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Research article

Sign-changing solutions for a class of p-Laplacian Kirchhoff-type problem with logarithmic nonlinearity

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Abstract: In this paper, we study the existence of ground state sign-changing solutions for following p-Laplacian Kirchhoff-type problem with logarithmic nonlinearity

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^p dx)\Delta_p u = |u|^{q-2} u \ln u^2, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $a, b > 0$ are constant, $4 \le 2p < q < p^*$ and $N > p$. By using constraint variational method, topological degree theory and the quantitative deformation lemma, we prove the existence of ground state sign-changing solutions with precisely two nodal domains.

Keywords: p-Laplacian Kirchhoff-type equation; nonlocal term; variation methods; sign-changing solutions
Mathematics Subject Classification: 35J20, 35J65

1. Introduction

In this article, we are consider the existence of the ground state sign-changing solution for the following p-Laplacian Kirchhoff-type equation

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^p dx)\Delta_p u = |u|^{q-2} u \ln u^2, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $a, b > 0$, $4 \le 2p < q < p^*$ and $N > p$, Δ_p denote the p-Laplacian operator defined by $\Delta_p u = div(|\nabla u|^{p-2} \cdot \nabla u)$.

Problem (1.1) stem from following Kirchhoff equations

$$-(a+b\int_\Omega |\nabla u|^2dx)\Delta u = f(x,u), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain or $\Omega = \mathbb{R}^N$, $a > 0, b > 0$ and u satisfies some boundary conditions.

Problem (1.2) is related to the following stationary analogue of the equation of Kirchhoff type

$$u_{tt} - (a+b\int_\Omega |\nabla u|^2dx)\Delta u = f(x,u), \tag{1.3}$$

which was introduced by Kirchhoff [1] as a generalization of the well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - (\frac{p_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2dx)\frac{\partial^2 u}{\partial x^2} = f(x,u), \tag{1.4}$$

for free vibration of elastic strings.

After the pioneer work of Lions [2], where a functional analysis approach was proposed to (1.3) with Dirichlet boundary condition, a lot of interesting results for (1.2) or similar problems are obtained in last decades.

Recently, many authors pay their attentions to find sign-changing solutions to problem (1.2) or similar equations, and indeed some interesting results were obtained, see for examples, [3–28] and the references therein.

On the other hand, the problem (1.1) derive from the following Logarithmic Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = |u|^{q-2}u\ln u^2, \ x \in \Omega \\ u \in H_0^1(\Omega). \end{cases} \tag{1.5}$$

Recently, there are many results about Logarithmic Schrödinger equation like (1.5), see [26,29–37] and references therein. Moreover, some scholars considered sign-changing solutions to Logarithmic Schrödinger equation like (1.5) [26,33].

Motivated by the works mentioned above, especially by [9,10,26], in the present paper, we consider the existence of ground state sign-changing solutions for problem (1.1).

Denote $W_0^{1,p}(\Omega)$ the usual Sobolev space equipped with the norm

$$||u||^p = \int_\Omega |\nabla u|^p dx.$$

The usual $L^p(\Omega)$ norm is denote by $|u|_p^p = \int_\Omega |u|^p dx$. And we define the energy functional of problem (1.1) as follow:

$$\Phi(u) = \frac{a}{p}\int_\Omega |\nabla u|^p dx + \frac{b}{2p}(\int_\Omega |\nabla u|^p dx)^2 + \frac{2}{q^2}\int_\Omega |u|^q dx - \frac{1}{q}\int_\Omega |u|^q \ln u^2 dx,$$

for any $u \in W_0^{1,p}(\Omega)$.

Moreover, under our conditions, $\Phi(u)$ belongs to C^1, and the Fréchet derivative of Φ is

$$\langle \Phi'(u), v\rangle = a\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx + b(\int_\Omega |\nabla u|^p dx)(\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx)$$
$$-\int_\Omega |u|^{q-2}uv\ln u^2 dx,$$

for any $u, v \in W_0^{1,p}(\Omega)$.

The solution of problem (1.1) is the critical point of the functional $\Phi(u)$. Furthermore, if $u \in W_0^{1,p}(\Omega)$ is a solution of problem (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution of problem (1.1), where

$$u^+ = \max\{u(x), 0\}, \quad u^- = \min\{u(x), 0\}.$$

It is noticed that

$$\Phi(u) = \Phi(u^+) + \Phi(u^+) + \frac{b}{p}\|u^+\|^p\|u^-\|^p, \quad \text{if } u^\pm \not\equiv 0,$$
$$\langle \Phi'(u), u^+\rangle = \langle \Phi'(u^+), u^+\rangle + b\|u^+\|^p\|u^-\|^p, \quad \text{if } u^\pm \not\equiv 0,$$
$$\langle \Phi'(u), u^-\rangle = \langle \Phi'(u^-), u^-\rangle + b\|u^+\|^p\|u^-\|^p, \quad \text{if } u^\pm \not\equiv 0.$$

The main results can be stated as follows.

Theorem 1.1. *The problem (1.1) has a sign-changing $u_0 \in \mathcal{M}$ with precisely two nodal domains such that $\Phi(u_0) = inf_{\mathcal{M}}\Phi := m$, where*

$$\mathcal{M} = \{u \in W_0^{1,p}(\Omega), u^\pm \neq 0, \text{ and } \langle \Phi'(u), u^+\rangle = \langle \Phi'(u), u^-\rangle = 0\}.$$

Theorem 1.2. *The problem (1.1) has a solutions $u_0 \in \mathcal{N}$ such that $\Phi(u_0) = inf_{\mathcal{N}}\Phi := c$, where $\mathcal{N} = \{u \in W_0^{1,p}(\Omega), u \neq 0$, and $\langle \Phi'(u), u\rangle = 0\}$. Moreover, $m \geq 2c$.*

Remark 1.1. *The Kirchhoff function $M(t) = a + bt^n (n > 0, a > 0, b \geq 0)$ can be regarded as the special case of the function $M : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:*
$(M1)$ $M \in C(\mathbb{R}^+)$ satisfies $\inf_{t\in\mathbb{R}^+} M(t) \geq m_0 > 0$, where m_0 is a constant;
$(M2)$ There exists $\theta \geq 1$ such that $\theta\mathbb{M} = \theta\int_0^t M(\tau)d\tau \geq M(t)t$ for any $t \geq 0$.
In this paper, we consider this problem under condition $M(t) = a + bt$. From details in proof of Lemma 2.1 (see later), we can not obtain result similar as Lemma 2.1 for the case of Kirchhoff equations of the general forms. However, Lemma 2.1 play an important role in proof of main resuls. So, for the case of Kirchhoff equations of the general forms, the methods in this paper seems not valid.

2. Technical lemmas

Lemma 2.1. *For all $u \in W_0^{1,p}(\Omega)$ and $s, t \geq 0$ there holds*

$$\Phi(u) \geq \Phi(su^+ + tu^-) + \frac{1-s^q}{q}\langle \Phi'(u), u^+\rangle + \frac{1-t^q}{q}\langle \Phi'(u), u^-\rangle + a(\frac{1-s^p}{p} - \frac{1-s^q}{q})\|u^+\|^p$$
$$+ a(\frac{1-t^p}{p} - \frac{1-t^q}{q})\|u^-\|^p + b[(\frac{1-s^{2p}}{2p} - \frac{1-s^q}{q})\|u^+\|^{2p} + (\frac{1-t^{2p}}{2p} - \frac{1-t^q}{q})\|u^-\|^{2p}]$$
$$+ b[\frac{s^{2p} + t^{2p} - 2s^pt^p}{2p}]\|u^+\|^p\|u^-\|^p. \quad (2.1)$$

Proof. Since (2.1) holds when $u = 0$, in the following, we always assume that $u \neq 0$. It is easy to see that

$$2(1 - \tau^q) + q\tau^q \ln \tau^2 > 0, \forall \tau \in (0, 1) \cup (1, \infty). \tag{2.2}$$

Let $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ and $\Omega^- = \{x \in \Omega : u(x) < 0\}$. For any $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we have that

$$\begin{aligned}
&\int_\Omega |su^+ + tu^-|^q \ln(su^+ + tu^-)^2 dx \\
&= \int_{\Omega^+} |su^+ + tu^-|^q \ln(su^+ + tu^-)^2 dx + \int_{\Omega^-} |su^+ + tu^-|^q \ln(su^+ + tu^-)^2 dx \\
&= \int_{\Omega^+} |su^+|^q \ln(su^+)^2 dx + \int_{\Omega^-} |su^-|^q \ln(su^-)^2 dx \\
&= \int_\Omega [|su^+|^q \ln(su^+)^2] + |tu^-|^q \ln(tu^-)^2 dx \\
&= \int_\Omega [|su^+|^q(\ln s^2 + \ln(u^+)^2) + |tu^-|^q(\ln t^2 + \ln(u^-)^2)]dx.
\end{aligned} \tag{2.3}$$

Combining (2.2) with (2.3), we obtain

$$\begin{aligned}
\Phi(u)-\Phi(su^+ + tu^-) &= \frac{a}{p}(\|u\|^p - \|su^+ + tu^-\|^p) + \frac{b}{2p}(\|u\|^{2p} - \|su^+ + tu^-\|^{2p}) \\
&+ \frac{2}{q^2}\int_\Omega |u|^q - |su^+ + tu^-|^q dx - \frac{1}{q}\int_\Omega |u|^q \ln u^2 - |su^+ + tu^-|^q \ln(su^+ + tu^-)^2 dx \\
&= \frac{a}{p}(\|u\|^p - s^p\|u^+\|^p - t^p\|u^-\|^p) + \frac{b}{2p}(\|u\|^{2p} - s^{2p}\|u^+\|^{2p} - t^{2p}\|u^-\|^{2p} \\
&- 2s^p t^p\|u^+\|^p\|u^-\|^p) + \frac{2}{q^2}\int_\Omega |u^+|^q + |u^-|^q - s^q|u^+|^q - t^q|u^-|^q dx - \frac{1}{q}\int_\Omega (|u^+|^q \ln(u^+)^2 \\
&+ |u^-|^q \ln(u^-)^2 - |su^+|^q \ln s^2 - |su^+|^q \ln(u^+)^2 - |tu^-|^q \ln t^2 - |tu^-|^q \ln(u^-)^2)dx \\
&= \frac{a}{p}(1 - s^p)\|u^+\|^p + \frac{a}{p}(1 - t^p)\|u^-\|^p + \frac{b}{2p}(1 - s^{2p})\|u^+\|^{2p} \\
&+ \frac{b}{2p}(1 - t^{2p})\|u^-\|^{2p} + \frac{b}{p}(1 - s^p t^p)\|u^+\|^p\|u^-\|^p + \frac{2}{q^2}\int_\Omega (|u^+|^q - |su^+|^q + |u^-|^q - |tu^-|^q)dx \\
&- \frac{1}{q}\int_\Omega (|u^+|^q \ln(u^+)^2 - |su^+|^q \ln(su^+)^2)dx - \frac{1}{q}\int_\Omega (|u^-|^q \ln(u^-)^2 - |tu^-|^q \ln(tu^-)^2)dx \\
&= \frac{1 - s^q}{q}\langle \Phi'(u), u^+\rangle + \frac{1 - t^q}{q}\langle \Phi'(u), u^-\rangle + a(\frac{1 - s^p}{p} - \frac{1 - s^q}{q})\|u^+\|^p \\
&+ a(\frac{1 - t^p}{p} - \frac{1 - t^q}{q})\|u^-\|^p + b((\frac{1 - s^{2p}}{2p} - \frac{1 - s^q}{q})\|u^+\|^{2p} + (\frac{1 - t^{2p}}{2p} - \frac{1 - t^q}{q})\|u^-\|^{2p}) \\
&+ b(\frac{1 - s^p t^p}{p} - \frac{1 - s^q}{q} - \frac{1 - t^q}{q})\|u^+\|^p\|u^-\|^p + \frac{2(1 - s^q) + qs^q \ln s^2}{q^2}|u^+|_q^q \\
&+ \frac{2(1 - t^q) + qt^q \ln t^2}{q^2}|u^-|_q^q
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1-s^q}{q}\langle \Phi'(u),u^+\rangle + \frac{1-t^q}{q}\langle \Phi'(u),u^-\rangle + a(\frac{1-s^p}{p} - \frac{1-s^q}{q})\|u^+\|^p \\
&+ a(\frac{1-t^p}{p} - \frac{1-t^q}{q})\|u^-\|^p + b((\frac{1-s^{2p}}{2p} - \frac{1-s^q}{q})\|u^+\|^{2p} + (\frac{1-t^{2p}}{2p} - \frac{1-t^q}{q})\|u^-\|^{2p}) \\
&+ b[\frac{s^{2p}+t^{2p}-2s^pt^p}{2p}]\|u^+\|^p\|u^-\|^p, \qquad (2.4)
\end{aligned}
$$

which implies that (2.2) holds. □

According to Lemma 2.1, we can obtain the following corollaries.

Corollary 2.1. *For all $u \in W_0^{1,p}(\Omega)$ and $t \geq 0$, we have that*

$$
\Phi(u) \geq \Phi(tu) + \frac{1-t^q}{q}\langle \Phi'(u),u\rangle + a(\frac{1-t^p}{p} - \frac{1-t^q}{q})\|u\|^p. \qquad (2.5)
$$

Corollary 2.2. *For any $u \in \mathcal{M}$, we have that*

$$
\Phi(u) = \max_{s,t\geq 0} \Phi(su^+ + tu^-).
$$

Corollary 2.3. *For any $u \in \mathcal{N}$, we have that*

$$
\Phi(u) = \max_{t\geq 0} \Phi(tu).
$$

Lemma 2.2. *For any $u \in W_0^{1,p}(\Omega)$ with $u^\pm \neq 0$, there exists an unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}$.*

Proof. Firstly, for any $u \in W_0^{1,p}(\Omega)$ with $u^\pm \neq 0$, we prove the existence of (s_u, t_u). Let

$$
\begin{aligned}
G(s,t) = \langle \Phi'(su^+ + tu^-), su^+\rangle &= as^p\|u^+\|^p + bs^{2p}\|u^+\|^{2p} \\
&+ bs^pt^p\|u^+\|^p\|u^-\|^p - \int_\Omega |su^+|^q \ln(su^+)^2 dx, \qquad (2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
H(s,t) = \langle \Phi'(su^+ + tu^-), tu^-\rangle &= at^p\|u^+\|^p + bt^{2p}\|u^-\|^{2p} \\
&+ bs^pt^p\|u^+\|^p\|u^-\|^p - \int_\Omega |tu^-|^q \ln(tu^-)^2 dx. \qquad (2.7)
\end{aligned}
$$

From assumptions, we have that

$$
\lim_{t\to 0} \frac{|t|^{q-1}\ln t^2}{|t|^{p-1}} = 0; \;\; \lim_{t\to\infty} \frac{|t|^{q-1}\ln t^2}{|t|^{r-1}} = 0, r \in (q, p^*).
$$

Then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
|t|^{q-1}\ln t^2 \leq \varepsilon|t|^{p-1} + C_\varepsilon|t|^{r-1}. \qquad (2.8)
$$

Since $4 \leq 2p < q < p^*$, it follows from (2.8) that

$$G(s,s) > 0 \text{ and } H(s,s) > 0 \text{ for } s > 0 \text{ small enough,}$$
$$G(t,t) > 0 \text{ and } H(t,t) > 0 \text{ for } t > 0 \text{ large enough.}$$

Thus, there exist $0 < \alpha < \beta$ such that

$$G(\alpha,\alpha) > 0,\ H(\alpha,\alpha) > 0;\ G(\beta,\beta) < 0,\ H(\beta,\beta) < 0. \tag{2.9}$$

Thanks to (2.6), (2.7) and (2.9), we have that

$$G(\alpha,t) > 0,\ G(\beta,t) < 0,\ \forall t \in [\alpha,\beta] \tag{2.10}$$

and

$$H(s,\alpha) > 0,\ H(s,\beta) < 0,\ \forall s \in [\alpha,\beta]. \tag{2.11}$$

So, combining (2.10), (2.11) with Miranda's Theorem [38], there exists some point (s_u, t_u) with $\alpha < s_u, t_u < \beta$ such that

$$G(s_u,t_u) = H(s_u,t_u) = 0.$$

That is, there exists a pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}$.

Secondly, we prove that (s_u, t_u) is unique.

Arguing by contradiction, we assume that there exist two pair (s_i, t_i), $i = 1, 2$ such that $s_1 u^+ + t_1 u^- \in \mathcal{M}$ and $s_2 u^+ + t_2 u^- \in \mathcal{M}$.

According to corollary 2.2, we have that

$$\Phi(s_1u^+ + t_1u^-) \geq \Phi(s_2u^+ + t_2u^-) + as_1^p\Big(\frac{1-(\frac{s_2}{s_1})^p}{p} - \frac{1-(\frac{s_2}{s_1})^q}{q}\Big)\|u^+\|^p, \tag{2.12}$$

$$\Phi(s_2u^+ + t_2u^-) \geq \Phi(s_1u^+ + t_1u^-) + as_2^p\Big(\frac{1-(\frac{s_1}{s_2})^p}{p} - \frac{1-(\frac{s_1}{s_2})^q}{q}\Big)\|u^+\|^p. \tag{2.13}$$

It is noticed that

$$h(x) = \frac{1-s^x}{x} \text{ is monotonically decreasing on } (0,\infty) \text{ for } s > 0 \text{ and } s \neq 1.$$

Therefore, by (2.12) and (2.13), we have that $(s_1, t_1) = (s_2, t_2)$. That is, (s_u, t_u) is unique. □

Lemma 2.3. *For any $u \in W_0^{1,p}(\Omega)$ with $u \neq 0$, there exists an unique $t_u > 0$ such that $t_u u \in \mathcal{N}$.*

Proof. Since the proof is similar to that of Lemma 2.2, we omit detail here. □

Through the standard discussions, we have following result.

Lemma 2.4. *The following minimax characterization hold*

$$c = \inf_{u\in W_0^{1,p}(\Omega),u\neq 0} \max_{t\geq 0} \Phi(tu),\ m = \inf_{u\in W_0^{1,p}(\Omega),u\neq 0} \max_{s,t\geq 0} \Phi(su^+ + tu^-).$$

Lemma 2.5. *$m > 0$ is achieved.*

Proof. For any $u \subset \mathcal{M}$, we have $\langle \Phi'(u), u \rangle = 0$ and then

$$\begin{aligned} a\|u\|^p \leq a\|u\|^p + b\|u\|^{2p} &= \int_\Omega |u|^q \ln u^2 dx \\ &\leq C_1\|u\|^q + C_2\|u\|^r. \end{aligned} \tag{2.14}$$

Since $q, r > p$, there exists a constant $\rho > 0$ such that $\|u\| \geq \rho$ for any $u \subset \mathcal{M}$.

Let $u_n \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$, then u_n is bounded in $W_0^{1,p}$. Thus, there exists u_0, in subsequence sense, such that

$$\begin{gathered} u_n^\pm \rightharpoonup u_0^\pm \text{ in } W_0^{1,p}(\Omega) \\ u_n^\pm \to u_0^\pm \text{ in } L^s(\Omega), p \leq s < p^* \end{gathered}$$

Since $u_n \subset \mathcal{M}$, one has $\langle \Phi'(u_n), u_n^\pm \rangle = 0$, that is

$$\begin{aligned} a\rho^p \leq a\|u_n^\pm\|^p \leq a\|u_n^\pm\|^p + b\|u_n^\pm\|^{2p} + b\|u_n^+\|^p\|u_n^-\|^p &= \int_\Omega |u_n^\pm|^q \ln(u_n^\pm)^2 dx \\ &\leq \varepsilon \int_\Omega |u_n^\pm|^q dx + C_\varepsilon \int_\Omega |u_n^\pm|^r dx \leq C_4 \int_\Omega |u_n^\pm|^r dx \end{aligned} \tag{2.15}$$

By the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, we get

$$C_5\rho^p \leq \int_\Omega |u_0^\pm|^r dx,$$

which implies $u_0^\pm \neq 0$.

By the Lebesgue dominated convergence theorem and the weak semicontinuity of norm, we have

$$\begin{aligned} a\|u_0^\pm\|^p + b\|u_0^\pm\|^{2p} + b\|u_0^+\|^p\|u_0^-\|^p &\leq \liminf_{n\to\infty}(a\|u_n^\pm\|^p + b\|u_n^+\|^{2p} + b\|u_n^+\|^p\|u_n^-\|^p) \\ &= \liminf_{n\to\infty} \int_\Omega |u_n^\pm|^q \ln(u_n^\pm)^2 dx \\ &= \int_\Omega |u_0^\pm|^q \ln(u_0^\pm)^2 dx, \end{aligned}$$

that is,

$$\langle \Phi'(u_0), u_0^+ \rangle \leq 0 \text{ and } \langle \Phi'(u_0), u_0^- \rangle \leq 0.$$

Since $u_0^\pm \neq 0$, it follows from Lemma 2.2 that there exist constants $s, t > 0$ such that $su_0^+ + tu_0^- \in \mathcal{M}$. From corollary 2.2, corollary 2.3 and the weak semicontinuity of norm, we have that

$$\begin{aligned} m &= \lim_{n\to\infty}[\Phi(u_n) - \frac{1}{q}\langle \Phi'(u_n), u_n \rangle] \\ &= \lim_{n\to\infty}[(\frac{a}{p} - \frac{a}{q})\|u_n\|^p + (\frac{b}{2p} - \frac{b}{q})\|u_n\|^{2p} + \frac{2}{q^2}|u_n|_q^q] \\ &\geq (\frac{a}{p} - \frac{a}{q})\|u_0\|^p + (\frac{b}{2p} - \frac{b}{q})\|u_0\|^{2p} + \frac{2}{q^2}|u_0|_q^q \end{aligned}$$

$$
\begin{aligned}
&= \Phi(u_0) - \frac{1}{q}\langle \Phi'(u_0, u_0\rangle \\
&\geq \Phi(su_0^+ + tu_0^-) + \frac{1-s^q}{q}\langle \Phi'(u_0), u_0^+\rangle + \frac{1-t^q}{q}\langle \Phi'(u_0), u_0^-\rangle - \frac{1}{q}\langle \Phi'(u_0, u_0\rangle \\
&\geq m - \frac{s^q}{q}\langle \Phi'(u_0), u_0^+\rangle - \frac{t^q}{q}\langle \Phi'(u_0), u_0^-\rangle \\
&\geq m,
\end{aligned}
$$

which asserts

$$\langle \Phi'(u_0), u_0^{\pm}\rangle = 0,\ \Phi(u_0) = m.$$

Furthermore, thanks to $u_0^{\pm} \neq 0$ and (2.1), we have that

$$m = \Phi(u_0) \geq a(\frac{1}{p} - \frac{1}{q})\|u_0^+\|^p + a(\frac{1}{p} - \frac{1}{q})\|u_0^-\|^p > 0.$$

□

3. Proof main results

Proof of Theorem 1.1 :

Proof. Firstly, thanks to Lemma 2.5, we prove the minimizer u_0 of $\inf_{\mathcal{M}} \Phi$ is critical point of Φ.

Arguing by contradiction, we assume that $\Phi'(u_0) \neq 0$. Then there exist $\delta > 0$ and $\varsigma > 0$ such that $\|(\Phi'(u)\| \geq \varsigma$, for all $\|u - u_0\| \leq 3\delta$ and $u \in W_0^{1,p}(\Omega)$.

Let $D := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$, by Lemma 2.1, one has

$$\epsilon := \max_{(s,t)\in\partial D} \Phi(su_0^+ + tu_0^-) < m. \tag{3.1}$$

For $\varepsilon := \min{(m-\epsilon)/3, \delta\varsigma/8}$ and $S_\delta := B(u_0, \delta)$, according to Lemma 2.3 in [39], there exists a deformation $\eta \in C([0,1] \times W_0^{1,p}(\Omega), W_0^{1,p}(\Omega))$ such that

(*a*) $\eta(1, v) = v$ if $v \notin \Phi^{-1}([m - 2\varepsilon, m + 2\varepsilon]) \cap S_{2\delta}$;

(*b*) $\eta(1, (\Phi^{m+\varepsilon} \cap S_\delta) \subset \Phi^{m-\varepsilon}$, where $\Phi^c = \{u \in W_0^{1,p}(\Omega) : \Phi(u) \leq c\}$;

(*c*) $\Phi(\eta(1, v)) \leq \Phi(v)$ for all $v \in W_0^{1,p}(\Omega)$.

By Lemma 2.1 and (*c*), we have

$$
\begin{aligned}
\Phi(\eta(1, su_0^+ + tu_0^-) &\leq \Phi(su_0^+ + tu_0^-) < \Phi(u_0) \\
&= m, \quad \forall s, t \geq 0, |s-1|^2 + |t-1|^2 \geq \delta^2/\|u_0\|^2.
\end{aligned} \tag{3.2}
$$

On the other hand, by Corollary 2.3, we can obtain that $\Phi(su_0^+ + tu_0^-) \leq \Phi(u_0) = m$ for $s, t > 0$. Then it follows from (*b*) that

$$\Phi(\eta(1, su_0^+ + tu_0^-) \leq m - \varepsilon, \quad \forall s, t \geq 0, |s-1|^2 + |t-1|^2 < \delta^2/\|u_0\|^2. \tag{3.3}$$

So, thanks to (3.2) and (3.3), one has

$$\max_{(s,t)\in\bar{D}} \Phi(\eta(1, su_0^+ + tu_0^-) < m. \tag{3.4}$$

Let $k(s,t) = su_0^+ + tu_0^-$, we now prove that $\eta(1, k(D)) \cap \mathcal{M} \neq \emptyset$. Let $\gamma(s,t) := \eta(1, k(s,t))$

$$\begin{aligned}\Psi_0(s,t) &:= (\langle \Phi'(k(s,t)), u_0^+\rangle, \langle \Phi'(k(s,t)), u_0^-\rangle)\\ &= (\langle \Phi'(su_0^+ + tu_0^-), u_0^+\rangle, \langle \Phi'(su_0^+ + tu_0^-), u_0^-\rangle)\\ &:= (h_1(s,t), h_2(s,t))\end{aligned}$$

and

$$\Psi_1(s,t) := (\frac{1}{s}\langle \Phi'(\gamma(s,t)), (\gamma(s,t))^+\rangle, \frac{1}{t}\langle \Phi'(\gamma(s,t)), (\gamma(s,t))^-\rangle).$$

Clearly, Ψ_0 is a C^1 functions and by a direct calculation, we have

$$\begin{aligned}\frac{\partial h_1(s,t)}{\partial s}|_{(1,1)} &= a(p-1)\|u_0^+\|^p + b(2p-1)\|u_0^+\|^{2p} + b(p-1)\|u_0^+\|^p\|u_0^-\|^p\\ &\quad - (q-1)\int_\Omega |u_0^+|^q \ln(u_0^+)^2 dx - 2\int_\Omega |u_0^+|^q dx\\ &= bp\|u_0^+\|^{2p} - (q-p)\int_\Omega |u_0^+|^q \ln(u_0^+)^2 dx - 2\int_\Omega |u_0^+|^p dx,\end{aligned}$$

$$\frac{\partial h_1(s,t)}{\partial t}|_{(1,1)} = bp\|u_0^+\|^p\|u_0^-\|^p.$$

Similarly, we have

$$\frac{\partial h_2(s,t)}{\partial t}|_{(1,1)} = bp\|u_0^-\|^{2p} - (q-p)\int_\Omega |u_0^-|^q \ln(u_0^-)^2 dx - 2\int_\Omega |u_0^-|^p dx,$$

$$\frac{\partial h_2(s,t)}{\partial s}|_{(1,1)} = bp\|u_0^+\|^p\|u_0^-\|^p.$$

Let

$$M = \begin{bmatrix} \frac{\partial h_1(s,t)}{\partial s}|_{(1,1)} & \frac{\partial h_2(s,t)}{\partial s}|_{(1,1)}\\ \frac{\partial h_1(s,t)}{\partial t}|_{(1,1)} & \frac{\partial h_2(s,t)}{\partial t}|_{(1,1)}\end{bmatrix},$$

then we have that

$$\det M = \frac{\partial h_1(s,t)}{\partial s}|_{(1,1)} \times \frac{\partial h_2(s,t)}{\partial t}|_{(1,1)} - \frac{\partial h_1(s,t)}{\partial t}|_{(1,1)} \times \frac{\partial h_2(s,t)}{\partial s}|_{(1,1)} \neq 0.$$

Therefore, by topological degree theory [40–42], we conclude that $\Psi_1(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$, so that $\eta(1, k(s_0, t_0)) = \gamma(s_0, t_0) \in \mathcal{M}$, which is contradicted to (3.4).

Next, we prove u_0 has two nodal domains.

We assume that

$$u_0 = u_1 + u_2 + u_3$$

where

$$u_1 \geq 0,\ u_2 \leq 0,\ \Omega_1 \cap \Omega_2 = \emptyset,\ u_1\ |_{\Omega\setminus\Omega_1\cup\Omega_2} = u_2\ |_{\Omega\setminus\Omega_1\cup\Omega_2} = u_3\ |_{\Omega_1\cup\Omega_2} = 0,$$

$$\Omega_1 := \{x \in \Omega, u_1(x) > 0\} \text{ and } \Omega_2 := \{x \in \Omega, u_2(x) < 0\}$$

are two connected open subsets of Ω.

Setting $v := u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \neq 0$. According to $\langle \Phi'(u_0), v^\pm \rangle = 0$, we have that

$$\langle \Phi'(v), v^\pm \rangle = -b\|v^\pm\|^p \|u_3\|^p. \tag{3.5}$$

Thanks to (2.1) and (3.5), we have that

$$\begin{aligned} m = \Phi(u_0) &= \Phi(u_0) - \frac{1}{q}\langle \Phi'(u_0), u_0 \rangle \\ &= \Phi(v) + \Phi(u_3) + \frac{b}{p}\|u_3\|^p\|v\|^p - \frac{1}{q}(\langle \Phi'(v), v\rangle + \langle \Phi'(u_3), u_3\rangle + 2b\|u_3\|^p\|v\|^p) \\ &\geq \sup_{s,t\geq 0}(\Phi(sv^+ + tv^-) + \frac{1-s^q}{q}\langle \Phi'(v), v^+\rangle + \frac{1-t^q}{q}\langle \Phi'(v), v^-\rangle) \\ &+ \Phi(u_3) - \frac{1}{q}\langle \Phi'(v), v\rangle - \frac{1}{q}\langle \Phi'(u_3), u_3\rangle \\ &\geq \sup_{s,t\geq 0}(\Phi(sv^+ + tv^-) + \frac{b}{q}s^p\|v^+\|^p\|u_3\|^p + \frac{b}{q}t^p\|v^-\|^p\|u_3\|^p) \\ &+ a(\frac{1}{p} - \frac{1}{q})\|u_3\|^p + b(\frac{1}{2p} - \frac{1}{q})\|u_3\|^{2p} + \frac{2}{q^2}|u_3|_q^q \\ &\geq m + a(\frac{1}{p} - \frac{1}{q})\|u_3\|^p, \end{aligned}$$

which implies $u_3 = 0$. That is, u_0 has two nodal domains. □

Proof of Theorem 1.2 :

Proof. Similar as the proof of Lemma 2.5, there exists $v_0 \in \mathcal{N}$ such that $\Phi(v_0) = c > 0$. By arguments similar to that of Theorem 1.1, the critical points of the functional Φ on $\mathcal{N}$ are critical points of Φ in $W_0^{1,p}(\Omega)$ and we obtain $(\Phi)'(v_0) = 0$. It follows from definitions of $c = inf_{\mathcal{N}}\Phi$ and $\mathcal{N} = \{u \in W_0^{1,p}(\Omega), u \neq 0, \text{ and } \langle \Phi'(u), u\rangle = 0\}$ that v_0 is a ground state solution of (1.1).

According to Theorem 1.1, there exists a $u_0 \in \mathcal{M}$ such that $\Phi(u_0) = m, \Phi'(u_0) = 0$. Therefore, by Corollary 2.2 and Lemma 2.4, we have that

$$\begin{aligned} m = \Phi(u_0) &= \sup_{s,t\geq 0}\Phi(su_0^+ + tu_0^-) \\ &= \sup_{s,t\geq 0}(\Phi(su_0^+) + \Phi(tu_0^-) + \frac{b}{p}s^p t^p\|u_0^+\|^p\|u_0^-\|^p) \\ &\geq \sup_{s\geq 0}\Phi(su_0^+) + \sup_{t\geq 0}\Phi(tu_0^-) \geq 2c > 0. \end{aligned}$$

The proof of Theorem 2.2 is completed. □

4. Conclusions

In this paper, by using constraint variational method, topological degree theory and the quantitative deformation lemma, we prove the existence of ground state sign-changing solutions with precisely two nodal domains for a class of p-Laplacian Kirchhoff-type problem (special form) with logarithmic nonlinearity. However, for the case of Kirchhoff equations of the general forms, the methods in this paper seems not valid. So, we will continuous discuss sign-changing solutions for general Kirchhoff equations with logarithmic nonlinearity in the follow-up work.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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