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*Research article*

# On analytic multivalent functions associated with lemniscate of Bernoulli

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**Abstract:** In this paper, we establish some sufficient conditions for analytic functions associated with lemniscate of Bernoulli. In particular, we determine conditions on $\alpha$ such that

$$1+\alpha\frac{z^{2+p(j-1)}g'(z)}{pg^{j}(z)},\ \text{for each } j=0,1,2,3,$$

are subordinated by Janowski function, then $\frac{g(z)}{z^p}\prec\sqrt{1+z}$, $(z\in\mathfrak{D})$. By choosing particular values of functions $g$, we obtain some sufficient conditions for multivalent starlike functions associated with lemniscate of Bernoulli.

**Keywords:** multivalent functions; subordination; lemniscate of Bernoulli; Janowski functions
**Mathematics Subject Classification:** 30C45, 30C50

## 1. Introduction and definitions

To understand in a clear way the notions used in our main results, we need to add here some basic literature of Geometric function theory. For this we start first with the notation $\mathcal{A}$ which denotes the class of holomorphic or analytic functions in the region $\mathfrak{D}=\{z\in\mathbb{C}:|z|<1\}$ and if a function $g\in\mathcal{A}$, then the relations $g(0)=g'(0)-1=0$ must hold. Also, all univalent functions will be in a subfamily $\mathcal{S}$ of $\mathcal{A}$. Next we consider to define the idea of subordinations between analytic functions $g_1$ and $g_2$, indicated by $g_1(z)\prec g_2(z)$, as; the functions $g_1,g_2\in\mathcal{A}$ are connected by the relation of subordination, if there exists an analytic function $w$ with the restrictions $w(0)=0$ and $|w(z)|<1$ such that $g_1(z)=g_2(w(z))$. Moreover, if the function $g_2\in\mathcal{S}$ in $\mathfrak{D}$, then we obtain:

$$g_1(z)\prec g_2(z)\Leftrightarrow[g_1(0)=g_2(0)\ \ \&\ \ g_1(\mathfrak{D})\subset g_1(\mathfrak{D})].$$

In 1992, Ma and Minda [16] considered a holomorphic function $\varphi$ normalized by the conditions $\varphi(0) = 1$ and $\varphi'(0) > 0$ with $Re\varphi > 0$ in $\mathfrak{D}$. The function $\varphi$ maps the disc $\mathfrak{D}$ onto region which is star-shaped about 1 and symmetric along the real axis. In particular, the function $\varphi(z) = (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$ maps $\mathfrak{D}$ onto the disc on the right-half plane with centre on the real axis and diameter end points $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$. This interesting familiar function is named as Janowski function [10]. The image of the function $\varphi(z) = \sqrt{1 + z}$ shows that the image domain is bounded by the right-half of the Bernoulli lemniscate given by $|w^2 - 1| < 1$, [25]. The function $\varphi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$ maps $\mathfrak{D}$ into the image set bounded by the cardioid given by $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0$, [21] and further studied in [23]. The function $\varphi(z) = 1 + \sin z$ was examined by Cho and his coauthors in [3] while $\varphi(z) = e^z$ is recently studied in [17] and [24]. Further, by choosing particular $\varphi$, several subclasses of starlike functions have been studied. See the details in [2, 4, 5, 11, 12, 14, 19].

Recently, Ali et al. [1] have obtained sufficient conditions on $\alpha$ such that

$$1 + zg'(z)/g^n(z) \prec \sqrt{1 + z} \quad \Rightarrow \quad g(z) \prec \sqrt{1 + z}, \quad \text{for } n = 0, 1, 2.$$

Similar implications have been studied by various authors, for example see the works of Halim and Omar [6], Haq et al [7], Kumar et al [13, 15], Paprocki and Sokól [18], Raza et al [20] and Sharma et al [22].

In 1994, Hayman [8] studied multivalent ($p$-valent) functions which is a generalization of univalent functions and is defined as: an analytic function $g$ in an arbitrary domain $\mathcal{D} \subset \mathbb{C}$ is said to be $p$-valent, if for every complex number $\omega$, the equation $g(z) = \omega$ has maximum $p$ roots in $\mathcal{D}$ and for a complex number $\omega_0$ the equation $g(z) = \omega_0$ has exactly $p$ roots in $\mathcal{D}$. Let $\mathcal{A}_p$ $(p \in \mathbb{N} = \{1, 2, \ldots\})$ denote the class of functions, say $g \in \mathcal{A}_p$, that are multivalent holomorphic in the unit disc $\mathfrak{D}$ and which have the following series expansion:

$$g(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ \ (z \in \mathfrak{D}). \tag{1.1}$$

Using the idea of multivalent functions, we now introduce the class $\mathcal{SL}_p^*$ of multivalent starlike functions associated with lemniscate of Bernoulli and as given below:

$$\mathcal{SL}_p^* = \left\{ g(z) \in \mathcal{A}_p : \frac{zg'(z)}{pg(z)} \prec \sqrt{1 + z}, \ \ (z \in \mathfrak{D}) \right\}.$$

In this article, we determine conditions on $\alpha$ such that for each

$$1 + \alpha \frac{z^{2+p(j-1)} g'(z)}{p g^j(z)}, \text{ for each } j = 0, 1, 2, 3,$$

are subordinated to Janowski functions implies $\frac{g(z)}{z^p} \prec \sqrt{1 + z}$, $(z \in \mathfrak{D})$. These results are then utilized to show that $g$ are in the class $\mathcal{SL}_p^*$.

### *1.1. Lemma*

Let $w$ be analytic non-constant function in $\mathfrak{D}$ with $w(0) = 0$. If

$$|w(z_0)| = \max\{|w(z)|, \quad |z| \leq |z_0|\}, \ \ z \in \mathfrak{D},$$

then there exists a real number $m$ $(m \geq 1)$ such that $z_0 w'(z_0) = m w(z_0)$.
This Lemma is known as Jack's Lemma and it has been proved in [9].

## 2. Main results

*2.1. Theorem*

Let $g \in \mathcal{A}_p$ and satisfying

$$1+\frac{\alpha z^{1-p} g'(z)}{p} \prec \frac{1+Az}{1+Bz},$$

with the restriction on $\alpha$ is

$$|\alpha| \geq \frac{2^{\frac{3}{2}} p(A-B)}{1-|B|-4p(1+|B|)}. \tag{2.1}$$

Then

$$\frac{g(z)}{z^p} \prec \sqrt{1+z}.$$

*Proof*

Let us define a function

$$p(z)=1+\frac{\alpha z^{1-p} g'(z)}{p}, \tag{2.2}$$

where the function $p$ is analytic in $\mathfrak{D}$ with $p(0)=1$. Also consider

$$\frac{g(z)}{z^p}=\sqrt{1+w(z)}. \tag{2.3}$$

Now to prove our result we will only require to prove that $|w(z)|<1$. Logarithmically differentiating (2.3) and then using (2.2), we get

$$p(z)=1+\frac{\alpha z w'(z)}{2p\sqrt{1+w(z)}}+\alpha\sqrt{1+w(z)},$$

and so

$$\begin{aligned}\left|\frac{p(z)-1}{A-Bp(z)}\right| &= \left|\frac{\frac{\alpha z w'(z)}{2p\sqrt{1+w(z)}}+\alpha\sqrt{1+w(z)}}{A-B\left(1+\frac{\alpha z w'(z)}{2p\sqrt{1+w(z)}}+\alpha\sqrt{1+w(z)}\right)}\right| \\ &= \left|\frac{\alpha z w'(z)+2p\alpha(1+w(z))}{2p(A-B)\sqrt{1+w(z)}-B(\alpha z w'(z)+2p\alpha(1+w(z)))}\right|.\end{aligned}$$

Now, we suppose that a point $z_0 \in \mathfrak{D}$ occurs such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1.$$

Also by Lemma 1.1, a number $m \geq 1$ exists with $z_0 w'(z_0)=m w(z_0)$. In addition, we also suppose that $w(z_0)=e^{i\theta}$ for $\theta \in [-\pi,\pi]$. Then we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| = \left|\frac{\alpha m w(z_0)-2p\alpha(1+w(z_0))}{2p(A-B)\sqrt{1+w(z_0)}-B(\alpha m w(z_0)+2p\alpha(1+w(z_0)))}\right|,$$

$$\geq \frac{|\alpha| m - 2p |\alpha| \left(\left|1 + e^{i\theta}\right|\right)}{2p (A - B) \sqrt{|1 + e^{i\theta}|} + |B| (|\alpha| m + 2p\alpha |1 + e^{i\theta}|)},$$

$$\geq \frac{|\alpha| m - 4p |\alpha|}{2^{\frac{3}{2}} p (A - B) + |B| |\alpha| (m + 4p)}.$$

Now if

$$\phi (m) = \frac{|\alpha| (m - 4p)}{2^{\frac{3}{2}} p (A - B) + |B| |\alpha| (m + 4p)},$$

then

$$\phi' (m) = \frac{2^{\frac{3}{2}} p (A - B) |\alpha| + 8 |\alpha|^2 p |B|}{\left(2^{\frac{3}{2}} p (A - B) + |B| |\alpha| (m + 4p)\right)^2} > 0,$$

which illustrates that the function $\phi (m)$ is increasing and hence $\phi (m) \geq \phi (1)$ for $m \geq 1$, so

$$\left|\frac{p (z_0) - 1}{A - Bp (z_0)}\right| \geq \frac{|\alpha| (1 - 4p)}{2^{\frac{3}{2}} p (A - B) + |B| |\alpha| (1 + 4p)}.$$

Now, by using (2.1) , we have

$$\left|\frac{p (z_0) - 1}{A - Bp (z_0)}\right| \geq 1$$

which contradicts the fact that $p (z) \prec \frac{1+Az}{1+Bz}$. Thus $|w (z)| < 1$ and so we get the desired result. Taking $g (z) = \frac{z^{p+1} f'(z)}{pf(z)}$ in the last result, we obtain the following Corollary:

*2.2. Corollary*

Let $f \in \mathcal{A}_p$ and satisfying

$$1 + \frac{\alpha z f' (z)}{p^2 f (z)} \left(p + 1 + \frac{z f'' (z)}{f' (z)} - \frac{z f' (z)}{f (z)}\right) \prec \frac{1 + Az}{1 + Bz}, \tag{2.4}$$

with the condition on $\alpha$ is

$$|\alpha| \geq \frac{2^{\frac{3}{2}} p (A - B)}{1 - |B| - 4p (1 + |B|)}.$$

Then $f \in \mathcal{SL}_p^*$.

*2.3. Theorem*

If $g \in \mathcal{A}_p$ such that

$$1 + \frac{\alpha}{p} \frac{z g' (z)}{g (z)} \prec \frac{1 + Az}{1 + Bz}, \tag{2.5}$$

with

$$|\alpha| \geq \frac{8p (A - B)}{1 - |B| - 4p (1 + |B|)}, \tag{2.6}$$

then

$$\frac{g (z)}{z^p} \prec \sqrt{1 + z}.$$

*Proof*

Let us choose a function $p$ by

$$p(z) = 1 + \alpha \frac{zg'(z)}{pg(z)},$$

in such a way that $p$ is analytic in $\mathfrak{D}$ with $p(0) = 1$. Also consider

$$\frac{g(z)}{z^p} = \sqrt{1 + w(z)}.$$

Using some simple calculations, we obtain

$$p(z) = 1 + \frac{\alpha z w'(z)}{2p(1 + w(z))} + \alpha,$$

and so

$$\begin{aligned} \left| \frac{p(z) - 1}{A - Bp(z)} \right| &= \left| \frac{\frac{\alpha z w'(z)}{2p(1+w(z))} + \alpha}{A - B\left(1 + \frac{\alpha z w'(z)}{2p(1+w(z))} + \alpha\right)} \right| \\ &= \left| \frac{\alpha z w'(z) + 2p\alpha(1 + w(z))}{2p(A - B)(1 + w(z)) - B(\alpha z w'(z) + 2p\alpha(1 + w(z)))} \right|. \end{aligned}$$

Let a point $z_0 \in \mathfrak{D}$ exists in such a way

$$\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, by virtue of Lemma 1.1, a number $m \ge 1$ occurs such that $z_0 w'(z_0) = m w(z_0)$. In addition, we set $w(z_0) = e^{i\theta}$, so we have

$$\begin{aligned} \left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| &= \left| \frac{\alpha m w(z_0) + 2p\alpha(1 + w(z_0))}{2p(A - B)(1 + w(z_0)) - B(\alpha m w(z_0) + 2p\alpha(1 + w(z_0)))} \right|, \\ &\ge \frac{|\alpha| m - 2p|\alpha| \left|1 + e^{i\theta}\right|}{2(A - B)|1 + e^{i\theta}| + |B||\alpha| m + 2p|B||\alpha||1 + e^{i\theta}|}, \\ &= \frac{|\alpha| m - 2p|\alpha| \sqrt{2 + 2\cos\theta}}{2(2(A - B) + |B||\alpha|)p\sqrt{2 + 2\cos\theta} + |B||\alpha| m}, \\ &\ge \frac{|\alpha|(m - 4p)}{4p(2(A - B) + |B||\alpha|) + |B||\alpha| m}. \end{aligned}$$

Now let

$$\phi(m) = \frac{|\alpha|(m - 4p)}{4p(2(A - B) + |B||\alpha|) + |B||\alpha| m},$$

it implies

$$\phi'(m) = \frac{|\alpha| 8p((A - B) + |\alpha||B|)}{(4p(2(A - B) + |B||\alpha|) + |B||\alpha| m)^2} > 0,$$

which illustrates that the function $\phi(m)$ is increasing and so $\phi(m) \ge \phi(1)$ for $m \ge 1$, hence

$$\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \ge \frac{|\alpha|(1 - 4p)}{4p(2(A - B) + |B||\alpha|) + |B||\alpha|}.$$

Now, by using (2.6) , we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

which contradicts (2.5) . Thus $|w(z)| < 1$ and so the desired proof is completed.
Putting $g(z) = \frac{z^{p+1} f'(z)}{p f(z)}$ in last Theorem, we get the following Corollary:

*2.4. Corollary*

If $f \in \mathcal{A}_p$ and satisfying

$$1 + \frac{\alpha}{p}\left(p + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz},$$

with

$$|\alpha| \geq \frac{8p(A-B)}{1-|B|-4p(1+|B|)},$$

then $f \in \mathcal{SL}_p^*$.

*2.5. Theorem*

If $g \in \mathcal{A}_p$ and satisfy the subordination relation

$$1 + \alpha \frac{z^{1-p} g'(z)}{p(g(z))^2} \prec \frac{1+Az}{1+Bz}, \tag{2.7}$$

with the condition on $\alpha$

$$|\alpha| \geq \frac{2^{\frac{5}{2}} p(A-B)}{1-|B|-4p(1+|B|)} \tag{2.8}$$

is true, then

$$\frac{g(z)}{z^p} \prec \sqrt{1+z}.$$

*Proof*

Let us define a function

$$p(z) = 1 + \alpha \frac{z^{1-p} g'(z)}{p(g(z))^2}.$$

Then $p$ is analytic in $\mathfrak{D}$ with $p(0) = 1$. Also let us consider

$$\frac{g(z)}{z^p} = \sqrt{1+w(z)}.$$

Using some simplification, we obtain

$$p(z) = 1 + \frac{\alpha z w'(z)}{2p(1+w(z))^{\frac{3}{2}}} + \frac{\alpha}{\sqrt{1+w(z)}},$$

and so

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{\frac{\alpha z w'(z)}{2p(1+w(z))^{\frac{3}{2}}}+\frac{\alpha}{\sqrt{1+w(z)}}}{A-B\left(1+\frac{\alpha z w'(z)}{2p(1+w(z))^{\frac{3}{2}}}+\frac{\alpha}{\sqrt{1+w(z)}}\right)}\right|$$
$$= \left|\frac{\alpha z w'(z)+2p\alpha(1+w(z))}{2p(A-B)(1+w(z))^{\frac{3}{2}}-B\alpha z w'(z)-2p\alpha B(1+w(z))}\right|.$$

Let us choose a point $z_0 \in \mathfrak{D}$ such a way that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1.$$

Then, by the consequences of Lemma 1.1, a number $m \geq 1$ occurs such that $z_0 w'(z_0) = m w(z_0)$ and also put $w(z_0) = e^{i\theta}$,for $\theta \in [-\pi, \pi]$ , we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| = \left|\frac{\alpha m w(z_0)+2p\alpha(1+w(z_0))}{2p(A-B)(1+w(z_0))^{\frac{3}{2}}-B\alpha m w(z_0)-2p\alpha B(1+w(z_0))}\right|,$$
$$\geq \frac{|\alpha| m-2p|\alpha|\left|1+e^{i\theta}\right|}{2p(A-B)\left|1+e^{i\theta}\right|^{\frac{3}{2}}+|B||\alpha| m+2p|\alpha||B|\left|1+e^{i\theta}\right|},$$
$$= \frac{|\alpha| m-4p|\alpha|}{2^{\frac{5}{2}}p(A-B)+|B||\alpha| m+4p|\alpha||B|},$$
$$\geq \frac{|\alpha|(m-4p)}{2^{\frac{5}{2}}p(A-B)+|B||\alpha| m+4p|\alpha||B|} = \phi(m) \text{ (say).}$$

Then

$$\phi'(m) = \frac{2^{\frac{5}{2}}p(A-B)+8|\alpha|^2|B|p}{\left(2^{\frac{5}{2}}p(A-B)+B|\alpha| m+4p|\alpha| B\right)^2} > 0,$$

which demonstrates that the function $\phi(m)$ is increasing and thus $\phi(m) \geq \phi(1)$ for $m \geq 1$, hence

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq \frac{|\alpha|(1-4p)}{2^{\frac{5}{2}}p(A-B)+|B||\alpha|+4p|\alpha||B|}.$$

Now, using (2.8) , we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

which contradicts (2.7). Thus $|w(z)| < 1$ and so we get the required proof.
If we set $g(z) = \frac{z^{p+1}f'(z)}{pf(z)}$ in last theorem, we easily have the following Corollary:

*2.6. Corollary*

Assume that

$$|\alpha| \geq \frac{2^{\frac{5}{2}}p(A-B)}{1-|B|-4p(1+|B|)},$$

and if $f \in \mathcal{A}_p$ satisfy

$$1+\frac{\alpha f(z)}{z^{2p+1}f'(z)}\left(p+1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec\frac{1+Az}{1+Bz},$$

then $f \in \mathcal{SL}_p^*$.

*2.7. Theorem*

If $g \in \mathcal{A}_p$ satisfy the subordination

$$1+\alpha\frac{z^{1-2p}g'(z)}{p(g(z))^3}\prec\frac{1+Az}{1+Bz},$$

with restriction on $\alpha$ is

$$|\alpha|\geq\frac{8p(A-B)}{1-|B|-4p(1+|B|)}. \tag{2.9}$$

then

$$\frac{g(z)}{z^p}\prec\sqrt{1+z}.$$

*Proof.* Let us define a function

$$p(z)=1+\alpha\frac{z^{1-2p}g'(z)}{p(g(z))^3},$$

where $p$ is analytic in $\mathfrak{D}$ with $p(0)-1=0$. Also let

$$\frac{g(z)}{z^p}=\sqrt{1+w(z)}.$$

Using some simple calculations, we obtain

$$p(z)=1+\frac{\alpha zw'(z)}{2p(1+w(z))^2}+\frac{\alpha}{1+w(z)},$$

and so

$$\begin{aligned}\left|\frac{p(z)-1}{A-Bp(z)}\right| &= \left|\frac{\frac{\alpha zw'(z)}{2p(1+w(z))^2}+\frac{\alpha}{1+w(z)}}{A-B\left(1+\frac{\alpha zw'(z)}{2p(1+w(z))^2}+\frac{\alpha}{1+w(z)}\right)}\right|\\ &= \left|\frac{\alpha zw'(z)+2p\alpha(1+w(z))}{2p(A-B)(1+w(z))^2-B\alpha zw'(z)-2p\alpha B(1+w(z))}\right|.\end{aligned}$$

Let us pick a point $z_0 \in \mathfrak{D}$ in such a way that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1.$$

Then, by using Lemma 1.1, a number $m \geq 1$ exists such that $z_0w'(z_0)=mw(z_0)$ and put $w(z_0)=e^{i\theta}$, for $\theta \in [-\pi,\pi]$, we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| = \left|\frac{\alpha mw(z_0)+2p\alpha(1+w(z_0))}{2p(A-B)(1+w(z_0))^2-B\alpha mw(z_0)-2p\alpha B(1+w(z_0))}\right|$$

$$\geq \frac{|\alpha| m - 2p |\alpha| \left|1 + e^{i\theta}\right|}{2p (A - B) |1 + e^{i\theta}|^2 + |B| |\alpha| m + 2p |\alpha| |B| |1 + e^{i\theta}|}$$
$$= \frac{|\alpha| m - 2p |\alpha| \sqrt{2 + 2\cos\theta}}{2p (A - B) \left(\sqrt{2 + 2\cos\theta}\right)^2 + |B| |\alpha| m + 2p |\alpha| |B| \sqrt{2 + 2\cos\theta}}$$
$$\geq \frac{|\alpha| (m - 4p)}{8p (A - B) + |B| |\alpha| m + 4p |\alpha| |B|},$$

Now let

$$\phi (m) = \frac{|\alpha| (m - 4p)}{8p (A - B) + |B| |\alpha| m + 4p |\alpha| |B|},$$

then

$$\phi' (m) = \frac{8p |\alpha| (A - B) + 8 |\alpha|^2 |B| p}{(8p (A - B) + |B| |\alpha| m + 4p |\alpha| |B|)^2} > 0$$

which shows that $\phi (m)$ is an increasing function and hence it will have its minimum value at $m = 1$, so

$$\left|\frac{p (z_0) - 1}{A - Bp (z_0)}\right| \geq \frac{|\alpha| (1 - 4p)}{8p (A - B) + |B| |\alpha| + 4p |\alpha| |B|}.$$

Using (2.9) , we easily obtain

$$\left|\frac{p (z_0) - 1}{A - Bp (z_0)}\right| \geq 1,$$

which is a contradiction to the fact that $p (z) \prec \frac{1+Az}{1+Bz}$, and so $|w (z)| < 1$. Hence we get the desired result. □

If we put $g (z) = \frac{z^{p+1} f'(z)}{p f(z)}$ in last Theorem, we achieve the following result:

*2.8. Corollary*

If $f \in \mathcal{A}_p$ and satisfy the condition

$$|\alpha| \geq \frac{8p (A - B)}{1 - |B| - 4p (1 + |B|)},$$

and

$$1 + \alpha \frac{p (f (z))^2}{z^{3p+2} (f' (z))^2} \left(p + 1 + \frac{zf'' (z)}{f' (z)} - \frac{zf' (z)}{f (z)}\right) \prec \frac{1 + Az}{1 + Bz},$$

then $f \in \mathcal{SL}_p^*$.

## Conflict of interest

All authors declare no conflict of interest in this paper.

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