{cases} {}^{ABC}D^{\alpha}u(x) + e^x u(x) = f(x), \quad x \in (0, 1], \\ u(0) = 1, \end{cases}$$

where $\alpha(x) = 0.5x + 0.1$, $f(x) = e^x(x^2 + x^3 + 1) + \frac{M(\alpha(x))}{1-\alpha(x)}2x^2E_{\alpha(x),3}(-\frac{\alpha(x)}{1-\alpha(x)}x^{\alpha(x)}) + +\frac{M(\alpha(x))}{1-\alpha(x)}6x^3E_{\alpha(x),4}(-\frac{\alpha(x)}{1-\alpha(x)}x^{\alpha(x)})$. The true solution of this equation is $u(x) = x^2 + x^3 + 1$.

Selecting $m = 8, N = 8$, $x_i = \frac{i}{N}, i = 1, 2, \ldots, N$, we apply our new method to Problem 4.1. The obtained numerical results are shown in Tables 1. The Mathematica codes for Problem 4.1 is provided as follows:

```
tru[x_] = x^2 + x^3 + 1; p[x_] = E^x; α[x_] = 0.5x + 0.1;
B[x_] = 1; a[x_] = 1/Gamma[2−α[x]]; K[x_, y_] = (xy + 1)^8;
R[x_, y_] = K[x, y] − K[x, 0] K[0, y]/K[0, 0]; w[x_, y_] = p[x] ∗ R[x, y];
v[x_, d_] = B[α[x]] ∗ Gamma[d + 1] ∗ x^d ∗ MittagLefflerE[2, d + 1, −α[x] ∗ x^α[x]/(1 − α[x])];
fu[x_, y_] = 8yv[x, 1] + 28y^2v[x, 2] + 56y^3v[x, 3] + 70y^4v[x, 4]
+56y^5v[x, 5] + 28y^6v[x, 6] + 8y^7v[x, 7] + y^8v[x, 8];
m = 8; xx = Table[0, {i, 1, m}]; A = Table[0, {i, 1, m}, {j, 1, m}];
For[i = 1, i ≤ m, i + +, xx[[i]] = i/m];
For[i = 1, i ≤ m, i + +, For[j = 1, j ≤ m, j + +, A[[i, j]] = w[xx[[i]], xx[[j]]] + fu[xx[[i]] + xx[[j]]]]]];
v[x_] = tru[0]; f0[x] = p[x] ∗ tru[x] + v[x, 2] + v[x, 3]; f[x] = f0[x] − p[x] ∗ v[x];
b = Table[f[xx[[k]]], {i, 1, m}]; c = LinearSolve[A, b];
u[x_] = Sum_{i=1}^{m} c[[i]] ∗ R[x, xx[[i]]]; u[x_] = u[x] + v[x];
```

Table 1. Errors of numerical results for Problem 4.1.

Nodes x	Exact solution	Absolute error	Relative error
0.10	1.011	1.88×10^{-13}	1.86×10^{-13}
0.20	1.048	2.57×10^{-13}	2.45×10^{-13}
0.30	1.117	9.50×10^{-14}	8.50×10^{-14}
0.40	1.224	6.35×10^{-13}	5.19×10^{-13}
0.50	1.375	0	0
0.60	1.576	2.17×10^{-14}	1.38×10^{-14}
0.70	1.833	7.65×10^{-13}	4.17×10^{-13}
0.80	2.152	8.65×10^{-13}	4.02×10^{-13}
0.90	2.539	2.40×10^{-13}	9.46×10^{-14}
1.00	3.000	9.09×10^{-13}	3.03×10^{-13}

Problem 4.2

Solve the variable order fractional linear terminal value problems

$$\begin{cases} {}^{ABC}D^{\alpha}u(x) + 2u(x) = f(x), \quad x \in [0, 1), \\ u(1) = 3, \end{cases}$$

where $\alpha(x) = \sin x$, $f(x) = 2(x^4+2)+\frac{M(\alpha(x))}{1-\alpha(x)}24x^4E_{\alpha(x),5}(-\frac{\alpha(x)}{1-\alpha(x)}x^{\alpha(x)})$. The exact solution is $u(x) = x^4+2$.

Selecting $m = 8, N = 8$, $x_i = \frac{i-1}{N}, i = 1, 2, \ldots, N$, the obtained absolute and relative errors of numerical results using our method are listed in Tables 2.

Table 2. Errors of numerical results for Problem 4.2.

Nodes x	Exact solution	Absolute error	Relative error
0.00	2.0000	2.75×10^{-10}	1.37×10^{-10}
0.10	2.0001	1.02×10^{-10}	5.14×10^{-11}
0.20	2.0016	9.96×10^{-11}	4.97×10^{-11}
0.30	2.0081	1.08×10^{-10}	5.39×10^{-11}
0.40	2.0256	1.12×10^{-10}	5.56×10^{-11}
0.50	2.0625	1.10×10^{-10}	5.37×10^{-11}
0.60	2.1296	1.05×10^{-10}	4.96×10^{-11}
0.70	2.2401	1.08×10^{-10}	4.83×10^{-11}
0.80	2.4096	9.36×10^{-11}	3.88×10^{-11}
0.90	2.6561	4.38×10^{-11}	1.64×10^{-11}

Problem 4.3

We apply our method to the nonlinear variable order fractional IVPs as follows

$$\begin{cases} {}^{ABC}D^{\alpha}u(x) + \sinh xu(x) + \sin(u) = f(x), \quad x \in (0,1], \\ u(0) = 1, \end{cases}$$

where $\alpha(x) = 0.5x + 0.1$, $f(x) = \sinh x(x + x^3 + 1) + \frac{M(\alpha(x))}{1-\alpha(x)} x E_{\alpha(x),2}(-\frac{\alpha(x)}{1-\alpha(x)} x^{\alpha(x)}) + \frac{M(\alpha(x))}{1-\alpha(x)} 6x^3 E_{\alpha(x),4}(-\frac{\alpha(x)}{1-\alpha(x)} x^{\alpha(x)})$. Its true solution is $u(x) = x + x^3 + 1$.

Choosing $m = 8, N = 8, x_i = \frac{i}{N}, i = 1, 2, \ldots, N$, we plot the absolute and relative errors in Figure 1.

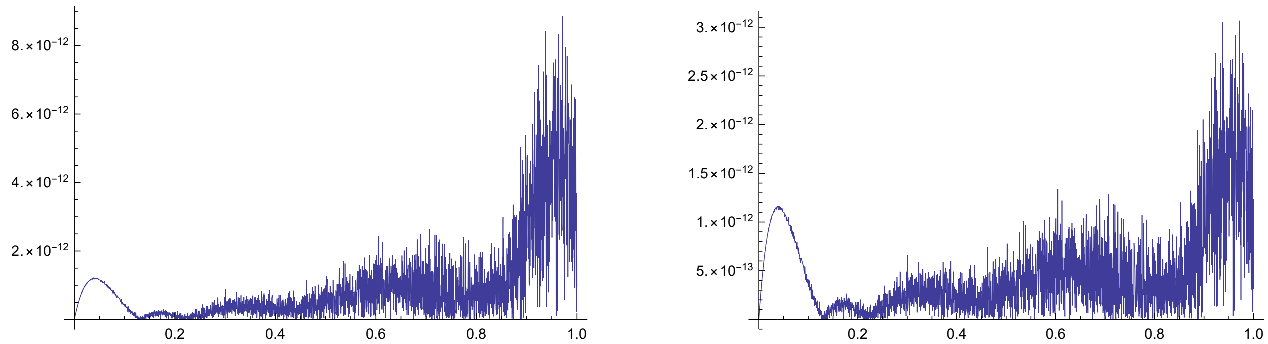

Figure 1. Absolute errors (left) and relative errors (right) for Problem 4.3

5. Conclusions

In this work, a new RKF based collocation technique is developed for Atangana-Baleanu variable order fractional problems. The proposed scheme is meshless and therefore it does not require any background meshes. From the numerical results, it is found that the accuracy of obtained approximate solutions is high and can reach to $O(10^{-10})$. Also, for nonlinear fractional problems, our method can yield highly accurate numerical solutions. Hence, our new method is very effective and easy to implement for the considered problems.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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