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Research article

On a subclass related to Bazilevič functions

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Abstract: The present paper introduces and studies a subclass of analytic functions defined by using the concept of Bazilevič and Janowski functions. Various properties such as coefficient estimates, Fekete-Szegö type inequalities, arc length problem and growth rate of coefficients are investigated for related functions.

Keywords: univalent functions; subordination; Bazilevič functions; Janowski functions; strongly starlike functions; Fekete-Szeö inequalities
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1. Introduction and definitions

Let $\mathfrak{A}$ denote the family of all functions f which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and satisfying the normalization

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

while by $\mathcal{S}$ we mean the class of all functions in $\mathfrak{A}$ which are univalent in $\mathcal{U}$. Also let $\mathcal{S}^*$ and C denote the familiar classes of starlike and convex functions, respectively. If f and g are analytic functions in $\mathcal{U}$, then we say that f is subordinate to g, denoted by $f \prec g$, if there exists an analytic Schwarz function w in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Moreover if the function g is univalent in $\mathcal{U}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

For arbitrary fixed numbers A, B and b such that A, B are real with $-1 \le B < A \le 1$ and $b \in \mathbb{C}\backslash\{0\}$, let $\mathcal{P}[b, A, B]$ denote the family of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.2}$$

analytic in $\mathcal{U}$ such that

$$1 + \frac{1}{b}\{p(z) - 1\} \prec \frac{1 + Az}{1 + Bz}.$$

Then, $p \in \mathcal{P}[b, A, B]$ can be written in terms of the Schwarz function w by

$$p(z) = \frac{b(1 + Aw(z)) + (1 - b)(1 + Bw(z))}{1 + Bw(z)}.$$

By taking $b = 1 - \sigma$ with $0 \leq \sigma < 1$, the class $\mathcal{P}[b, A, B]$ coincides with $\mathcal{P}[\sigma, A, B]$, defined by Polatoğ lu [17, 18] (see also [2]) and if we take $b = 1$, then $\mathcal{P}[b, A, B]$ reduces to the familiar class $\mathcal{P}[A, B]$ defined by Janowski [10]. Also by taking $A = 1$, $B = -1$ and $b = 1$ in $\mathcal{P}[b, A, B]$, we get the most valuable and familiar set $\mathcal{P}$ of functions having positive real part. Let $\mathcal{S}^*[A, B, b]$ denote the class of univalent functions g of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.3}$$

in $\mathcal{U}$ such that

$$1 + \frac{1}{b}\left\{\frac{zg'(z)}{g(z)} - 1\right\} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in \mathcal{U}.$$

Then $S^*[A, B] := S^*[A, B, 1]$ and the subclass $\mathcal{S}^*[1, -1, 1]$ coincides with the usual class of starlike functions.

The set of Bazilevič functions in $\mathcal{U}$ was first introduced by Bazilevič [7] in 1955. He defined the Bazilevič function by the relation

$$f(z) = \left\{(\alpha + i\beta)\int_0^z g^{\alpha}(t)p(t)t^{i\beta - 1}dt\right\}^{\frac{1}{\alpha + i\beta}},$$

where $p \in \mathcal{P}$, $g \in \mathcal{S}^*$, β is real and $\alpha > 0$. In 1979, Campbell and Pearce [8] generalized the Bazilevič functions by means of the differential equation

$$1 + \frac{zf''(z)}{f'(z)} + (\alpha + i\beta - 1)\frac{zf'(z)}{f(z)} = \alpha\frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)} + i\beta,$$

where $\alpha + i\beta \in \mathbb{C} - \{\text{negative integers}\}$. They associate each generalized Bazilevič functions with the quadruple (α, β, g, p).

Now we define the following subclass.

Definition 1.1. Let g be in the class $\mathcal{S}^*[A, B]$ and let $p \in \mathcal{P}[b, A, B]$. Then a function f of the form (1.1) is said to belong to the class of generalized Bazilevič function associated with the quadruple (α, β, g, p) if f satisfies the differential equation

$$1 + \frac{zf''(z)}{f'(z)} + (\alpha + i\beta - 1)\frac{zf'(z)}{f(z)} = \alpha\frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)} + i\beta.$$

where $\alpha + i\beta \in \mathbb{C} - \{\text{negative integers}\}$.

The above differential equation can equivalently be written as

$$\frac{zf'(z)}{f(z)} = \left(\frac{g(z)}{z}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta} p(z),$$

or

$$\frac{z^{1-i\beta}f'(z)}{f^{1-(\alpha+i\beta)}g^{\alpha}(z)} = p(z),\ z \in \mathcal{U}.$$

Since $p \in \mathcal{P}[b, A, B]$, it follows that

$$1 + \frac{1}{b}\left\{\frac{z^{1-i\beta}f'(z)}{f^{1-(\alpha+i\beta)}g^{\alpha}(z)} - 1\right\} \prec \frac{1 + Az}{1 + Bz},$$

where $g \in \mathcal{S}^*[A, B]$.

Several research papers have appeared recently on classes related to the Janowski functions, Bazilevič functions and their generalizations, see [3–5, 13, 16, 21, 22].

2. Lemmas

The following are some results that would be useful in proving the main results.

Lemma 2.1. *Let* $p \in \mathcal{P}[b, A, B]$ *with* $b \neq 0, -1 \leq B < A \leq 1$, *and has the form* (1.2) . *Then for* $z = re^{i\theta}$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|p(re^{i\theta})\right|^2 d\theta \leq \frac{1 + \left[|b|^2 (A - B)^2 - 1\right] r^2}{1 - r^2}.$$

Proof. The proof of this lemma is straightforward but we include it for the sake of completeness. Since $p \in \mathcal{P}[b, A, B]$, we have

$$p(z) = b\tilde{p}(z) + (1 - b), \quad \tilde{p} \in \mathcal{P}[A, B].$$

Let $\tilde{p}(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then

$$1 + \sum_{n=1}^{\infty} p_n z^n = b\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) + (1 - b).$$

Comparing the coefficients of z^n, we have

$$p_n = bc_n.$$

Since $|c_n| \leq A - B$ [20], it follows that $|p_n| \leq |b| (A - B)$ and so

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|p(re^{i\theta})\right|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=0}^{\infty} p_n r^n e^{in\theta}\right|^2 d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{\infty} |p_n|^2 r^{2n}\right) d\theta \\ &= \sum_{n=0}^{\infty} |p_n|^2 r^{2n}\end{aligned}$$

$$\begin{aligned}
&\leq 1+|b|^2(A-B)^2\sum_{n=1}^{\infty}r^{2n}\\
&=1+|b|^2(A-B)^2\frac{r^2}{1-r^2}\\
&=\frac{1+\left(|b|^2(A-B)^2-1\right)r^2}{1-r^2}.
\end{aligned}$$

Thus the proof is complete. □

Lemma 2.2. *[1] Let Ω be the family of analytic functions ω on $\mathcal{U}$, normalized by $\omega(0)=0$, satisfying the condition $|\omega(z)|<1$. If $\omega\in\Omega$ and*

$$\omega(z)=\omega_1 z+\omega_2 z^2+\cdots, \quad (z\in\mathcal{U}),$$

then for any complex number t,

$$\left|\omega_2-t\omega_1^2\right|\leq\max\{1,|t|\}.$$

The above inequality is sharp for $\omega(z)=z$ or $\omega(z)=z^2$.

Lemma 2.3. *Let $p(z)=1+\sum\limits_{n=1}^{\infty}p_n z^n\in\mathcal{P}[b,A,B]$, $b\in\mathbb{C}\backslash\{0\}$, $-1\leq B<A\leq 1$. Then for any complex number μ,*

$$\begin{aligned}
\left|p_2-\mu p_1^2\right| &\leq |b|(A-B)\max\{1,|\mu b(A-B)+B|\}\\
&=\begin{cases}|b|(A-B), & \text{if } |\mu b(A-B)+B|\leq 1,\\ |b|(A-B)|\mu b(A-B)+B|, & \text{if } |\mu b(A-B)+B|\geq 1.\end{cases}
\end{aligned}$$

This result is sharp.

Proof. Let $p\in\mathcal{P}[b,A,B]$. Then we have

$$1+\frac{1}{b}\{p(z)-1\}\prec\frac{1+Az}{1+Bz},$$

or, equivalently

$$p(z)\prec\frac{1+[bA+(1-b)B]z}{1+Bz}=1+b(A-B)\sum_{n=1}^{\infty}(-B)^{n-1}z^n,$$

which would further give

$$\begin{aligned}
1+p_1z+p_2z^2+\cdots &= 1+b(A-B)\omega(z)+b(A-B)(-B)\omega^2(z)+\cdots\\
&=1+b(A-B)\left(\omega_1 z+\omega_2 z^2+\cdots\right)\\
&\quad +b(A-B)(-B)\left(\omega_1 z+\omega_2 z^2+\cdots\right)^2+\cdots\\
&=1+b(A-B)\omega_1 z+b(A-B)\left\{\omega_2-B\omega_1^2\right\}z^2+\cdots.
\end{aligned}$$

Comparing the coefficients of z and z^2, we obtain

$$p_1=b(A-B)\omega_1$$

$$p_2 = b(A-B)\omega_2 - b(A-B)B\omega_1^2.$$

By a simple computation,

$$\left|p_2 - \mu p_1^2\right| = |b|\,(A-B)\left|\omega_2 - (\mu b(A-B)+B)\,\omega_1^2\right|.$$

Now by using Lemma 2.2 with $t = \mu b(A-B)+B$, we get the required result. Equality holds for the functions

$$p_\circ(z) = \frac{1+(bA+(1-b)B)\,z^2}{1+Bz^2} = 1 + b(A-B)z^2 + b(A-B)(-B)z^4 + \cdots,$$
$$p_1(z) = \frac{1+(bA+(1-b)B)\,z}{1+Bz} = 1 + b(A-B)z + b(A-B)(-B)z^2 + \cdots. \qquad \square$$

Now we prove the following result by using a method similar to the one in Libera [12].

Lemma 2.4. *Suppose that N and D are analytic in $\mathcal{U}$ with $N(0) = D(0) = 0$ and D maps $\mathcal{U}$ onto a many sheeted region which is starlike with respect to the origin. If $\frac{N'(z)}{D'(z)} \in \mathcal{P}[b, A, B]$, then*

$$\frac{N(z)}{D(z)} \in \mathcal{P}[b, A, B].$$

Proof. Let $\frac{N'(z)}{D'(z)} \in \mathcal{P}[b, A, B]$. Then by using a result due to Attiya [6], we have

$$\left|\frac{N'(z)}{D'(z)} - c(r)\right| \le d\,(r), \qquad |z| < r, \quad 0 < r < 1,$$

where $c(r) = \frac{1-B[B+b(A-B)]r^2}{1-B^2r^2}$ and $d\,(r) = \frac{|b|(A-B)r^2}{1-B^2r^2}$. We choose $A\,(z)$ such that $|A\,(z)| < d\,(r)$ and

$$A\,(z)\,D'(z) = N'(z) - c\,(r)\,D'(z).$$

Now for a fixed z_0 in $\mathcal{U}$, consider the line segment L joining 0 and $D\,(z_0)$ which remains in one sheet of the starlike image of $\mathcal{U}$ by D. Suppose that L^{-1} is the pre-image of L under D. Then

$$\begin{aligned}\left|N(z_0) - c\,(r)\,D(z_0)\right| &= \left|\int_0^{z_0}\Big(N'(t) - c(r)D'(t)\Big)dt\right| \\ &= \left|\int_{L^{-1}} A(t)D'(t)\,dt\right| \\ &< d(r)\int_{L^{-1}} |dD(t)| \\ &= d(r)D(z_0).\end{aligned}$$

This implies that

$$\left|\frac{N(z_0)}{D(z_0)} - c\,(r)\right| < d\,(r).$$

Therefore

$$\frac{N(z)}{D(z)} \in \mathcal{P}[b, A, B]. \qquad \square$$

For $A = -B = b = 1$, we have the following result due to Libera [12].

Lemma 2.5. *If N and D are analytic in $\mathcal{U}$ with $N(0) = D(0) = 0$ and D maps $\mathcal{U}$ onto a many sheeted region which is starlike with respect to the origin, then*

$$\frac{N'(z)}{D'(z)} \in \mathcal{P} \text{ implies } \frac{N(z)}{D(z)} \in \mathcal{P}.$$

Lemma 2.6. *[14] If $-1 \leq B < A \leq 1, \beta_1 > 0$ and the complex number γ satisfies $Re\{\gamma\} \geq -\beta_1 (1-A) / (1-B)$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta_1 q(z) + \gamma} = \frac{1+Az}{1+Bz}, \quad z \in \mathcal{U},$$

has a univalent solution in $\mathcal{U}$ given by

$$q(z) = \begin{cases} \frac{z^{\beta_1+\gamma}(1+Bz)^{\beta_1(A-B)/B}}{\beta_1 \int_0^z t^{\beta_1+\gamma-1}(1+Bt)^{\beta_1(A-B)/B}dt} - \frac{\gamma}{\beta_1}, & B \neq 0, \\ \frac{z^{\beta_1+\gamma} e^{\beta_1 Az}}{\beta_1 \int_0^z t^{\beta_1+\gamma-1} e^{\beta_1 At}dt} - \frac{\gamma}{\beta_1}, & B = 0. \end{cases}$$

If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in $\mathcal{U}$ and satisfies

$$p(z) + \frac{zp'(z)}{\beta_1 p(z) + \gamma} \prec \frac{1+Az}{1+Bz},$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

and $q(z)$ is the best dominant.

3. Some auxiliary results

Before proving the results for the generalized Bazilevič functions, let us discuss a few results related to the function $g \in S^*[A, B]$.

Theorem 3.1. *Let $g \in \mathcal{S}^*[A, B]$ and of the form* (1.3)*. Then for any complex number μ,*

$$\left|b_3 - \mu b_2^2\right| \leq \frac{(A-B)}{2} \max\{1, |2(A-B)\mu - (A-2B))|\}.$$

Proof. The proof of the result is the same as of Lemma 2.3. The result is sharp and equality holds for the function defined by

$$g_1(z) = \begin{cases} z(1+Bz^2)^{\frac{A-B}{2B}} = z + \frac{1}{2}(A-B) z^3 + \cdots, & B \neq 0, \\ ze^{\frac{A}{2}z^2} = z + \frac{A}{2}z^3 + \cdots, & B = 0, \end{cases}$$

or

$$g_2(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}}, & B \neq 0, \\ ze^{Az}, & B = 0, \end{cases}$$

$$= \begin{cases} z + (A-B) z^2 + \frac{1}{2}(A-B)(A-2B) z^3 + \cdots, & B \neq 0, \\ z + Az^2 + \frac{1}{2}A^2 z^3 + \cdots, & B = 0. \end{cases}$$

□

Theorem 3.2. *Let $g \in \mathcal{S}^*[A,B]$. Then for $c > 0$, $\alpha > 0$ and β any real number,*

$$G^{\alpha}(z) = \frac{c+\alpha+i\beta}{z^{c+i\beta}} \int_0^z t^{c+i\beta-1} g^{\alpha}(t)\, dt, \tag{3.1}$$

is in $\mathcal{S}^*[A,B]$. In addition

$$Re \frac{zG'(z)}{G(z)} > \delta = \min_{|z|=1} Re\, q(z),$$

where

$$q(z) = \begin{cases} \frac{1}{\alpha} \frac{\alpha+i\beta+c}{{}_2F_1\left(1;\, \alpha\left(1-\frac{A}{B}\right);\, \alpha+i\beta+c+1;\, \frac{Bz}{1+Bz}\right)} - (c+i\beta), & B \neq 0, \\ \frac{1}{\alpha} \frac{\alpha+i\beta+c}{{}_1F_1(1;\, \alpha+i\beta+c+1;\, -\alpha Az)} - (c+i\beta), & B = 0. \end{cases}$$

Proof. From (3.1) , we have

$$z^{c+i\beta} G^{\alpha}(z) = (c+\alpha+i\beta) \int_0^z t^{c+i\beta-1} g^{\alpha}(t) dt.$$

Differentiating and rearranging gives

$$(c+\alpha+i\beta) \frac{g^{\alpha}(z)}{G^{\alpha}(z)} = (c+i\beta) + \alpha p(z), \tag{3.2}$$

where $p(z) = \frac{zG'(z)}{G(z)}$. Then differentiating (3.2) logarithmically, we have

$$\frac{zg'(z)}{g(z)} = p(z) + \frac{zp'(z)}{\alpha p(z) + (c+i\beta)}.$$

Since $g \in \mathcal{S}^*[A,B]$, it follows that

$$p(z) + \frac{zp'(z)}{\alpha p(z) + (c+i\beta)} \prec \frac{1+Az}{1+Bz}.$$

Now by using Lemma 2.6 , for $\beta_1 = \alpha$ and $\gamma = c + i\beta$, we obtain

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where

$$q(z) = \begin{cases} \frac{z^{c+\alpha+i\beta}(1+Bz)^{\alpha(A-B)/B}}{\alpha \int_0^z t^{c+\alpha+i\beta-1}(1+Bt)^{\alpha(A-B)/B} dt} - \frac{c+i\beta}{\alpha}, & B \neq 0, \\ \frac{z^{c+\alpha+i\beta} e^{\alpha Az}}{\alpha \int_0^z t^{c+\alpha+i\beta-1} e^{\alpha At} dt} - \frac{c+i\beta}{\alpha}, & B = 0. \end{cases}$$

Now by using the properties of the familiar hypergeometric functions proved in [15], we have

$$q(z) = \begin{cases} \frac{1}{\alpha} \frac{\alpha+i\beta+c}{{}_2F_1\left(1;\, \alpha\left(1-\frac{A}{B}\right);\, \alpha+i\beta+c+1;\, \frac{Bz}{1+Bz}\right)} - (c+i\beta), & B \neq 0, \\ \frac{\alpha+i\beta+c}{{}_1F_1(1;\, \alpha+i\beta+c+1;\, -\alpha Az)} - (c+i\beta), & B = 0. \end{cases}$$

This implies that

$$p(z) \prec q(z) = \begin{cases} \frac{1}{\alpha} \frac{\alpha+i\beta+c}{{}_2F_1\left(1;\, \alpha\left(1-\frac{A}{B}\right);\, \alpha+i\beta+c+1;\, \frac{Bz}{1+Bz}\right)} - (c+i\beta), & B \neq 0, \\ \frac{1}{\alpha} \frac{\alpha+i\beta+c}{{}_1F_1(1;\, \alpha+i\beta+c+1;\, -\alpha Az)} - (c+i\beta), & B = 0, \end{cases}$$

and

$$Re \frac{zG'(z)}{G(z)} = Re\, p(z) > \delta = \min_{|z|=1} Re\, q(z).$$

□

Theorem 3.3. *Let $g \in \mathcal{S}^*[A, B]$. Then*

$$S(z) = \int_0^z t^{c+i\beta-1} g^{\alpha}(t)\, dt,$$

is $(\alpha + c)$-valent starlike, where $\alpha > 0$, $c > 0$ and β is a real number.

Proof. Let $D_1(z) = zS'(z) = z^{c+i\beta} g^{\alpha}(z)$ and $N_1(z) = S(z)$. Then

$$\begin{aligned} Re\frac{zD_1'(z)}{D_1(z)} &= Re\left\{(c + i\beta) + \alpha\frac{zg'(z)}{g(z)}\right\} \\ &= c + \alpha\, Re\frac{zg'(z)}{g(z)}. \end{aligned}$$

Since $g \in \mathcal{S}^*[A, B] \subset \mathcal{S}^*\left(\frac{1-A}{1-B}\right)$, see [10], it follows that

$$Re\frac{zD_1'(z)}{D_1(z)} > c + \alpha\left(\frac{1 - A}{1 - B}\right) > 0.$$

Also

$$Re\frac{D_1'(z)}{N_1'(z)} = Re\left\{(c + i\beta) + \alpha\frac{zg'(z)}{g(z)}\right\} > c + \alpha\left(\frac{1 - A}{1 - B}\right) > 0.$$

Now by using Lemma 2.5, we have

$$Re\frac{D_1(z)}{N_1(z)} > 0 \text{ or } Re\frac{zS'(z)}{S(z)} > 0.$$

By the mean value theorem for harmonic functions,

$$Re\frac{zS'(z)}{S(z)}\bigg|_{z=0} = \frac{1}{2\pi}\int_0^{2\pi} Re\frac{re^{i\theta}S'(re^{i\theta})}{S(re^{i\theta})}\, d\theta.$$

Therefore

$$\begin{aligned} \int_0^{2\pi} Re\frac{re^{i\theta}S'(re^{i\theta})}{S(re^{i\theta})}\, d\theta &= 2\pi Re\left\{c + i\beta + \alpha\frac{zg'(z)}{g(z)}\right\}\bigg|_{z=0} \\ &= 2\pi(c + \alpha). \end{aligned}$$

Now by using a result due to [9, p 212] , we have that S is $(c + \alpha)$-valent starlike function. □

4. Main results

Now we are ready to discuss some results related to the defined generalized Bazilevič functions.

Theorem 4.1. *Let f be a generalized Bazilevič function associated by the quadruple (α, β, g, p), where $g \in \mathcal{S}^*[A, B]$ of the form (1.3) and $p \in \mathcal{P}[b, A, B]$ of the form (1.2). Then for $c > 0$,*

$$F(z) = \left[\frac{c + \alpha + i\beta}{z^c}\int_0^z t^{c-1} f^{\alpha+i\beta}(t)dt\right]^{\frac{1}{\alpha+i\beta}} \tag{4.1}$$

is a generalized Bazilevič function associated by the quadruple (α, β, G, p), where $G \in \mathcal{S}^*[A, B, \delta]$, as defined by (3.1).

Proof. From (4.1), we have

$$F^{\alpha+i\beta}(z) = \frac{c+\alpha+i\beta}{z^c}\int_0^z t^{c-1}(f(t))^{\alpha+i\beta}dt.$$

This implies that

$$z^cF^{\alpha+i\beta}(z) = (c+\alpha+i\beta)\int_0^z t^{c-1}(f(t))^{\alpha+i\beta}dt.$$

Differentiate both sides and rearrange, we get

$$cz^{c-1}F^{\alpha+i\beta}(z) + (\alpha+i\beta)z^cF^{\alpha+i\beta-1}(z)F'(z) = (c+\alpha+i\beta)\, z^{c-1}(f(z))^{\alpha+i\beta},$$

and

$$\frac{z^{1-i\beta}F'(z)}{F^{1-(\alpha+i\beta)}(z)} = \frac{1}{\alpha+i\beta}\left\{(c+\alpha+i\beta)z^{-i\beta}f^{\alpha+i\beta}(z) - cz^{-i\beta}F^{\alpha+i\beta}(z)\right\}.$$

Now from (3.1), we have

$$\begin{aligned}&\frac{z^{1-i\beta}F'(z)}{F^{1-(\alpha+i\beta)}G^{\alpha}(z)}\\ &= \frac{\frac{1}{\alpha+i\beta}\left\{(c+\alpha+i\beta)z^{-i\beta}f^{\alpha+i\beta}(z) - cz^{-i\beta-c}(c+\alpha+i\beta)\int_0^z t^{c-1}(f(t))^{\alpha+i\beta}dt\right\}}{\frac{(c+\alpha+i\beta)}{z^{c+i\beta}}\int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt}\\ &= \frac{\frac{1}{\alpha+i\beta}\left\{(z^cf^{\alpha+i\beta}(z) - c\int_0^z t^{c-1}(f(t))^{\alpha+i\beta}dt\right\}}{\int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt}\\ &:= \frac{N(z)}{D(z)}.\end{aligned}$$

With this, note that

$$\begin{aligned}\frac{N'(z)}{D'(z)} &= \frac{\frac{1}{\alpha+i\beta}\left\{(cz^{c-1}f^{\alpha+i\beta}(z) + (\alpha+i\beta)z^cf^{\alpha+i\beta-1}(z)f'(z) - cz^{c-1}(f(z))^{\alpha+i\beta}\right\}}{z^{c+i\beta-1}g^{\alpha}(z)}\\ &= \frac{z^{1-i\beta}f'(z)}{f^{1-(\alpha+i\beta)}(z)g^{\alpha}(z)},\end{aligned}$$

which implies $\frac{N'(z)}{D'(z)} \in \mathcal{P}[b, A, B]$. By Theorem 3.3, we know that $D(z) = \int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt$ is $(\alpha+c)$-valent starlike. Therefore by using Lemma 2.4, we obtain

$$\frac{z^{1-i\beta}F'(z)}{F^{1-(\alpha+i\beta)}(z)G^{\alpha}(z)} \in \mathcal{P}[b, A, B].$$

This is the equivalent form of Definition 1.1. Hence the result follows. □

Corollary 4.2. *Let* $A = 1, B = -1$ *and* $\beta = 0$ *in Theorem 4.1. Then*

$$G^{\alpha}(z) = \frac{(\alpha+c)}{z^c}\int_0^z t^{c-1}g^{\alpha}(t)dt$$

belong to $S^*(\delta_1)$, where

$$\delta_1 = \frac{-(1+2c) + \sqrt{(1+2c)^2 + 8\alpha}}{4\alpha}, (\textit{see [16]}).$$

Hence G is starlike when $g \in \mathcal{S}^*$, and

$$F^\alpha(z) = \frac{(\alpha + c)}{z^c} \int_0^z t^{c-1} g^\alpha(t) dt$$

belongs to the class of Bazilevič functions associated by the quadruple $(\alpha, 0, G, p)$.

Theorem 4.3. *Let f of the form (1.1) be a generalized Bazilevič function associated by the quadruple (α, β, g, p), with $g \in \mathcal{S}^*[A, B]$ of the form (1.3) and $p \in \mathcal{P}[b, A, B]$ of the form (1.2). Then*

$$\left| a_3 - \frac{3 + \alpha + i\beta}{2(2 + \alpha + i\beta)} a_2^2 \right| \le \frac{A - B}{2\,|2 + \alpha + i\beta|} \left[\alpha + |b| \max\{2, |b(A - B) + 2B|\}\right].$$

This inequality is sharp.

Proof. Since f is a generalized Bazilevič function associated by the quadruple (α, β, g, p), we have

$$1 + \frac{zf''(z)}{f'(z)} + (\alpha + i\beta - 1)\frac{zf'(z)}{f(z)} = \alpha \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)} + i\beta. \tag{4.2}$$

As f, g and p respectively have the form (1.1), (1.3) and (1.2), it is easy to get

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= 1 + 2a_2 z + \left(6a_3 - 4a_2^2\right) z^2 + \cdots, \\ \frac{zf'(z)}{f(z)} &= 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \cdots, \\ \frac{zg'(z)}{g(z)} &= 1 + b_2 z + \left(2b_3 - b_2^2\right) z^2 + \cdots, \\ \frac{zp'(z)}{p(z)} &= p_1 z + \left(2p_2 - p_1^2\right) z^2 + \cdots. \end{aligned}$$

Putting these values in (4.2) and comparing the coefficients of z, we obtain

$$(1 + \alpha + i\beta) a_2 = \alpha b_2 + p_1. \tag{4.3}$$

Similarly by comparing the coefficients of z^2 and rearranging, we have

$$2(2 + \alpha + i\beta) a_3 = \alpha(2b_3 - b_2^2) + 2p_2 - p_1^2 + a_2^2(3 + \alpha + i\beta). \tag{4.4}$$

From (4.4), we have

$$\left| a_3 - \frac{3 + \alpha + i\beta}{2(2 + \alpha + i\beta)} a_2^2 \right| = \left| \frac{\alpha\left(b_3 - \frac{1}{2}b_2^2\right) + (p_2 - \frac{1}{2}p_1^2)}{2 + \alpha + i\beta} \right|$$

$$\leq \frac{\alpha\left|b_3 - \frac{1}{2}b_2^2\right|}{|2+\alpha+i\beta|} + \frac{\left|p_2 - \frac{1}{2}p_1^2\right|}{|2+\alpha+i\beta|}.$$

Now by using Theorem 3.1 and Lemma 2.3, both with $\mu = \frac{1}{2}$, we obtain

$$\left|b_3 - \frac{1}{2}b_2^2\right| \leq \frac{A-B}{2}\max\{1, |B|\} = \frac{A-B}{2},$$

and

$$\left|p_2 - \frac{1}{2}p_1^2\right| \leq |b|\,(A-B)\max\left\{1, \frac{1}{2}\Big|b(A-B)+2B\Big|\right\}.$$

Therefore, we have

$$\left|a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| \leq \frac{A-B}{2\,|2+\alpha+i\beta|}\left[\alpha + 2\,|b|\max\left\{1,\ \frac{1}{2}\Big|b(A-B)+2B\Big|\right\}\right].$$

The equality

$$\left|b_3 - \frac{1}{2}b_2^2\right| = \frac{A-B}{2}$$

for $B \neq 0$ can be obtained for

$$g(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}} = z + (A-B)\,z^2 + \frac{1}{2}\,(A-B)\,(A-2B)\,z^3 + \cdots, \\ z(1+Bz^2)^{\frac{A-B}{2B}} = z + \frac{1}{2}\,(A-B)\,z^3 + \cdots. \end{cases}$$

Similarly, the equality

$$\left|b_3 - \frac{1}{2}b_2^2\right| = \frac{A}{2}$$

for $B = 0$ can be obtained for the function $g_*(z) = ze^{\frac{A}{2}z^2} = z + \frac{A}{2}z^3 + \cdots$. Also equality for the functional $\left|p_2 - \frac{1}{2}p_1^2\right|$ can be obtained by the functions

$$p_\circ(z) = \frac{1 + (bA + (1-b)B)\,z}{1+Bz} \quad \text{or} \quad p_1(z) = \frac{1 + (bA + (1-b)B)\,z^2}{1+Bz^2}. \qquad \square$$

Corollary 4.4. *For $A = 1$, $B = -1$ and $b = 1$, we have the result proved in [8]:*

$$\left|a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| \leq \frac{\alpha+2}{|2+\alpha+i\beta|}.$$

For $\alpha = 1, \beta = 0$, we have $f \in \mathcal{K}$, the class of close-to-convex functions, and

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leq 1.$$

The latter result has been proved in [11].

Theorem 4.5. *Let f of the form* (1.1) *be a generalized Bazilevič function associated by the quadruple (α, β, g, p), with $g \in \mathcal{S}^*[A, B]$ and of the form* (1.3) *and $p \in \mathcal{P}[b, A, B]$ of the form* (1.2)*. Then*
(i)

$$|a_2| \leq \frac{(A-B)(\alpha+|b|)}{|1+\alpha+i\beta|}.$$

(ii) If $\alpha = 0$, *then*

$$|a_3| \leq \frac{|b|\,(A-B)}{|2+i\beta|} \max\left\{1, \left|\frac{b(A-B)}{2}\left(1-\frac{(3+i\beta)}{(1+i\beta)^2}\right)+B\right|\right\}.$$

Both the above inequalities are sharp.

Proof. (i) From (4.3), we have

$$(1+\alpha+i\beta)a_2 = \alpha b_2 + p_1.$$

This implies that

$$|a_2| \leq \frac{\alpha\,|b_2| + |p_1|}{|1+\alpha+i\beta|}.$$

By using the coefficient bound for $\mathcal{S}^*[A,B]$ along with the coefficient bound of $\mathcal{P}[b,A,B]$, we have

$$|b_2| \leq A-B \text{ and } |p_1| \leq |b|\,(A-B).$$

This implies that

$$|a_2| \leq \frac{(\alpha+|b|)\,(A-B)}{|1+\alpha+i\beta|}.$$

Equality can be obtained by the functions

$$g_\circ(z) = z(1+Bz)^{\frac{A-B}{B}},\ B\neq 0 \text{ and } p_\circ(z) = \frac{1+[bA+(1-b)B]z}{1+Bz}.$$

(ii) Let $\alpha = 0$. Then from (4.3) and (4.4), we have

$$\begin{aligned}(2+i\beta)a_3 &= p_2 - \frac{1}{2}p_1^2 + \frac{(3+i\beta)p_1^2}{2\,(1+i\beta)^2}\\ &= p_2 - \frac{1}{2}\left(1-\frac{(3+i\beta)}{(1+i\beta)^2}\right)p_1^2.\end{aligned}$$

This implies

$$|a_3| = \frac{1}{|2+i\beta|}\left|p_2 - \mu p_1^2\right|,$$

where $\mu = \frac{1}{2} - \frac{(3+i\beta)}{2(1+i\beta)^2}$. Now by using Lemma 2.3, we obtain

$$|a_3| \leq \frac{|b|\,(A-B)}{|2+i\beta|} \max\left\{1, \left|\frac{b(A-B)}{2}\left(\frac{-2-\beta^2+i\beta}{(1+i\beta)^2}\right)+B\right|\right\}.$$

Sharpness can be attained by the functions

$$\begin{aligned}p_0(z) &= \frac{1+(bA+(1-b)B)\,z^2}{1+Bz^2} = 1 + b(A-B)z^2 + b(A-B)(-B)z^4 + \cdots,\\ p_1(z) &= \frac{1+(bA+(1-b)B)\,z}{1+Bz} = 1 + b(A-B)z + b(A-B)(-B)z^2 + \cdots.\end{aligned}$$

□

Corollary 4.6. *For $A = 1$, $B = -1$ and $b = 1$, we have*

$$|a_2| \leq \frac{2(\alpha + 1)}{|1 + \alpha + i\beta|},$$

and

$$|a_3| = \frac{2}{|2 + i\beta|} \max\left\{1, \left|\frac{(3 + i\beta)}{(1 + i\beta)^2}\right|\right\}.$$

In the final part of this paper, we look at some results for the generalized Bazilevič functions associated with $\beta = 0$.

Let C_r denote the closed curve which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$, and $L_r(f(z))$ denote the length of C_r. Also let $M(r) = \max_{|z|=r} |f(z)|$ and $m(r) = \min_{|z|=r} |f(z)|$. We now prove the following result.

Theorem 4.7. *Let f be a generalized Bazilevič function associated by the quadruple $(\alpha, 0, g, p)$. Then for $B \neq 0$,*

$$L_r(f(z)) \leq \begin{cases} C(\alpha, b, A, B) M^{1-\alpha}(r)\left[1 - (1 - r)^{\alpha\left(\frac{A-B}{B}\right)}\right], & 0 < \alpha \leq 1, \\ C(\alpha, b, A, B) m^{1-\alpha}(r)\left[1 - (1 - r)^{\alpha\left(\frac{A-B}{B}\right)}\right], & \alpha > 1, \end{cases}$$

where

$$C(\alpha, b, A, B) = 2\pi|b|B\left[(A - B) + \frac{1}{\alpha}\right].$$

Proof. As f is a generalized Bazilevič function associated by the quadruple $(\alpha, 0, g, p)$, we have

$$zf'(z) = f^{1-\alpha}(z)g^{\alpha}(z)p(z),$$

where $g \in \mathcal{S}^*[A, B]$ and $p \in \mathcal{P}[b, A, B]$. Since for $z = re^{i\theta}$, $0 < r < 1$,

$$L_r(f(z)) = \int_0^{2\pi} |zf'(z)|\, d\theta,$$

we have for $0 < \alpha \leq 1$,

$$\begin{aligned} &L_r(f(z)) \\ &= \int_0^{2\pi} \left|f^{1-\alpha}(z)g^{\alpha}(z)p(z)\right| d\theta, \\ &\leq M^{1-\alpha}(r) \int_0^{2\pi} \int_0^r \left|\alpha g'(z)\, g^{\alpha-1}(z)\, p(z) + g^{\alpha}(z)\, p'(z)\right| ds\, d\theta, \\ &\leq M^{1-\alpha}(r) \left\{\int_0^{2\pi} \int_0^r \frac{\alpha|g^{\alpha}(z)|}{s} \left|h(z)\, p(z)\right| ds\, d\theta + \int_0^{2\pi} \int_0^r \frac{|g^{\alpha}(z)|}{s} \left|zp'(z)\right| ds\, d\theta\right\}, \end{aligned}$$

where $\frac{zg'(z)}{g(z)} = h(z) \in \mathcal{P}[A, B]$. Now by using the distortion theorem for Janowski starlike functions when $B \neq 0$ (see [10]) and the Cauchy-Schwarz inequality, we have

$$L_r(f(z)) \leq M^{1-\alpha}(r)$$

$$\times \int_0^r \frac{s^{\alpha-1}}{(1-|B|\,s)^{\alpha\frac{B-A}{B}}} \left\{ \alpha \sqrt{\int_0^{2\pi} |h\,(z)|^2\, d\theta} \sqrt{\int_0^{2\pi} |p(z)|^2\, d\theta} + \int_0^{2\pi} |zp'(z)|\, d\theta \right\} ds.$$

Now by using Lemma 2.1 for both the classes $\mathcal{P}[b, A, B]$ and $\mathcal{P}[A, B]$, along with the result

$$\int_0^{2\pi} |zp'(z)| \leq \frac{|b|\,(A-B)\,r}{1-B^2r^2},$$

for $p \in \mathcal{P}[b, A, B]$ (see [19]), we can write

$$L_r(f(z)) \leq 2\pi M^{1-\alpha}(r)$$
$$\times \int_0^r \frac{s^{\alpha-1}}{(1-|B|\,s)^{\alpha\frac{B-A}{B}}} \left\{ \alpha \sqrt{\frac{1+[(A-B)^2-1]\,s^2}{1-s^2}} \sqrt{\frac{1+\left[|b|^2\,(A-B)^2-1\right] s^2}{1-s^2}} + \frac{|b|\,(A-B)\,s}{1-B^2s^2} \right\} ds.$$

Since $1-|B|\,r \geq 1-r$ and $1-B^2r^2 \geq 1-r^2$,

$$\begin{aligned} L_r\,(f(z)) &\leq 2\pi M^{1-\alpha}(r) \left[|b|\,(A-B)^2 + |b|\,(A-B)\right] \int_0^r \frac{1}{(1-s)^{\alpha\frac{B-A}{B}+1}} ds \\ &= C(\alpha, b, A, B) M^{1-\alpha}(r) \left[1-(1-r)^{\alpha\left(\frac{A-B}{B}\right)}\right], \end{aligned}$$

where $C(\alpha, b, A, B) = 2\pi|b|[(A-B)+(1/\alpha)]B$.

When $\alpha > 1$, we can prove similarly as above to get

$$L_r\,(f(z)) \leq C(\alpha, b, A, B) m^{1-\alpha}(r) \left[1-(1-r)^{\alpha\left(\frac{A-B}{B}\right)}\right]. \qquad \square$$

Corollary 4.8. *For $g \in \mathcal{S}^*$ and $p \in \mathcal{P}(b)$, we have*

$$L_r(f(z)) \leq \begin{cases} 2\pi|b|\left(2+\frac{1}{\alpha}\right) M^{1-\alpha}(r) \left[\frac{1}{(1-r)^{2\alpha}} - 1\right], & 0 < \alpha \leq 1, \\ 2\pi|b|\left(2+\frac{1}{\alpha}\right) m^{1-\alpha}(r) \left[\frac{1}{(1-r)^{2\alpha}} - 1\right], & \alpha > 1. \end{cases}$$

Theorem 4.9. *Let f be a generalized Bazilevič function associated by the quadruple $(\alpha, 0, g, p)$, where $g \in \mathcal{S}^*[A, B]$ and $p \in \mathcal{P}[b, A, B]$. Then for $B \neq 0$,*

$$|a_n| \leq \begin{cases} \frac{1}{n}|b|B\left(A-B+\frac{1}{\alpha}\right) \lim_{r\to 1^-} M^{1-\alpha}(r), & 0 < \alpha \leq 1, \\ \frac{1}{n}|b|B\left(A-B+\frac{1}{\alpha}\right) \lim_{r\to 1^-} m^{1-\alpha}(r), & \alpha > 1. \end{cases}$$

Proof. By Cauchy's theorem for $z = re^{i\theta}$, $n \geq 2$, we have

$$na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z)e^{-in\theta} d\theta.$$

Therefore

$$n\,|a_n| \le \frac{1}{2\pi r^n}\int_0^{2\pi}|zf'(z)|\,d\theta,$$
$$= \frac{1}{2\pi r^n}L_r(f(z)).$$

By using Theorem 4.7 for the case $0 < \alpha \le 1$, we have

$$n\,|a_n| \le \frac{1}{2\pi r^n}\left(2\pi|b|B\left(A - B + \frac{1}{\alpha}\right)M^{1-\alpha}(r)\left[1-(1-r)^{\alpha\frac{A-B}{B}}\right]\right).$$

Hence, by taking r approaches 1^-,

$$|a_n| \le \frac{1}{n}|b|B\left(A - B + \frac{1}{\alpha}\right)\lim_{r\to 1^-}M^{1-\alpha}(r).$$

For $\alpha > 1$, we have

$$|a_n| \le \frac{1}{n}|b|B\left(A - B + \frac{1}{\alpha}\right)\lim_{r\to 1^-}m^{1-\alpha}(r).$$ □

Theorem 4.10. *Let f be a generalized Bazilevič function represented by the quadruple $(\alpha, 0, g, p)$, where $g \in \mathcal{S}^*[A, B]$ and $p \in \mathcal{P}[b, A, B]$. Then for $B \neq 0$,*

$$|f(z)|^\alpha \le \alpha\frac{(1-B^2)+(A-B)\,(|b| - B\,Re(b))}{1-B}r^\alpha\,{}_2F_1\left(\alpha\left(1-\frac{A}{B}\right)+1;\alpha;\alpha+1;|B|\,r\right).$$

Proof. Since f is a generalized Bazilevič function associated by the quadruple $(\alpha, 0, g, p)$, by definition, we have

$$\frac{zf'(z)}{f^{1-\alpha}(z)g^\alpha(z)} = p(z),$$

where $g \in \mathcal{S}^*[A, B]$ and $p \in \mathcal{P}[b, A, B]$. This implies that

$$f^\alpha(z) = \alpha\int_0^z t^{-1}g^\alpha(t)p(t)dt,$$

and so

$$|f(z)|^\alpha \le \alpha\int_0^{|z|}|t^{-1}||g^\alpha(t)||p(t)|d|t|,$$
$$= \alpha\int_0^r s^{-1}|g^\alpha(t)||p(t)|ds.$$

Now by using the results

$$|g(z)| \le r(1-|B|\,r)^{\frac{A-B}{B}}, B \neq 0,\ \text{(see [10])},$$

and

$$|p(z)| \le \frac{1+|b|\,(A-B)r - B\,[(A-B)Re\{b\}+B]\,r^2}{1-|B|^2\,r^2},\ \text{(see [6])},$$

we have

$$|f(z)|^{\alpha} \leq \alpha \int_0^r s^{-1} \frac{s^{\alpha}}{(1-|B|\,s)^{\alpha\left(1-\frac{A}{B}\right)}} \frac{1+|b|(A-B)s - B\left[(A-B)Re\{b\}+B\right]s^2}{1-|B|^2\,s^2} ds$$
$$\leq \alpha \frac{(1-B^2)+(A-B)\left(|b|-BRe\{b\}\right)}{1+|B|} \int_0^r s^{\alpha-1}(1-|B|\,s)^{-\alpha\left(1-\frac{A}{B}\right)-1} ds.$$

Putting $s = ru$, we have

$$|f(z)|^{\alpha} \leq \alpha \frac{(1-B^2)+(A-B)\left(|b|-BRe\{b\}\right)}{1-B} r^{\alpha} \int_0^1 u^{\alpha-1}(1-|B|ru)^{-\alpha\left(1-\frac{A}{B}\right)-1} du$$
$$= \frac{(1-B^2)+(A-B)\left(|b|-BRe\{b\}\right)}{1-B} r^{\alpha} \, {}_2F_1\left(\alpha\left(1-\frac{A}{B}\right)+1; \alpha; \alpha+1; |B|\,r\right),$$

where ${}_2F_1\,(a; b; c; z)$ is the hypergeometric function. □

Corollary 4.11. *For $g \in \mathcal{S}^*$ and $p \in \mathcal{P}$, we have*

$$|f(z)|^{\alpha} \leq 2\alpha r^{\alpha} \, {}_2F_1\,(2\alpha+1; \alpha; \alpha+1; r)\,.$$

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Conflict of interest

The authors declare that they have no conflict of interests.

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