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Research article

Perturbed mixed variational-like inequalities

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Abstract: Perturbed mixed variational-like inequalities is a newly developed class of variational-like inequalities. The existence of a unique solution of perturbed mixed variational-like inequalities is being conducted by using auxiliary principle technique. As special cases, many known and new results are obtained from our results. It is an interesting problem to develop some efficient numerical techniques for solving perturbed variational-like inequalities.

Keywords: perturbed mixed variational-like inequality; auxiliary principle technique; existence
Mathematics Subject Classification: 90C40, 26D15, 26D10, 90C23

1. Introduction

Variational inequalities introduced by Stampacchia [1] have played an important role in studying various problems which arise in transportation, elasticity, economics, operational research, optimization and in the field of pure and applied sciences, see [1–16] and references therein. The minimum of the differentiable convex function on convex set can be characterized by variational inequalities.

In recent years, the concept of convexity has been generalized and extended in many directions, which has potential and important applications in many fields. A significant generalization of convex functions is that of invex functions, which was introduced by Hanson [4]. In mathematical programming, Hanson [4] showed many results involving convex functions actually hold for invex functions. A generalized class of convex functions, preinvex functions was introduced by Ben-Israel and Mond [2] and Weir and Jeyakumar [16]. In optimization problems, Weir and Mond [17] studied that the convex functions can be replaced by preinvex functions. Noor [8] and Yang and Chen [18] proved that the optimal conditions of differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities, known as variational-like inequalities, which were introduced and studied by Parida and Sen [13].

In this paper, we introduce and consider a new class of variational-like inequalities which is known as perturbed mixed variational-like inequalities. The existence of the solution of perturbed mixed variational-like inequalities cannot be studied by resolvent method and projection method as these perturbed mixed variational-like inequalities involve the nonlinear terms and bifunction $\eta(.,.)$. This fact motivated us to use the auxiliary principle technique to develop the existence theory for variational-like inequalities. Auxiliary principle technique was introduced and developed by Lions and Stampacchia [5] and Glowinski et al. [3]. To study the existence of a solution of perturbed mixed variational-like inequalities, we use the auxiliary principle technique. Some special cases are also considered. Our results show a significant refinement and improvement of the already existing results.

2. Preliminaries

Let H be a real Hilbert space. The inner product and norm are denoted by $\langle .,. \rangle$ and $\| . \|$ respectively. Let K_η be a nonempty invex set in H and $\eta(.,.) : H \times H \to R$ be a continuous bifunction.

First, we recall the well-known basic definitions and concepts.

Definition 2.1: [7] Let K be a nonempty closed set in H. The set K is said to be a convex set, if

$$u + t(v - u) \in K, \ \forall u, v \in K, t \in [0, 1].$$

Definition 2.2: [7] A function $F : K \to R$ is said to be a convex function, if

$$F(u + t(v - u)) \leq (1 - t)F(u) + tF(v), \ \forall u, v \in K, t \in [0, 1].$$

Definition 2.3: [15] A function $F : K \to R$ is said to be a strongly convex function, if there exists a constant $\mu > 0$, such that

$$F(u + t(v - u)) \leq (1 - t)F(u) + tF(v) - \mu t(1 - t) \| v - u \|^2, \ \forall u, v \in K, t \in [0, 1].$$

Definition 2.4: [16, 17] A set K_η is said to be an invex set, if there exists a bifunction $\eta(.,.)$, such that

$$u + t\eta(v, u) \in K_\eta, \ \forall u, v \in K_\eta, t \in [0, 1].$$

Definition 2.5: [6, 17] A function $F : K_\eta \to R$ is said to be a preinvex function, if there exists a bifunction $\eta(.,.)$, such that

$$F(u + t\eta(v, u)) \leq (1 - t)F(u) + tF(v), \ \forall u, v \in K_\eta, t \in [0, 1].$$

Definition 2.6: [10, 11] A function $F : K_\eta \to R$ is said to be a strongly preinvex function with respect to a bifunction $\eta(.,.)$, if there exists a constant $\mu > 0$, such that

$$F(u + t\eta(v, u)) \leq (1 - t)F(u) + tF(v) - \mu t(1 - t) \| \eta(v, u) \|^2, \ \forall u, v \in K_\eta, t \in [0, 1].$$

We note that, if $\eta(v, u) = v - u$, the invex set K_η reduces to the convex set K, preinvex functions and strongly preinvex functions reduce to convex functions and strongly convex functions.

Now we consider the functional $I[v]$, defined as

$$I[v] = F(v) + \Phi(v), \tag{2.1}$$

where F is a differentiable preinvex function and ϕ is a nondifferentiable strongly preinvex function. The minimum $u \in K_\eta$ of the functional $I[v]$, defined by (2.1), can be characterized by a class of variational-like inequalities.

$$\langle F'(u)\ ,\ \eta(v,u)\rangle + \Phi(v) - \Phi(u) + \mu \parallel \eta(v,u) \parallel^2 \geq 0\ ,\ \forall v \in K_{\eta.} \tag{2.2}$$

The inequality (2.2) is called perturbed mixed variational-like inequality.

It is well-known that in many significant applications, variational-like inequalities do not arise as a result of minimization problem. This motivated us to consider a more general class of variational-like inequalities, of which (2.2) is a special cases.

Given an operator $T : H \to R$ and a function $\Phi : H \to R$, we consider the problem of finding $u \in H$ and constant $\mu > 0$, such that

$$\langle Tu\ ,\ \eta(v,u)\rangle + \Phi(v) - \Phi(u) + \mu \parallel \eta(v,u) \parallel^2 \geq 0\ ,\ \forall v \in H. \tag{2.3}$$

The inequality (2.3) is called perturbed mixed variational-like inequality (PMVLI).

If $Tu = F'(u)$, then PMVLI (2.3) includes inequality (2.2) as a special case. These perturbed mixed variational-like inequalities are more general and include classical mixed variational-like inequalities and related optimization problems as special cases. For the applications of PMVLI (2.3), see [2,3,5,9].

We now discuss some special cases of PMVLI (2.3).

(I) For $\eta(v,u) = v - u$, problem (2.3) reduces to

$$\langle Tu\ ,\ v-u\rangle + \Phi(v) - \Phi(u) + \mu \parallel v-u \parallel^2 \geq 0\ ,\ \forall v \in H, \tag{2.4}$$

which is known as strongly mixed variational inequality, introduced and studied by Noor [8].

(II) For $\mu = 0$, problem (2.3) is equivalent to finding $u \in H$, such that

$$\langle Tu\ ,\ \eta(v,u)\rangle + \Phi(v) - \Phi(u) \geq 0\ ,\ \forall v \in H, \tag{2.5}$$

which is known as the mixed variational-like inequality (MVLI), introduced by Noor [9]. For the formulation of numerical methods and applications of (2.5), see [9,10,11,12].

(III) For $\eta(v,u) = v - u$, problem (2.5) is equivalent to finding $u \in H$, such that

$$\langle Tu\ ,\ v-u\rangle + \Phi(v) - \Phi(u) \geq 0\ ,\ \forall v \in H, \tag{2.6}$$

which is known as mixed variational inequality (MVI), introduced and studied by Lions and Stampacchia [4]. Mixed variational inequalities have applications in elasticity, structural engineering and electronic network, see [3,9,10,11] and the references therein.

(**IV**) If $\Phi(.)$ is an indicator function of a closed invex set K_η in H, that is

$$I_\Phi(u) = \begin{cases} 0, & u \in K_\eta. \\ \infty, & \text{otherwise} \end{cases}$$

then problem (2.5) is equivalent to finding $u \in K_\eta$, such that

$$\langle Tu\ ,\ \eta(v,u)\rangle \geq 0\ ,\ \forall v \in K_\eta, \tag{2.7}$$

which is known as the variational-like inequality (VLI), introduced by Parida and Sen [13]. For the applications of variational-like inequalities and optimization problems, see [9,10,12].

(**V**) For $\eta(v,u) = v - u$, problem (2.7) reduces to

$$\langle Tu\ ,\ v-u\rangle \geq 0\ ,\ \forall v \in K, \tag{2.8}$$

which is known as variational inequality, proposed and studied by Stampacchia [1] and Lions and Stampacchia [5]. Several equilibrium and obstacle problems in finance, economics, mathematical and engineering sciences can be studied by variational inequality of type (2.8), see [3, 5, 8, 13, 16, 18] and the references therein.

For a proper and suitable choice of T, η, Φ and the invex set, one can get a number of known and unknown variational inequalities and complementary problems. It is evident that the problems (2.4)–(2.8) are special cases of the PMVLI (2.3). In fact, PMVLI (2.3) is the most general and unifying one, which is the inspirational force of our work.

We also need the following assumption regarding the bifunction $\eta(.,.)$ to obtain the main results.

Assumption 2.1: The bifunction $\eta(.,.) : H \times H \to H$ satisfies the following condition

$$\eta(u,v) = \eta(u,z) + \eta(z,v)\ ,\ \forall u,v,z \in H.$$

We assume that $\eta(v,u)$ is skew symmetric, that is, $\eta(v,u) = -\eta(u,v)$, $\forall u,v \in H$, unless otherwise.

The assumption 2.1 played an important part in the study of the existence of a solution of variational-like inequalities, see [9, 10].

Definition 2.6: [9] An operator $T : H \to H$ is said to be:

(i) strongly $\eta-$monotone, if there exists a constant $\alpha > 0$ such that

$$\langle T(u) - T(v)\ ,\ \eta(u,v)\rangle \geq \alpha \parallel \eta(u,v) \parallel^2\ ,\ \forall u,v \in H,$$

(ii) $\eta-$ monotone, if

$$\langle T(u) - T(v)\ ,\ \eta(u,v)\rangle \geq 0\ ,\ \forall u,v \in H,$$

(iii) partially relaxed strongly $\eta-$monotone, if there exists a constant $\alpha > 0$ such that

$$\langle T(u) - T(v)\ ,\ \eta(z,v)\rangle \geq -\alpha \parallel \eta(u,z) \parallel^2\ ,\ \forall u,v,z \in H,$$

(iv) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\| T(u) - T(v) \| \leq \beta \| v - u \| , \ \forall u, v \in H.$$

We note that for $z = u$, partially relaxed strongly monotonicity reduces to monotonicity.

Definition 2.7: [11] The bifunction $\eta(.,.) : H \times H \rightarrow H$ is said to be:

(a) **Strongly monotone:** if there exists a constant $\sigma > 0$, such that

$$\langle \eta(v, u), v - u \rangle \geq \sigma \| v - u \|^2 , \ \forall u, v \in H.$$

(b) **Lipschitz continuous:** if there exists a constant $\delta > 0$, such that

$$\| \eta(v, u) \| \leq \delta \| v - u \| , \ \forall u, v \in H.$$

From (a) and (b), it is observed that $\sigma \leq \delta$.

If $\eta(v, u) = Tv - Tu$, then definition (2.7) reduces to the strongly monotonicity and Lipschitz continuity of the nonlinear operator T.

3. Existence results

In this section, we consider the conditions under which the perturbed mixed variational-like inequality (2.3) has an unique solution. We use the auxiliary principle technique of Glowinski et al. [3] to study the existence of the solution of perturbed mixed variational-like inequality (2.3).

Theorem 3.1: Let the operator $T : H \rightarrow H$ be strongly monotone and Lipschitz continuous with constants $\alpha > 0$, $\beta > 0$ and the bifunction $\eta(.,.) : H \times H \rightarrow H$ be strongly monotone and Lipschitz continuous with constants $\sigma > 0$, $\delta > 0$, respectively. If assumption 2.1 holds and there is a constant $\rho > 0$, such that

$$0 < \rho < 2\frac{\alpha - (\gamma + 2\mu\delta^2)}{\beta^2 - (\gamma + 2\mu\delta^2)^2}, \quad \rho < \frac{1}{(\gamma + 2\mu\delta^2)}, \quad \alpha > \gamma + 2\mu\delta^2 \tag{3.1}$$

where

$$\gamma = \beta\sqrt{1 - 2\sigma + \delta^2}, \tag{3.2}$$

then there exists a unique solution of (2.3).

Proof (a) Uniqueness: Let $u_1 \neq u_2 \in H$, be two solutions of problem (2.3). Then, we have

$$\langle Tu_1 , \ \eta(v, u_1) \rangle + \Phi(v) - \Phi(u_1) \geq -\mu \| \eta(v, u_1) \|^2 , \ \forall v \in H. \tag{3.3}$$

$$\langle Tu_2 , \ \eta(v, u_2) \rangle + \Phi(v) - \Phi(u_2) \geq -\mu \| \eta(v, u_2) \|^2 , \ \forall v \in H. \tag{3.4}$$

Replacing v by u_2 in (3.3) and v by u_1 in (3.4), adding the resultants and using the Assumption 2.1 , we have

$$\langle Tu_1 - Tu_2 \ , \ \eta(u_1, u_2)\rangle \leq 2\mu \parallel \eta(u_1, u_2) \parallel^2, \tag{3.5}$$

which can be written as

$$\langle Tu_1 - Tu_2 \ , \ u_1 - u_2\rangle \leq \langle Tu_1 - Tu_2 \ , \ u_1 - u_2 - \eta(u_1, u_2)\rangle + 2\mu \parallel \eta(u_1, u_2) \parallel^2 .$$

Since T is a strongly monotone and Lipschitz continuous with constants $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}\alpha \parallel u_1 - u_2 \parallel^2 &\leq \langle Tu_1 - Tu_2 \ , \ u_1 - u_2\rangle \\ &\leq \parallel Tu_1 - Tu_2 \parallel\parallel u_1 - u_2 - \eta(u_1, u_2) \parallel + 2\mu \parallel \eta(u_1, u_2) \parallel^2 \\ &\leq \beta \parallel u_1 - u_2 \parallel\parallel u_1 - u_2 - \eta(u_1, u_2) \parallel + 2\mu \parallel \eta(u_1, u_2) \parallel^2 \\ &\leq \beta \parallel u_1 - u_2 \parallel\parallel u_1 - u_2 - \eta(u_1, u_2) \parallel + 2\mu\delta^2 \parallel u_1 - u_2 \parallel^2 .\end{aligned} \tag{3.6}$$

Since $\eta(.,.)$ is strongly monotone and Lipschitz continuous with constants $\sigma > 0$ and $\delta > 0$, we have

$$\begin{aligned}\parallel u_1 - u_2 - \eta(u_1, u_2) \parallel^2 &\leq \parallel u_1 - u_2 \parallel^2 - 2\langle u_1 - u_2 \ , \ \eta(u_1, u_2)\rangle + \parallel \eta(u_1, u_2) \parallel^2 \\ &\leq (1 - 2\sigma + \delta^2) \parallel u_1 - u_2 \parallel^2 .\end{aligned} \tag{3.7}$$

From (3.2), (3.6) and (3.7), we get

$$\begin{aligned}\alpha \parallel u_1 - u_2 \parallel^2 &\leq (\beta \sqrt{1 - 2\sigma + \delta^2} + 2\mu\delta^2) \parallel u_1 - u_2 \parallel^2 \\ &= (\gamma + 2\mu\delta^2) \parallel u_1 - u_2 \parallel^2,\end{aligned} \tag{3.8}$$

It follows that

$$(\alpha - \gamma - 2\mu\delta^2) \parallel u_1 - u_2 \parallel^2 \leq 0,$$

Thus, for $\alpha > \gamma + 2\mu\delta^2$, we get

$$\parallel u_1 - u_2 \parallel^2 \leq 0,$$

which implies that $u_1 = u_2$ is the uniqueness of the solution of (2.3).

Existence: We use the auxiliary principle technique to prove the existence of a solution of problem (2.3), which is mainly due to Glowinski et al. [3] as developed by Noor [10].

For a given $u \in H$ satisfying (2.3), we consider the problem of finding $w \in H$, such that

$$\langle \rho Tu \ , \ \eta(v, w)\rangle + \langle w - u, v - w\rangle + \rho\, \Phi(v) - \rho\, \Phi(w) + \rho\, \mu \parallel \eta(v, w) \parallel^2 \geq 0, \ \forall v \in H, \tag{3.9}$$

where $\rho > 0$ is the constant.

The inequality (3.9) is called auxiliary perturbed mixed variational-like inequality. The relation (3.9) defines a mapping $w = w(u)$ between the problem (2.3) and (3.9). In order to prove the existence of a solution of problem (2.3), it is enough to show that the connecting mapping is a contraction mapping and consequently has a unique fixed point.

For $w_1 \neq w_2 \in H$ (corresponding to $u_1 \neq u_2 \in H$), consider two solutions of (3.9). Then

$$\langle \rho T u_1 \,,\ \eta(v, w_1)\rangle + \langle w_1 - u_1, v - w_1\rangle + \rho\, \Phi(v) - \rho\, \Phi(w_1) + \rho\, \mu \parallel \eta(v, w_1) \parallel^2 \geq 0\,,\ \forall v \in H. \tag{3.10}$$

and

$$\langle \rho T u_2 \,,\ \eta(v, w_2)\rangle + \langle w_2 - u_2, v - w_2\rangle + \rho\, \Phi(v) - \rho\, \Phi(w_2) + \rho\, \mu \parallel \eta(v, w_2) \parallel^2 \geq 0\,,\ \forall v \in H. \tag{3.11}$$

By taking $v = w_2$ in (3.10) and $v = w_1$ in (3.11) and by adding the resultants and using the Assumption 2.1, we get

$$\begin{aligned}\langle w_1 - w_2, w_1 - w_2\rangle &\leq \langle u_1 - u_2, w_1 - w_2\rangle - \rho\langle T u_1 - T u_2\,,\ \eta(w_1, w_2)\rangle\\ &\quad + 2\rho\, \mu \parallel \eta(w_1, w_2) \parallel^2\\ &= \langle u_1 - u_2 - \rho(T u_1 - T u_2), w_1 - w_2\rangle\\ &\quad + \rho\langle T u_1 - T u_2\,,\ w_1 - w_2 - \eta(w_1, w_2)\rangle + 2\rho\, \mu \parallel \eta(w_1, w_2) \parallel^2 .\end{aligned}$$

Thus, we have

$$\begin{aligned}\parallel w_1 - w_2 \parallel^2 &\leq \parallel u_1 - u_2 - \rho(T u_1 - T u_2) \parallel\ \parallel w_1 - w_2 \parallel\\ &\quad + \rho \parallel T u_1 - T u_2 \parallel \parallel w_1 - w_2 - \eta(w_1, w_2) \parallel + 2\rho\, \mu\delta^2 \parallel w_1 - w_2 \parallel^2 .\end{aligned} \tag{3.12}$$

Since T is strongly monotone and Lipschitz continuous with constants $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}\parallel u_1 - u_2 - \rho(T u_1 - T u_2) \parallel^2 &\leq \parallel u_1 - u_2 \parallel^2 - 2\rho\langle T u_1 - T u_2\,,\ u_1 - u_2\rangle\\ &\quad + \rho^2 \parallel T u_1 - T u_2 \parallel^2\\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2) \parallel u_1 - u_2 \parallel^2 .\end{aligned} \tag{3.13}$$

By combining (3.2), (3.7), (3.12) and (3.13) and using the Lipschitz continuity of T, we have

$$\begin{aligned}\parallel w_1 - w_2 \parallel &\leq \{ \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\beta \sqrt{1 - 2\sigma + \delta^2}\} \parallel u_1 - u_2 \parallel\\ &\quad + 2\rho\mu\delta^2 \parallel w_1 - w_2 \parallel\\ &\leq (t(\rho) + \rho\gamma) \parallel u_1 - u_2 \parallel + 2\rho\mu\delta^2 \parallel w_1 - w_2 \parallel .\end{aligned}$$

where

$$t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\ ,$$

From above, it follows that

$$\parallel w_1 - w_2 \parallel \leq \{\frac{t(\rho) + \rho\gamma}{1 - 2\rho\mu\delta^2}\} \parallel u_1 - u_2 \parallel = \theta \parallel u_1 - u_2 \parallel,$$

where

$$\theta = \frac{t(\rho) + \rho\gamma}{1 - 2\rho\mu\delta^2},$$

From (3.1), it follows that $\theta < 1$, so the mapping w defined by (3.9) is a contraction mapping and consequently, it has a fixed point $w(u) = u \in H$ which satisfy the perturbed mixed variational-like inequality (2.3). □

Special cases:

(I) If $\eta(v, u) = v - u$, Theorem (3.1) reduces to:

Theorem 3.2: [8] Let the operator $T : H \to H$ be strongly monotone and Lipschitz continuous with constants $\alpha > 0$, $\beta > 0$ respectively. If there is a constant $\rho > 0$, such that

$$0 < \rho < 2\frac{\alpha - 2\mu}{\beta^2}, \quad \rho < \frac{1}{4\mu}, \quad \mu < \frac{\alpha}{2},$$

then there exists a unique solution $u \in H$ satisfying strongly mixed variational inequality (2.4).
(II) If $\mu = 0$, Theorem (3.1) reduces to:

Theorem 3.3: [9] Let the operator $T : H \to H$ be strongly monotone and Lipschitz continuous with constants $\alpha > 0$, $\beta > 0$ and the bifunction $\eta(.,.) : H \times H \to H$ be strongly monotone and Lipschitz continuous with constants $\sigma > 0$, $\delta > 0$, respectively. If assumption 2.1 holds and there is a constant $\rho > 0$, such that

$$0 < \rho < 2\frac{\alpha - \gamma}{\beta^2 - \gamma^2}, \quad \gamma < \alpha,$$

then there exists a unique solution $u \in H$ satisfying mixed variational-like inequality (2.5).

For different choices of bifunction $\eta(.,.)$ and invex set K_η, one can obtain various known and new results as special cases of Theorem (3.1). This shows that our results are more general and unifying one.

It can be shown that the solution of the auxiliary perturbed mixed variational-like inequality (3.9) can be characterized as a minimum of the functional $I[w]$ on the invex set K_η in H, where

$$I[w] = \frac{1}{2}\langle w - u, w - u\rangle - \langle \rho Tu\, ,\ \eta(w, u)\rangle - \rho\, \Phi(w).$$

The functional $I[w]$ can be used to construct the class of gap (merit) functions for variational-like inequality (2.3). This functional can be used to propose and analyze many iterative methods for solving perturbed mixed variational-like inequalities. For more details, see [9,10,13] and the references therein.

4. Conclusions

In this paper, we have introduced and considered a new class of variational-like inequalities, which is called the perturbed mixed variational-like inequalities. We have studied the existence of the solution of PMVLI by using the auxiliary principle technique. Some special cases have been also considered under some suitable conditions. Further research is required to develop some numerical schemes to find the approximate solution of these inequalities, which will be quite interesting problem for new researchers.

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Conflict of interest

The authors declare no conflict of interest.

References

1. G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris., **258** (1964), 4413–4416.
2. A. Ben-Israel, B. Mond, *What is invexity?* ANZIAM J., **28** (1986), 1–9.
3. R. Glowinski, J. L. Lions, R. Tremolieres, *Numerical analysis of variational inequalities*, North-Holland, Amsterdam, 1981.
4. M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl., **80** (1981), 545–550.
5. J. L. Lions, G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math., **20** (1967), 493–519.
6. S. R. Mohan, S. K. Neogy, *On invex set and preinvex functions*, J. Math. Anal. Appl., **189** (1995), 901–908.
7. C. P. Niculescu, L. E. Persson, *Convex Functions and their applications*, Springer-Verlag, New York, 2018.
8. M. A. Noor, *On a class of variational inequalities*, J. Math. Anal. Appl., **128** (1987), 138–155.
9. M. A. Noor, *Variational-like inequalities*, Optimization, **30** (1994), 323–330.
10. M. A. Noor, *Fundamentals of mixed quasi variational inequalities*, Inter. J. Pure. Appl. Math., **15** (2004), 137–258.
11. M. A. Noor, K. I. Noor, *Some characterizations of strongly preinvex functions*, J. Math. Anal. Appl., **316** (2006), 697–706.
12. M. A. Noor, K. I. Noor, M. I. Baloch, *Auxiliary Principle Technique for strongly mixed variational-like inequalities*, U. P. B. Sci. Bull., Series A, **80** (2018), 93–100.
13. J. Parida, A. Sen, *A variational-like inequality for multifunctions with applications*, J. Math. Anal. Appl., **124** (1987), 73–81.
14. M. Patriksson, *Nonlinear programming and variational inequality problems: A unified approach*, Kluwer Academic Publishers, Drodrecht, 1999.
15. B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl., **7** (1966), 72–75.

16. T. Weir, V. Jeyakumar, *A class of nonconvex functions and mathematical programming*, Bull. Austral. Math. Soc., **38** (1988), 177–189.

17. T. Weir, B. Mond, *Preinvex functions in multiple objective optimization*, J. Math. Anal. Appl., **136** (1988), 29–38.

18. X. Q. Yang, G. Y. Chen, *A class of nonconvex functions and variational inequalities*, J. Math. Anal. Appl., **169** (1992), 359–373.

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