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Research article

Weighted Ostrowski, trapezoid and midpoint type inequalities for Riemann-Liouville fractional integrals

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Abstract: Our first aim is to establish two new identities for differentiable function involving Riemann-Liouville fractional integrals. Then, we obtain some new weighted versions of fractional trapezoid and Ostrowski type inequalities. Moreover, we give some weighted fractional midpoint type inequalities as special cases.

Keywords: trapezoid inequality; Ostrowski inequality; fractional integral operators; convex function; concave function
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1. Introduction

In recent years, the Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [9, 30, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1.1). For some examples, please refer to ([2, 4, 7, 9, 10, 23, 33, 34, 36, 37, 44])

The overall structure of the paper takes the form of four sections including introduction. The remainder of this work is organized as follows: we first give weighted version of (1.1) and definitions of Riemann-Liouville fractional integral operators. We also mention fractional Hermite-Hadamard type inequalities obtained in earlier works. In Section 2, we establish two important weighted equalities for differentiable functions involving fractional integrals. Using an identity given Section 2, we obtain some weighted fractional trapezoid type inequalities in Section 3. In Section 4, we first mention classical and fractional Ostrowski inequalities for bounded variation. Then, utilizing other identity given Section 2, we obtain some fractional Ostrowski type inequalities for convex functions. We also give some fractional midpoint type inequalities.

The weighted version of the inequalities (1.1), so-called Hermite-Hadamard-Fejér inequalities, was given by Fejer in [13] as follow:

Theorem 1. $f:[a,b]\to\mathbb{R}$*, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g:[a,b]\to\mathbb{R}$ *is non-negative, integrable, and symmetric about* $x=\frac{a+b}{2}$ *(i.e.,* $g(x)=g(a+b-x)$*).*

Tseng et al. give the following Lemma and by using this Lemma they obtain several weighed inequalities in [51].

Lemma 1. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *and let* $g:[a,b]\to\mathbb{R}$*. If* $f',g\in L[a,b]$*, then for all* $x\in[a,b]$ *we have the following equality for fractional integrals*

$$f(a)\int_a^x g(t)dt + f(b)\int_x^b g(t)dt - \int_a^b f(t)g(t)dt = \int_a^b \left(\int_x^t g(s)ds\right) f'(t)dt. \tag{1.3}$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. *Let* $f\in L_1[a,b]$*. The Riemann-Liouville integrals* $J_{a+}^{\alpha}f$ *and* $J_{b-}^{\alpha}f$ *of order* $\alpha>0$ *with* $a\ge 0$ *are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\quad x<b$$

respectively. Here, $\Gamma(\alpha)$ *is the Gamma function and* $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$*.*

For more information about fraction calculus please refer to [14, 21, 27, 35].

In [40], Sarikaya et al. first gave the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha>0$.

On the other hand, Işcan gave following Lemma and using this Lemma he proved the following Fejer type inequalities for Riemann-Liouville fractional integrals in [18].

Lemma 2. *If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $(a+b)/2$ with $a<b$, then*

$$J_{a+}^{\alpha}g(b)=J_{b-}^{\alpha}g(a)=\frac{1}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]$$

with $\alpha>0$.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be convex function with $0\leq a<b$ and $f\in L_1[a,b]$. If $g:[a,b]\to\mathbb{R}$ is non-negative,integrable and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] &\leq \left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \end{aligned} \tag{1.5}$$

with $\alpha>0$

Whereupon Sarikaya et al. obtained the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, please see ([3, 8, 12, 17–20, 22, 28, 32, 41–43, 46, 47, 52, 53, 56]).

2. Some weighted fractional equalities

In this section, we prove two identities which will be used frequently in Section 3 and Section 4 to obtain weighted fractional inequalities.

Lemma 3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f',g\in L[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$ we have the following equality for fractional integrals*

$$f(a)J_{a+}^{\alpha}g(x)+f(b)J_{b-}^{\alpha}g(x)-\left[J_{a+}^{\alpha}(fg)(x)+J_{b-}^{\alpha}(fg)(x)\right] \tag{2.1}$$

$$=\frac{1}{\Gamma(\alpha)}\left[\int_a^x\left(\int_x^t(x-s)^{\alpha-1}g(s)ds\right)f'(t)dt+\int_x^b\left(\int_x^t(s-x)^{\alpha-1}g(s)ds\right)f'(t)dt\right].$$

Proof. Integrating by parts, we have

$$\int_a^x \left(\int_x^t (x-s)^{\alpha-1} g(s)ds \right) f'(t)dt \tag{2.2}$$

$$= \left(\int_x^t (x-s)^{\alpha-1} g(s)ds \right) f(t) \Big|_a^x - \int_a^x (x-t)^{\alpha-1} g(t)f(t)dt$$

$$= \left(\int_a^x (x-s)^{\alpha-1} g(s)ds \right) f(a) - \int_a^x (x-t)^{\alpha-1} g(t)f(t)dt$$

$$= \Gamma(\alpha) \left[f(a) J_{a+}^{\alpha} g(x) - J_{a+}^{\alpha} (fg)(x) \right].$$

Similarly, we get

$$\int_x^b \left(\int_x^t (s-x)^{\alpha-1} g(s)ds \right) f'(t)dt = \Gamma(\alpha) \left[f(b) J_{b-}^{\alpha} g(x) - J_{b-}^{\alpha} (fg)(x) \right]. \tag{2.3}$$

By the identities (2.2) and (2.3), we obtain the required result (2.1). □

Remark 1. *If we choose $\alpha = 1$ in Lemma 3, then the identity (2.1) reduces to identity (1.3) proved by Tseng et al. in [51].*

Corollary 1. *In Lemma 3, let g be symmetric to $\frac{a+b}{2}$ and let $x = \frac{a+b}{2}$. Then (2.1) can be written as*

$$\left[J_{a+}^{\alpha} g\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} g\left(\frac{a+b}{2}\right) \right] \frac{f(a)+f(b)}{2}$$

$$- \left[J_{a+}^{\alpha} (fg)\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} (fg)\left(\frac{a+b}{2}\right) \right]$$

$$= \frac{1}{\Gamma(\alpha)} \left[\int_a^{\frac{a+b}{2}} \left(\int_{\frac{a+b}{2}}^t \left(\frac{a+b}{2} - s \right)^{\alpha-1} g(s)ds \right) f'(t)dt + \int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^t \left(s - \frac{a+b}{2} \right)^{\alpha-1} g(s)ds \right) f'(t)dt \right].$$

Proof. If we take $x = \frac{a+b}{2}$ in Lemma 3, since g is symmetric to $\frac{a+b}{2}$, then we have

$$J_{a+}^{\alpha} g\left(\frac{a+b}{2}\right) = \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - s \right)^{\alpha-1} g(s)ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \left(u - \frac{a+b}{2} \right)^{\alpha-1} g(a+b-u)ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} \left(u - \frac{a+b}{2}\right)^{\alpha-1} g(u)ds$$

$$= J_{b-}^{\alpha} g\left(\frac{a+b}{2}\right).$$

This completes the proof. □

Corollary 2. *Under assumptions of Lemma 3, we have the following equality*

$$f(a)J_{a+}^{\alpha}g(b) + f(b)J_{b-}^{\alpha}g(a) - \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right]$$

$$= \frac{1}{\Gamma(\alpha)}\left[\int_a^b \left(\int_a^t (s-a)^{\alpha-1} g(s)ds\right) f'(t)dt + \int_a^b \left(\int_b^t (b-s)^{\alpha-1} g(s)ds\right) f'(t)dt\right].$$

Proof. If we write the equality (2.1) for $x = a$ and $x = b$, then we have the identities

$$f(b)J_{b-}^{\alpha}g(a) - J_{b-}^{\alpha}(fg)(a) = \frac{1}{\Gamma(\alpha)} \int_a^b \left(\int_a^t (s-a)^{\alpha-1} g(s)ds\right) f'(t)dt \tag{2.4}$$

and

$$f(a)J_{a+}^{\alpha}g(b) - J_{a+}^{\alpha}(fg)(b) = \frac{1}{\Gamma(\alpha)} \int_a^b \left(\int_b^t (b-s)^{\alpha-1} g(s)ds\right) f'(t)dt, \tag{2.5}$$

respectively. By adding the equalities (2.4) and (2.5), we obtain desired result. □

Corollary 3. *In Lemma 3, let $g(t) = 1$ for all $t \in [a, b]$. Then we have the following identity*

$$(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b) - \Gamma(\alpha+1)\left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x)\right]$$

$$= -\left[\int_a^x (x-t)^{\alpha} f'(t)dt + \int_x^b (t-x)^{\alpha} f'(t)dt\right].$$

Lemma 4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f', g \in L[a, b]$, then for all $x \in [a, b]$ and $\alpha > 0$ we have the following equality for fractional integrals*

$$\left[J_{a+}^{\alpha}g(x) + J_{b-}^{\alpha}g(x)\right] f(x) - \left[J_{a+}^{\alpha}(fg)(x) + J_{b-}^{\alpha}(fg)(x)\right] \tag{2.6}$$

$$= \frac{1}{\Gamma(\alpha)}\left[\int_a^x \left(\int_a^t (x-s)^{\alpha-1} g(s)ds\right) f'(t)dt + \int_x^b \left(\int_b^t (s-x)^{\alpha-1} g(s)ds\right) f'(t)dt\right].$$

Proof. Integrating by parts, we have

$$\int_a^x \left(\int_a^t (x-s)^{\alpha-1} g(s)ds \right) f'(t)dt \tag{2.7}$$

$$= \left(\int_a^t (x-s)^{\alpha-1} g(s)ds \right) f(t) \Big|_a^x - \int_a^x (x-t)^{\alpha-1} g(t) f(t)dt$$

$$= \left(\int_a^x (x-s)^{\alpha-1} g(s)ds \right) f(x) - \int_a^x (x-t)^{\alpha-1} g(t) f(t)dt$$

$$= \Gamma(\alpha) \left[f(x) J_{a+}^{\alpha} g(x) - J_{a+}^{\alpha} (fg)(x) \right].$$

Similarly, we get

$$\int_x^b \left(\int_b^t (s-x)^{\alpha-1} g(s)ds \right) f'(t)dt = \Gamma(\alpha) f(x) \left[J_{b-}^{\alpha} g(x) - J_{b-}^{\alpha} (fg)(x) \right]. \tag{2.8}$$

By the identities (2.7) and (2.8), we obtain the required result (2.6). □

Remark 2. *If we choose* $\alpha = 1$ *in Lemma 4, then we have*

$$f(x) \int_a^b g(t)dt - \int_a^b f(t)g(t)dt = \int_a^x \left(\int_a^t g(s)ds \right) f'(t)dt + \int_x^b \left(\int_b^t g(s)ds \right) f'(t)dt.$$

Corollary 4. *If we choose* $g(t) = 1$ *for all* $t \in [a, b]$ *in Lemma 4, then we have*

$$\left(\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{b-a} \left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) \right]$$

$$= \frac{1}{b-a} \left[\int_a^x \left((x-a)^{\alpha} - (x-t)^{\alpha} \right) f'(t)dt + \int_x^b \left((b-x)^{\alpha} - (t-x)^{\alpha} \right) f'(t)dt \right].$$

Corollary 5. *Under assumption of Lemma 4, let* g *be symmetric to* $\frac{a+b}{2}$*. Then we have the following identity*

$$\frac{f(a)+f(b)}{2} \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a) \right] - \left[J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a) \right] \tag{2.9}$$

$$= \frac{1}{\Gamma(\alpha)} \left[\int_a^b \left(\int_a^t (b-s)^{\alpha-1} g(s)ds \right) f'(t)dt + \int_a^b \left(\int_b^t (s-a)^{\alpha-1} g(s)ds \right) f'(t)dt \right].$$

Proof. If we apply the identity (2.6) for $x = a$ and $x = b$, then we obtain the equalities

$$f(a)J_{b-}^{\alpha}g(a) - J_{b-}^{\alpha}(fg)(a) = \frac{1}{\Gamma(\alpha)}\int_a^b \left(\int_b^t (s-a)^{\alpha-1} g(s)ds\right) f'(t)dt \tag{2.10}$$

and

$$f(b)J_{a+}^{\alpha}g(b) - J_{a+}^{\alpha}(fg)(b) = \frac{1}{\Gamma(\alpha)}\int_a^b \left(\int_a^t (b-s)^{\alpha-1} g(s)ds\right) f'(t)dt \tag{2.11}$$

respectively. If we add the equalities (2.10) and (2.11), then by using Lemma 2, we establish the desired result (2.9). □

The inequality (2.9) is the same result which is given by İşcan in [18].

3. Weighted fractional trapezoid type inequalities

In this section, we obtain some weighted trapezoid inequalities involving Riemann-Liouviille fractional integral operators. We also give some special cases of our results.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is continuous and if $|f'|$ is convex on $[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$ we have the following inequality for fractional integrals,*

$$\begin{aligned} &\left|f(a)J_{a+}^{\alpha}g(x) + f(b)J_{b-}^{\alpha}g(x) - \left[J_{a+}^{\alpha}(fg)(x) + J_{b-}^{\alpha}(fg)(x)\right]\right| \qquad (3.1) \\ \leq\ & \frac{1}{(b-a)\Gamma(\alpha+3)} \\ &\times\Big\{\Big[(x-a)^{\alpha+1}\left[(\alpha+2)(b-x)+(\alpha+1)(x-a)\right]\|g\|_{[a,x],\infty} + (b-x)^{\alpha+2}\|g\|_{[x,b],\infty}\Big]|f'(a)| \\ &+\Big[(x-a)^{\alpha+2}\|g\|_{[a,x],\infty} + (b-x)^{\alpha+1}\left[(\alpha+1)(b-x)+(\alpha+2)(x-a)\right]\|g\|_{[x,b],\infty}\Big]|f'(b)|\Big\} \\ \leq\ & \frac{\|g\|_{\infty}}{(b-a)\Gamma(\alpha+3)}\Big\{\Big[\Big((x-a)^{\alpha+1}\left[(\alpha+2)(b-x)+(\alpha+1)(x-a)\right] + (b-x)^{\alpha+2}\Big)|f'(a)| \\ &+\Big((x-a)^{\alpha+2} + (b-x)^{\alpha+1}\left[(\alpha+1)(b-x)+(\alpha+2)(x-a)\right]|f'(b)|\Big)\Big\}. \end{aligned}$$

Proof. By taking modulus in Lemma 3, we have

$$\begin{aligned} &\left|f(a)J_{a+}^{\alpha}g(x) + f(b)J_{b-}^{\alpha}g(x) - \left[J_{a+}^{\alpha}(fg)(x) + J_{b-}^{\alpha}(fg)(x)\right]\right| \qquad (3.2) \\ \leq\ & \frac{1}{\Gamma(\alpha)}\left[\left|\int_a^x \left(\int_x^t (x-s)^{\alpha-1} g(s)ds\right) f'(t)dt\right| + \left|\int_x^b \left(\int_x^t (s-x)^{\alpha-1} g(s)ds\right) f'(t)dt\right|\right] \end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \left|\int_x^t (x-s)^{\alpha-1} g(s)ds\right| |f'(t)|\, dt + \int_x^b \left|\int_x^t (s-x)^{\alpha-1} g(s)ds\right| |f'(t)|\, dt\right]$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\|g\|_{[a,x],\infty}\int_a^x \left|\int_x^t (x-s)^{\alpha-1}\, ds\right| |f'(t)|\, dt + \|g\|_{[x,b],\infty}\int_x^b \left|\int_x^t (s-x)^{\alpha-1}\, ds\right| |f'(t)|\, dt\right]$$

$$= \frac{1}{\Gamma(\alpha+1)}\left[\|g\|_{[a,x],\infty}\int_a^x (x-t)^{\alpha} |f'(t)|\, dt + \|g\|_{[x,b],\infty}\int_x^b (t-x)^{\alpha} |f'(t)|\, dt\right].$$

Since $|f'|$ is convex on $[a,b]$, we get

$$|f'(t)| = \left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right| \leq \frac{b-t}{b-a}|f'(a)| + \frac{t-a}{b-a}|f'(b)|. \tag{3.3}$$

By using (3.3) in (3.2), we obtain

$$\left|f(a)J_{a+}^{\alpha}g(x) + f(b)J_{b-}^{\alpha}g(x) - \left[J_{a+}^{\alpha}(fg)(x) + J_{b-}^{\alpha}(fg)(x)\right]\right| \tag{3.4}$$

$$\leq \frac{\|g\|_{[a,x],\infty}}{(b-a)\Gamma(\alpha+1)}\int_a^x (x-t)^{\alpha}\left[(b-t)|f'(a)| + (t-a)|f'(b)|\right] dt$$

$$+\frac{\|g\|_{[x,b],\infty}}{(b-a)\Gamma(\alpha+1)}\int_x^b (t-x)^{\alpha}\left[(b-t)|f'(a)| + (t-a)|f'(b)|\right] dt.$$

By simple calculations, one can establish

$$\int_a^x (x-t)^{\alpha}(b-t)\, dt = \frac{(x-a)^{\alpha+1}\left[(\alpha+2)(b-x) + (\alpha+1)(x-a)\right]}{(\alpha+1)(\alpha+2)}, \tag{3.5}$$

$$\int_a^x (x-t)^{\alpha}(t-a)\, dt = \frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}, \tag{3.6}$$

$$\int_x^b (t-x)^{\alpha}(b-t)\, dt = \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \tag{3.7}$$

and

$$\int_x^b (t-x)^{\alpha}(t-a)\, dt = \frac{(b-x)^{\alpha+1}\left[(\alpha+1)(b-x) + (\alpha+2)(x-a)\right]}{(\alpha+1)(\alpha+2)}. \tag{3.8}$$

By substituting the equalities (3.5)–(3.8) in (3.4), we obtain

$$\left|f(a)J_{a+}^{\alpha}g(x) + f(b)J_{b-}^{\alpha}g(x) - \left[J_{a+}^{\alpha}(fg)(x) + J_{b-}^{\alpha}(fg)(x)\right]\right|$$

$$\leq \frac{\|g\|_{[a,x],\infty}}{(b-a)\Gamma(\alpha+3)}\left[(x-a)^{\alpha+1}\left[(\alpha+2)(b-x)+(\alpha+1)(x-a)\right]|f'(a)|+(x-a)^{\alpha+2}|f'(b)|\right]$$

$$+\frac{\|g\|_{[x,b],\infty}}{(b-a)\Gamma(\alpha+3)}\left[(b-x)^{\alpha+2}|f'(a)|+(b-x)^{\alpha+1}\left[(\alpha+1)(b-x)+(\alpha+2)(x-a)\right]|f'(b)|\right]$$

$$= \frac{1}{(b-a)\Gamma(\alpha+3)}$$

$$\times\left\{\left[(x-a)^{\alpha+1}\left[(\alpha+2)(b-x)+(\alpha+1)(x-a)\right]\|g\|_{[a,x],\infty}+(b-x)^{\alpha+2}\|g\|_{[x,b],\infty}\right]|f'(a)|\right.$$

$$\left.+\left[(x-a)^{\alpha+2}\|g\|_{[a,x],\infty}+(b-x)^{\alpha+1}\left[(\alpha+1)(b-x)+(\alpha+2)(x-a)\right]\|g\|_{[x,b],\infty}\right]|f'(b)|\right\}$$

which completes the proof of first inequality in (3.1).

From the facts that

$$\|g\|_{[a,x],\infty}\leq\|g\|_{[a,b],\infty}=\|g\|_{\infty} \text{ and } \|g\|_{[x,b],\infty}\leq\|g\|_{[a,b],\infty}=\|g\|_{\infty} \quad (3.9)$$

for all $x \in [a,b]$, the proof of the second inequality in (3.1) is obvious. □

Remark 3. *If we choose $\alpha = 1$ in Theorem 4, then we have the weighted inequality*

$$\left|f(a)\int_a^x g(t)dt+f(b)\int_x^b g(t)dt-\int_a^b f(t)g(t)dt\right|$$

$$\leq \left[(x-a)^2\left(\frac{3b-2a-x}{6(b-a)}\right)\|g\|_{[a,x],\infty}+\frac{(b-x)^3}{6(b-a)}\|g\|_{[x,b],\infty}\right]|f'(a)|$$

$$+\left[\frac{(x-a)^3}{6(b-a)}\|g\|_{[a,x],\infty}+(b-x)^2\left(\frac{2b-3a+x}{6(b-a)}\right)\|g\|_{[x,b],\infty}\right]|f'(b)|$$

$$\leq \left[\left(\frac{(x-a)^2(3(b-x)+2(x-a))+(b-x)^3}{6(b-a)}\right)|f'(a)|\right.$$

$$\left.+\left(\frac{(x-a)^3+(b-x)^2(2(b-x)+3(x-a))}{6(b-a)}|f'(b)|\right)\right]\|g\|_{\infty}$$

which is proved by Tseng et. al in [51].

Corollary 6. *In Theorem 4, let g be symmetric to $\frac{a+b}{2}$ and let $x = \frac{a+b}{2}$. Then we have the following "weighted fractional trapezoid inequality"*

$$\left|\left[J^{\alpha}_{a+}g\left(\frac{a+b}{2}\right)+J^{\alpha}_{b-}g\left(\frac{a+b}{2}\right)\right]\frac{f(a)+f(b)}{2}-\left[J^{\alpha}_{a+}(fg)\left(\frac{a+b}{2}\right)+J^{\alpha}_{b-}(fg)\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+2}\Gamma(\alpha+3)}$$

$$\times\left[\left((2\alpha+3)\|g\|_{[a,x],\infty}+\|g\|_{[x,b],\infty}\right)|f'(a)|+\left(\|g\|_{[a,x],\infty}+(2\alpha+3)\|g\|_{[x,b],\infty}\right)|f'(b)|\right]$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha}\Gamma(\alpha+2)}\frac{|f'(a)|+|f'(b)|}{2}.$$

Corollary 7. *Under assumptions of Theorem 4, we have the following inequality*

$$\left|f(a)J_{a+}^{\alpha}g(b)+f(b)J_{b-}^{\alpha}g(a)-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha}\Gamma(\alpha+2)}\left[|f'(a)|+|f'(b)|\right].$$

Proof. If we write the inequality (3.1) for $x=a$ and $x=b$, then we have the inequalities

$$\left|f(b)J_{b-}^{\alpha}g(a)-J_{b-}^{\alpha}(fg)(a)\right|\leq \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+3)}\left[|f'(a)|+(\alpha+1)|f'(b)|\right] \quad (3.10)$$

and

$$\left|f(a)J_{a+}^{\alpha}g(b)-J_{a+}^{\alpha}(fg)(b)\right|\leq \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+3)}\left[(\alpha+1)|f'(a)|+|f'(b)|\right] \quad (3.11)$$

respectively. By adding the inequalities (3.10) and (3.11) and using triangle inequality, we obtain

$$\left|f(a)J_{a+}^{\alpha}g(b)+f(b)J_{b-}^{\alpha}g(a)-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+3)}\left[|f'(a)|+(\alpha+1)|f'(b)|\right]+\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+3)}\left[(\alpha+1)|f'(a)|+|f'(b)|\right]$$

$$= \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{2^{\alpha}\Gamma(\alpha+2)}\left[|f'(a)|+|f'(b)|\right]$$

which completes the proof. □

Remark 4. *If we choose $\alpha=1$ in Corollary 7, then we have the following weighted trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(t)dt-\int_a^b f(t)g(t)dt\right|\leq \frac{|f'(a)|+|f'(b)|}{8}\|g\|_{\infty}$$

which is given by Tseng et al. in [51].

Corollary 8. *In Theorem 4, let $g(t)=1$ for all $t\in[a,b]$. Then we have the following identity*

$$\left|(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)-\Gamma(\alpha+1)\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)\right]\right|$$

$$\leq \frac{1}{(b-a)(\alpha+1)(\alpha+2)}$$
$$\times\left[\left((x-a)^{\alpha+1}\left[(\alpha+2)(b-x)+(\alpha+1)(x-a)\right]+(b-x)^{\alpha+2}\right)|f'(a)|\right.$$
$$\left.+\left((x-a)^{\alpha+2}+(b-x)^{\alpha+1}\left[(\alpha+1)(b-x)+(\alpha+2)(x-a)\right]|f'(b)|\right)\right).$$

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is continuous and if $|f'|^q$, $q > 1$, is convex on $[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$ we have the following inequality for fractional integrals,*

$$\begin{aligned}
&\left|f(a)J_{a+}^{\alpha}g(x) + f(b)J_{b-}^{\alpha}g(x) - \left[J_{a+}^{\alpha}(fg)(x) + J_{b-}^{\alpha}(fg)(x)\right]\right| \qquad (3.12)\\
\leq\ & \frac{1}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\\
&\times\left[\|g\|_{[a,x],\infty}(x-a)^{\alpha+\frac{1}{p}}\left(\left(\frac{x-a}{b-a}\right)\left(b-\frac{a+x}{2}\right)|f'(a)|^q + \frac{(x-a)^2}{2(b-a)}|f'(b)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\|g\|_{[x,b],\infty}(b-x)^{\alpha+\frac{1}{p}}\left(\frac{(b-x)^2}{2(b-a)}|f'(a)|^q + \left(\frac{b-x}{b-a}\right)\left(\frac{x+b}{2}-a\right)|f'(b)|^q\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[(x-a)^{\alpha+\frac{1}{p}}\left(\left(\frac{x-a}{b-a}\right)\left(b-\frac{a+x}{2}\right)|f'(a)|^q + \frac{(x-a)^2}{2(b-a)}|f'(b)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+(b-x)^{\alpha+\frac{1}{p}}\left(\frac{(b-x)^2}{2(b-a)}|f'(a)|^q + \left(\frac{b-x}{b-a}\right)\left(\frac{x+b}{2}-a\right)|f'(b)|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using the well-known Hölder inequality in (3.2), we have

$$\begin{aligned}
&\left|f(a)J_{a+}^{\alpha}g(x) + f(b)J_{b-}^{\alpha}g(x) - \left[J_{a+}^{\alpha}(fg)(x) + J_{b-}^{\alpha}(fg)(x)\right]\right| \qquad (3.13)\\
\leq\ & \frac{1}{\Gamma(\alpha+1)}\left[\|g\|_{[a,x],\infty}\int_a^x (x-t)^{\alpha}|f'(t)|\,dt + \|g\|_{[x,b],\infty}\int_x^b (t-x)^{\alpha}|f'(t)|\,dt\right]\\
\leq\ & \frac{1}{\Gamma(\alpha+1)}\left[\|g\|_{[a,x],\infty}\left(\int_a^x (x-t)^{p\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_a^x |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\|g\|_{[x,b],\infty}\left(\int_x^b (t-x)^{p\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_x^b |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right]\\
=\ & \frac{1}{\Gamma(\alpha+1)}\left[\|g\|_{[a,x],\infty}\frac{(x-a)^{\alpha+\frac{1}{p}}}{(p\alpha+1)^{\frac{1}{p}}}\left(\int_a^x |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\|g\|_{[x,b],\infty}\frac{(b-x)^{\alpha+\frac{1}{p}}}{(p\alpha+1)^{\frac{1}{p}}}\left(\int_x^b |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|f'|^q$ is convex on $[a,b]$, we get

$$|f'(t)|^q = \left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right|^q \le \frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q . \tag{3.14}$$

Then it follows that

$$\begin{aligned}
&\left|f(a)J_{a+}^{\alpha}g(x) + f(b)J_{b-}^{\alpha}g(x) - \left[J_{a+}^{\alpha}(fg)(x) + J_{b-}^{\alpha}(fg)(x)\right]\right| \\
\le\ & \frac{1}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \\
&\times\left[\|g\|_{[a,x],\infty}(x-a)^{\alpha+\frac{1}{p}}\left(\int_a^x\left(\frac{(b-t)|f'(a)|^q + (t-a)|f'(b)|^q}{b-a}\right)dt\right)^{\frac{1}{q}}\right. \\
&\left.\|g\|_{[x,b],\infty}(b-x)^{\alpha+\frac{1}{p}}\left(\int_x^b\left(\frac{(b-t)|f'(a)|^q + (t-a)|f'(b)|^q}{b-a}\right)dt\right)^{\frac{1}{q}}\right] \\
\le\ & \frac{1}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \\
&\times\left[\|g\|_{[a,x],\infty}(x-a)^{\alpha+\frac{1}{p}}\left(\left(\frac{x-a}{b-a}\right)\left(b-\frac{a+x}{2}\right)|f'(a)|^q + \frac{(x-a)^2}{2(b-a)}|f'(b)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\|g\|_{[x,b],\infty}(b-x)^{\alpha+\frac{1}{p}}\left(\frac{(b-x)^2}{2(b-a)}|f'(a)|^q + \left(\frac{b-x}{b-a}\right)\left(\frac{x+b}{2}-a\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof of first inequality in (3.12). The proof of second inequality is obvious from the inequalities (3.9). □

Remark 5. *If we choose $\alpha = 1$ in Theorem 5, then we obtain the following weighted inequality*

$$\begin{aligned}
&\left|f(a)\int_a^x g(t)dt + f(b)\int_x^b g(t)dt - \int_a^b f(t)g(t)dt\right| \\
\le\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\|g\|_{[a,x],\infty}(x-a)^{1+\frac{1}{p}}\left(\left(\frac{x-a}{b-a}\right)\left(b-\frac{a+x}{2}\right)|f'(a)|^q + \frac{(x-a)^2}{2(b-a)}|f'(b)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\|g\|_{[x,b],\infty}(b-x)^{1+\frac{1}{p}}\left(\frac{(b-x)^2}{2(b-a)}|f'(a)|^q + \left(\frac{b-x}{b-a}\right)\left(\frac{x+b}{2}-a\right)|f'(b)|^q\right)^{\frac{1}{q}}\right] \\
\le\ & \left(\frac{1}{p1+1}\right)^{\frac{1}{p}}\left[(x-a)^{1+\frac{1}{p}}\left(\left(\frac{x-a}{b-a}\right)\left(b-\frac{a+x}{2}\right)|f'(a)|^q + \frac{(x-a)^2}{2(b-a)}|f'(b)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+(b-x)^{1+\frac{1}{p}}\left(\frac{(b-x)^2}{2(b-a)}|f'(a)|^q + \left(\frac{b-x}{b-a}\right)\left(\frac{x+b}{2}-a\right)|f'(b)|^q\right)^{\frac{1}{q}}\right]\|g\|_{\infty} .
\end{aligned}$$

Corollary 9. *In Theorem 5, let g be symmetric to $\frac{a+b}{2}$ and let $x = \frac{a+b}{2}$. Then we have the following weighted fractional trapezoid inequality*

$$\left|\left[J_{a+}^{\alpha}g\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}g\left(\frac{a+b}{2}\right)\right]\frac{f(a)+f(b)}{2}-\left[J_{a+}^{\alpha}(fg)\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}(fg)\left(\frac{a+b}{2}\right)\right]\right| \tag{3.15}$$

$$\leq \frac{1}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{2}\right)^{\alpha+1}$$

$$\times\left[\|g\|_{[a,\frac{a+b}{2}],\infty}\left(\frac{3|f'(a)|^q+|f'(b)|^q\,dt}{4}\right)^{\frac{1}{q}}+\|g\|_{[x,b],\infty}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{2}\right)^{\alpha+1}$$

$$\times\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q\,dt}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(\frac{4}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{2}\right)^{\alpha+1}\left[|f'(a)|+|f'(b)|\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The proof of the first and second inequalities in (3.15) is obvious. For the proof of third inequality, let $a_1 = 3|f'(a)|^q$, $b_1 = |f'(b)|^q$, $a_2 = |f'(a)|^q$ and $b_2 = 3|f'(b)|^q$. Using the facts that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s,\ 0\leq s<1$$

and $3^{\frac{1}{q}}+1\leq 4$, the desired result can be obtained straightforwardly. □

Corollary 10. *Under assumptions of Theorem 4, we have the following equality*

$$\left|f(a)J_{a+}^{\alpha}g(b)+f(b)J_{b-}^{\alpha}g(a)-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq \frac{2\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}(b-a)^{\alpha+1}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. If we write the equality (3.1) for $x = a$ and $x = b$, then we have the inequalities

$$\left|f(b)J_{b-}^{\alpha}g(a)-J_{b-}^{\alpha}(fg)(a)\right| \leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}(b-a)^{\alpha+1}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \tag{3.16}$$

and

$$\left|f(a)J_{a+}^{\alpha}g(b)-J_{a+}^{\alpha}(fg)(b)\right| \leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}(b-a)^{\alpha+1}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}, \tag{3.17}$$

respectively. By adding the inequalities (3.16) and (3.17) and using triangle inequality, we obtain

$$\left|J_{a+}^{\alpha}g\,(b)\,f(a)+J_{b-}^{\alpha}g\,(a)\,f(b)-\left[J_{a+}^{\alpha}\,(fg)\,(b)+J_{b-}^{\alpha}\,(fg)\,(a)\right]\right|$$

$$\leq \frac{2\,\|g\|_{\infty}}{\Gamma\,(\alpha+1)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}(b-a)^{\alpha+1}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 6. *If we choose $\alpha = 1$ in Corollary 10, then we have the following weighted trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_{a}^{b}g(t)dt-\int_{a}^{b}f(t)g(t)dt\right|\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(b-a)^{2}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}\|g\|_{\infty}$$

which is given by Tseng et al. in [51].

Corollary 11. *In Theorem 5, let $g(t) = 1$ for all $t \in [a, b]$. Then we have the following inequality*

$$\left|(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)-\Gamma\,(\alpha+1)\left[J_{a+}^{\alpha}f\,(x)+J_{b-}^{\alpha}f\,(x)\right]\right|$$

$$\leq \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[(x-a)^{\alpha+\frac{1}{p}}\left(\left(\frac{x-a}{b-a}\right)\left(b-\frac{a+x}{2}\right)|f'(a)|^{q}+\frac{(x-a)^{2}}{2\,(b-a)}\,|f'(b)|^{q}\,dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\,(b-x)^{\alpha+\frac{1}{p}}\left(\frac{(b-x)^{2}}{2\,(b-a)}\,|f'(a)|^{q}+\left(\frac{b-x}{b-a}\right)\left(\frac{x+b}{2}-a\right)|f'(b)|^{q}\right)^{\frac{1}{q}}\right].$$

4. Weighted fractional Ostrowski and midpoint type inequalities

In this section, we obtain some weighted fractional Ostrowski type inequalities and we give some weighted fractional midpoint type inequalities as special cases.

First, we give the classical and fractional Ostrowski inequalities:

Ostrowski [31] gave the following classical integral inequality associated with the differentiable mappings:

Theorem 6. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_{\infty} = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^{2}}{(b-a)^{2}}\right](b-a)\,\|f'\|_{\infty} \tag{4.1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

In [45], Set obtained the following Ostrowski inequality for fractional integrals:

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is convex on $[a,b]$ and $|f'(x)\le M|$, $x\in[a,b]$ then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{M}{b-a}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}\right].$$

In recent years, several papers have devoted to Ostrowski type inequalities for several type fractional integrals, for some of them please see [1,5,6,11,15,16,24–26,29,38,39,48–50,54,55].

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is continuous and if $|f'|$ is convex on $[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$ we have the following "weighted fractional Ostrowski inequality"*

$$\left|\left[J_{a+}^{\alpha}g(x)+J_{b-}^{\alpha}g(x)\right]f(x)-\left[J_{a+}^{\alpha}(fg)(x)+J_{b-}^{\alpha}(fg)(x)\right]\right| \tag{4.2}$$
$$\leq \frac{\|g\|_{\infty}}{(b-a)\Gamma(\alpha+1)}\left[(x-a)^{\alpha+1}\left(\left(b-\frac{a+x}{2}\right)-\frac{(\alpha+2)(b-x)+(\alpha+1)(x-a)}{(\alpha+1)(\alpha+2)}\right)\right.$$
$$\left.+(b-x)^{\alpha+2}\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\right]|f'(a)|$$
$$+\left[(b-x)^{\alpha+1}\left(\left(\frac{x+b}{2}-a\right)-\frac{(\alpha+1)(b-x)+(\alpha+2)(x-a)}{(\alpha+1)(\alpha+2)}\right)\right.$$
$$\left.+(x-a)^{\alpha+2}\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\right]|f'(b)|.$$

Proof. From Lemma 4, we have

$$\left|\left[J_{a+}^{\alpha}g(x)+J_{b-}^{\alpha}g(x)\right]f(x)-\left[J_{a+}^{\alpha}(fg)(x)+J_{b-}^{\alpha}(fg)(x)\right]\right| \tag{4.3}$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\left|\int_a^x\left(\int_a^t(x-s)^{\alpha-1}g(s)ds\right)f'(t)dt\right|+\left|\int_x^b\left(\int_b^t(s-x)^{\alpha-1}g(s)ds\right)f'(t)dt\right|\right]$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x\left|\int_a^t(x-s)^{\alpha-1}g(s)ds\right||f'(t)|\,dt+\int_x^b\left|\int_b^t(s-x)^{\alpha-1}g(s)ds\right||f'(t)|\,dt\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\left[\|g\|_{[a,x],\infty}\int_a^x\left((x-a)^{\alpha}-(x-t)^{\alpha}\right)|f'(t)|\,dt\right.$$
$$\left.+\|g\|_{[x,b],\infty}\int_x^b\left((b-x)^{\alpha}-(t-x)^{\alpha}\right)|f'(t)|\,dt\right]$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left[\int_a^x \left((x-a)^\alpha-(x-t)^\alpha\right)|f'(t)|\,dt+\int_x^b \left((b-x)^\alpha-(t-x)^\alpha\right)|f'(t)|\,dt\right].$$

Since $|f'|$ is convex on $[a,b]$, then we get

$$\begin{aligned}
&\int_a^x \left((x-a)^\alpha-(x-t)^\alpha\right)|f'(t)|\,dt \qquad (4.4)\\
\leq\ & \frac{1}{b-a}\int_a^x \left((x-a)^\alpha-(x-t)^\alpha\right)\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt\\
=\ & \frac{|f'(a)|}{b-a}\int_a^x \left((x-a)^\alpha-(x-t)^\alpha\right)(b-t)\,dt\\
&+\frac{|f'(b)|}{b-a}\int_a^x \left((x-a)^\alpha-(x-t)^\alpha\right)(t-a)\,dt\\
=\ & \frac{(x-a)^{\alpha+1}}{b-a}\left[\left(b-\frac{a+x}{2}\right)-\frac{(\alpha+2)(b-x)+(\alpha+1)(x-a)}{(\alpha+1)(\alpha+2)}\right]|f'(a)|\\
&+\frac{(x-a)^{\alpha+2}}{b-a}\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]|f'(b)|.
\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}
&\int_x^b \left((b-x)^\alpha-(t-x)^\alpha\right)|f'(t)|\,dt \qquad (4.5)\\
\leq\ & \frac{1}{b-a}\int_a^x \left((b-x)^\alpha-(t-x)^\alpha\right)\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt\\
=\ & \frac{(b-x)^{\alpha+2}}{b-a}\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]|f'(a)|\\
&+\frac{(b-x)^{\alpha+1}}{b-a}\left[\left(\frac{x+b}{2}-a\right)-\frac{(\alpha+1)(b-x)+(\alpha+2)(x-a)}{(\alpha+1)(\alpha+2)}\right]|f'(b)|.
\end{aligned}$$

If we substitute the equalities (4.4) and (4.5) in (4.3), then we obtain the required inequality (4.2). □

Remark 7. *If we choose $\alpha = 1$ in Theorem 8, then we have the following weighted Ostrowski inequality*

$$\left|f(x)\int_a^b g(t)dt-\int_a^b f(t)g(t)dt\right|$$

$$\leq \frac{\|g\|_\infty}{(b-a)\,\Gamma(\alpha+1)}\left[\frac{(x-a)^2}{2}\left(b-\frac{a+2x}{3}\right)+\frac{(b-x)^3}{3}\right]|f'(a)|$$
$$+\left[\frac{(b-x)^{\alpha+1}}{2}\left(\frac{2x+b}{3}-a\right)+\frac{(x-a)^{\alpha+2}}{3}\right]|f'(b)|.$$

Corollary 12. *If we choose $x=\frac{a+b}{2}$ in Theorem 8, then we have the following "weighted fractional midpoint inequality"*

$$\left\|\left[J^\alpha_{a+}g\left(\frac{a+b}{2}\right)+J^\alpha_{b-}g\left(\frac{a+b}{2}\right)\right]f\left(\frac{a+b}{2}\right)-\left[J^\alpha_{a+}(fg)\left(\frac{a+b}{2}\right)+J^\alpha_{b-}(fg)\left(\frac{a+b}{2}\right)\right]\right\|$$
$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(\frac{b-a}{2}\right)^{\alpha+1}\left[|f'(a)|+|f'(b)|\right].$$

Corollary 13. *If we choose $g(t)=1$ for all $t\in[a,b]$ in Theorem 8, then we have the following new version of fractional Ostrowski type inequality*

$$\left|\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{a+}f(x)+J^\alpha_{b-}f(x)\right]\right|$$
$$\leq \frac{1}{(b-a)^2\,\Gamma(\alpha+1)}\left[(x-a)^{\alpha+1}\left(\left(b-\frac{a+x}{2}\right)-\frac{(\alpha+2)(b-x)+(\alpha+1)(x-a)}{(\alpha+1)(\alpha+2)}\right)\right.$$
$$\left.+(b-x)^{\alpha+2}\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\right]|f'(a)|$$
$$+\left[(b-x)^{\alpha+1}\left(\left(\frac{x+b}{2}-a\right)-\frac{(\alpha+1)(b-x)+(\alpha+2)(x-a)}{(\alpha+1)(\alpha+2)}\right)\right.$$
$$\left.+(x-a)^{\alpha+2}\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\right]|f'(b)|.$$

Corollary 14. *Under assumption of Theorem 8, let g be symmetric to $\frac{a+b}{2}$. Then we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2}\left[J^\alpha_{a+}g(b)+J^\alpha_{b-}g(a)\right]-\left[J^\alpha_{a+}(fg)(b)+J^\alpha_{b-}(fg)(a)\right]\right| \tag{4.6}$$
$$\leq \frac{(b-a)^{\alpha+1}\|g\|_\infty}{\Gamma(\alpha+1)}\left(1-\frac{1}{\alpha+1}\right)\left[|f'(a)|+|f'(b)|\right]$$

Proof. If we apply the inequality (4.2) for $x=a$ and $x=b$, then we obtain the inequalities

$$\left|f(a)J^\alpha_{b-}g(a)-J^\alpha_{b-}(fg)(a)\right| \tag{4.7}$$
$$\leq \frac{(b-a)^{\alpha+1}\|g\|_\infty}{\Gamma(\alpha+1)}\left[\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)|f'(a)|+\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)|f'(b)|\right]$$

and

$$\left|f(b)J_{a+}^{\alpha}g(b)-J_{a+}^{\alpha}(fg)(b)\right| \tag{4.8}$$

$$\leq \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+1)}\left[\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)|f'(a)|+\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)|f'(b)|\right]$$

respectively. If we add the inequalities (4.7) and (4.8), then by using triangle inequality and Lemma 2, we have

$$\left|\frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+1)}\left[\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)|f'(a)|+\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)|f'(b)|\right]$$

$$+\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+1)}\left[\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)|f'(a)|+\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)|f'(b)|\right]$$

$$=\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(1-\frac{1}{\alpha+1}\right)\left[|f'(a)|+|f'(b)|\right]$$

which completes the proof. □

Theorem 9. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is continuous and if $|f'|^q$, $q>1$, is convex on $[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$ we have the following inequality for fractional integrals,*

$$\left|\left[J_{a+}^{\alpha}g(x)+J_{b-}^{\alpha}g(x)\right]f(x)-\left[J_{a+}^{\alpha}(fg)(x)+J_{b-}^{\alpha}(fg)(x)\right]\right| \tag{4.9}$$

$$\leq \frac{\|g\|_{\infty}}{(b-a)^{\frac{1}{q}}\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[(x-a)^{\alpha+1}\left(\left(b-\frac{a+x}{2}\right)|f'(a)|^q+\frac{x-a}{2}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^{\alpha+1}\left(\frac{b-x}{2}|f'(a)|^q+\left(\frac{x+b}{2}-a\right)|f'(b)|^q\right)^{\frac{1}{q}}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Hölder inequality in (4.3) and by using the fact that

$$(A-B)^p\leq A^p-B^p$$

for any $A>B\geq 0$ and $p\geq 1$, we obtain

$$\left|\left[J_{a+}^{\alpha}g(x)+J_{b-}^{\alpha}g(x)\right]f(x)-\left[J_{a+}^{\alpha}(fg)(x)+J_{b-}^{\alpha}(fg)(x)\right]\right| \tag{4.10}$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left[\left(\int_a^x ((x-a)^\alpha-(x-t)^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_a^x |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_x^b ((b-x)^\alpha-(t-x)^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_x^b |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left[\left(\int_a^x ((x-a)^{p\alpha}-(x-t)^{p\alpha})\,dt\right)^{\frac{1}{p}}\left(\int_a^x |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_x^b ((b-x)^{p\alpha}-(t-x)^{p\alpha})\,dt\right)^{\frac{1}{p}}\left(\int_x^b |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right].$$

By simple calculations, we establish

$$\int_a^x ((x-a)^{p\alpha}-(x-t)^{p\alpha})\,dt=(x-a)^{p\alpha+1}\left(1-\frac{1}{p\alpha+1}\right) \tag{4.11}$$

and

$$\int_x^b ((b-x)^{p\alpha}-(t-x)^{p\alpha})\,dt=(b-x)^{p\alpha+1}\left(1-\frac{1}{p\alpha+1}\right). \tag{4.12}$$

Since $|f'|^q$ is convex on $[a,b]$, then we get

$$\begin{aligned}\int_a^x |f'(t)|^q\,dt &\leq \frac{1}{b-a}\int_a^x ((b-t)\,|f'(a)|^q+(t-a)\,|f'(b)|^q) \\ &\leq \frac{x-a}{b-a}\left[\left(b-\frac{a+x}{2}\right)|f'(a)|^q+\frac{x-a}{2}\,|f'(b)|^q\right]\end{aligned} \tag{4.13}$$

and similarly

$$\int_x^b |f'(t)|^q\,dt\leq\frac{b-x}{b-a}\left[\frac{b-x}{2}\,|f'(a)|^q+\left(\frac{x+b}{2}-a\right)|f'(b)|^q\right]. \tag{4.14}$$

If we put (4.11)–(4.11) in (4.10), then we have

$$\left|\left[J_{a+}^\alpha g\,(x)+J_{b-}^\alpha g\,(x)\right]f(x)-\left[J_{a+}^\alpha\,(fg)\,(x)+J_{b-}^\alpha\,(fg)\,(x)\right]\right|$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left[\left((x-a)^{p\alpha+1}\left(1-\frac{1}{p\alpha+1}\right)\right)^{\frac{1}{p}}\left(\frac{x-a}{b-a}\left[\left(b-\frac{a+x}{2}\right)|f'(a)|^q+\frac{x-a}{2}\,|f'(b)|^q\right]\right)^{\frac{1}{q}}\right.$$

$$+\left((b-x)^{p\alpha+1}\left(1-\frac{1}{p\alpha+1}\right)\right)^{\frac{1}{p}}\left(\frac{b-x}{b-a}\left[\frac{b-x}{2}\left|f'(a)\right|^q+\left(\frac{x+b}{2}-a\right)\left|f'(b)\right|^q\right]\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{\|g\|_\infty}{(b-a)^{\frac{1}{q}}\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\Bigg[(x-a)^{\alpha+1}\left(\left(b-\frac{a+x}{2}\right)\left|f'(a)\right|^q+\frac{x-a}{2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$+(b-x)^{\alpha+1}\left(\frac{b-x}{2}\left|f'(a)\right|^q+\left(\frac{x+b}{2}-a\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Bigg]$$

which completes the proof. □

Remark 8. *If we choose $\alpha=1$ in Theorem 9, then we have the following weighted Ostrowski inequality*

$$\left|f(x)\int_a^b g(t)dt-\int_a^b f(t)g(t)dt\right|$$

$$\leq\frac{\|g\|_\infty}{(b-a)^{\frac{1}{q}}}\left(1-\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg[(x-a)^2\left(\left(b-\frac{a+x}{2}\right)\left|f'(a)\right|^q+\frac{x-a}{2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$+(b-x)^2\left(\frac{b-x}{2}\left|f'(a)\right|^q+\left(\frac{x+b}{2}-a\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Bigg].$$

Corollary 15. *If we choose $x=\frac{a+b}{2}$ in Theorem 9, then we have the following "weighted fractional midpoint inequality"*

$$\left|\left[J_{a+}^\alpha g\left(\frac{a+b}{2}\right)+J_{b-}^\alpha g\left(\frac{a+b}{2}\right)\right]f\left(\frac{a+b}{2}\right)-\left[J_{a+}^\alpha(fg)\left(\frac{a+b}{2}\right)+J_{b-}^\alpha(fg)\left(\frac{a+b}{2}\right)\right]\right| \tag{4.15}$$

$$\leq\frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{2}\right)^{\alpha+1}$$

$$\times\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(4-\frac{4}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{2}\right)^{\alpha+1}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Proof. The proof of the first inequality in (4.15) is obvious. For the proof of second inequality, let $a_1=3\left|f'(a)\right|^q$, $b_1=\left|f'(b)\right|^q$, $a_2=\left|f'(a)\right|^q$ and $b_2=3\left|f'(b)\right|^q$. Using the facts that

$$\sum_{k=1}^n(a_k+b_k)^s\leq\sum_{k=1}^n a_k^s+\sum_{k=1}^n b_k^s,\ 0\leq s<1$$

and $3^{\frac{1}{q}}+1\leq4$, the desired result can be obtained straightforwardly. □

Corollary 16. *If we choose $g(t) = 1$ for all $t \in [a, b]$ in Theorem 9, then we have*

$$\left|\left(\frac{(x-s)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J\right]_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)\right|$$

$$\leq \frac{\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}}{(b-a)^{1+\frac{1}{q}}\Gamma(\alpha+1)}\left[(x-a)^{\alpha+1}\left(\left(b-\frac{a+x}{2}\right)|f'(a)|^{q}+\frac{x-a}{2}|f'(b)|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^{\alpha+1}\left(\frac{b-x}{2}|f'(a)|^{q}+\left(\frac{x+b}{2}-a\right)|f'(b)|^{q}\right)^{\frac{1}{q}}\right]$$

Corollary 17. *Under assumption of Theorem 9, let g be symmetric to $\frac{a+b}{2}$. Then we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \tag{4.16}$$

$$\leq \frac{2(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}\|g\|_{\infty}.$$

Proof. If we write the inequality (4.9) for $x = a$ and $x = b$, then we obtain the inequalities

$$\left|f(a)J_{b-}^{\alpha}g(a)-J_{b-}^{\alpha}(fg)(a)\right| \leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}(b-a)^{\alpha+1}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}} \tag{4.17}$$

and

$$\left|f(b)J_{a+}^{\alpha}g(a)-J_{a+}^{\alpha}(fg)(b)\right| \leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}(b-a)^{\alpha+1}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}} \tag{4.18}$$

respectively. If we add the inequalities (4.7) and (4.8), then by using triangle inequality and Lemma 2, we have

$$\left|\frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}(b-a)^{\alpha+1}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}$$

$$+\frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}(b-a)^{\alpha+1}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}$$

$$= \frac{2(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(1-\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}\|g\|_{\infty}.$$

This completes the proof. □

Conflict of interest

The authors declare no conflict of interest.

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