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Research article

Divisibility among determinants of power matrices associated with integer-valued arithmetic functions

Long Chen and Shaofang Hong*

Mathematical College, Sichuan University, Chengdu 610064, P.R. China

* **Correspondence:** Email: sfhong@scu.edu.cn; Tel: +862885412720; Fax: +862885471501.

Abstract: Let a, b and n be positive integers and $S = \{x_1, ..., x_n\}$ be a set of n distinct positive integers. The set S is called a divisor chain if there is a permutation σ of $\{1, ..., n\}$ such that $x_{\sigma(1)}|...|x_{\sigma(n)}$. We say that the set S consists of two coprime divisor chains if we can partition S as $S = S_1 \cup S_2$, where S_1 and S_2 are divisor chains and each element of S_1 is coprime to each element of S_2. For any arithmetic function f, we define the function f^a for any positive integer x by $f^a(x) := (f(x))^a$. The matrix $(f^a(S))$ is the $n \times n$ matrix having f^a evaluated at the the greatest common divisor of x_i and x_j as its (i, j)-entry and the matrix $(f^a[S])$ is the $n \times n$ matrix having f^a evaluated at the least common multiple of x_i and x_j as its (i, j)-entry. In this paper, when f is an integer-valued arithmetic function and S consists of two coprime divisor chains with $1 \notin S$, we establish the divisibility theorems between the determinants of the power matrices $(f^a(S))$ and $(f^b(S))$, between the determinants of the power matrices $(f^a[S])$ and $(f^b[S])$ and between the determinants of the power matrices $(f^a(S))$ and $(f^b[S])$. Our results extend Hong's theorem obtained in 2003 and the theorem of Tan, Lin and Liu gotten in 2011.

Keywords: divisibility; two coprime divisor chains; greatest-type divisor; power matrix; integer-valued function; multiplicative function
Mathematics Subject Classification: 11C20, 11A05, 15B36

1. Introduction

We denote by (x, y) (resp. $[x, y]$) the greatest common divisor (resp. least common multiple) of any given integers x and y. Let a, b and n be positive integers and $S = \{x_1, ..., x_n\}$ be a set of n distinct positive integers. Let f be an arithmetic function and we denote by $(f(S))$ (resp. $(f[S])$) the $n \times n$ matrix having f evaluated at (x_i, x_j) (resp. $[x_i, x_j]$) as its (i, j)-entry. Particularly, the $n \times n$ matrix $(S^a) = ((x_i, x_j)^a)$, having the ath power $(x_i, x_j)^a$ as its (i, j) -entry, is called the *ath power GCD matrix on* S. The $n \times n$ matrix $[S^a] = ([x_i, x_j]^a)$, having the ath power $[x_i, x_j]^a$ as its (i, j)-entry, is called the *ath power LCM matrix* on S. These are simply called the *GCD matrix* and *LCM matrix* respectively if

$a = 1$. The set S is said to be *factor closed (FC)* if it contains every divisor of x for any $x \in S$. The set S is said to be *gcd closed* (resp. *lcm closed*) if for all i and j, (x_i, x_j) (resp. $[x_i, x_j]$) is in S. Evidently, an FC set is gcd closed but not conversely. In 1875, Smith [33] published his famous theorem stating that the determinant of the GCD matrix (S) defined on the set $S = \{1, ..., n\}$ is the product $\prod_{k=1}^{n} \varphi(k)$, where φ is Euler's totient function. Since then many interesting generalizations of Smith's determinant and related results have been published (see, for example, [1, 3–32] and [34–42]).

Divisibility is an important topic in the field of power GCD matrices and power LCM matrices. Bourque and Ligh [5] showed that if S is an FC set, then the GCD matrix (S) divides the LCM matrix $[S]$ in the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices over the set $\mathbf{Z}$ of integers. That is, there exists an $A \in M_n(\mathbf{Z})$ such that $[S] = (S) A$ or $[S] = A (S)$. Hong [19] showed that such factorization is no longer true in general if S is gcd closed. Bourque and Ligh [8] extended their own result showing that if S is factor closed, then the power GCD matrix (S^a) divides the power LCM matrix $[S^a]$ in the ring $M_n(\mathbf{Z})$. The set S is called a *divisor chain* if there exists a permutation σ on $\{1, ..., n\}$ such that $x_{\sigma(1)}|...|x_{\sigma(n)}$. Obviously a divisor chain is gcd closed but the converse is not true. For $x, y \in S$, and $x < y$, if $x|y$ and the conditions $x|d|y$ and $d \in S$ imply that $d \in \{x, y\}$, then we say that x is a *greatest-type divisor* of y in S. For $x \in S$, we denote by $G_S(x)$ the set of all greatest-type divisors of x in S. By [19], we know that there is gcd-closed set S with $\max_{x \in S} \{|G_S(x)|\} = 2$ such that $(S)^{-1} [S] \notin M_n(\mathbf{Z})$. In [26], Hong, Zhao and Yin showed that if S is gcd closed and $\max_{x \in S} \{|G_S(x)|\} = 1$, then the GCD matrix (S) divides the LCM matrix $[S]$ in $M_n(\mathbf{Z})$. In [20], Hong showed that $(f(S))|(f[S])$ holds in the ring $M_n(\mathbf{Z})$ when S is a divisor chain and f is an integer-valued multiplicative function satisfying that $f(\min(S))|f(x)$ for any $x \in S$.

Hong [22] initiated the investigation of divisibility among power GCD matrices and among power LCM matrices. In fact, Hong [22] proved that the power GCD matrix (S^a) divides the power GCD matrix (S^b) if $a|b$ and S is a divisor chain. Hong also showed that the power LCM matrix $[S^a]$ divides the power LCM matrix $[S^b]$ if $a|b$ and S is a divisor chain. But such factorizations are not true if $a \nmid b$ and $\gcd(S) = 1$ as well $|S| \geq 2$, where by $|S|$ and $\gcd(S)$ we denote the cardinality of the set S and the greatest common divisor of all the elements in S, respectively. We say that the set S consists of *two coprime divisor chains* if we can partition S as $S = S_1 \cup S_2$, where S_1 and S_2 are divisor chains and each element of S_1 is coprime to each element of S_2. Later on, Hong's results were extended by Tan et al. These results confirm partially Conjectures 4.2-4.4 of [22]. It was proved in [36] that if $a|b$, then $(S^a)|(S^b)$, $[S^a]|[S^b]$ and $(S^a)|[S^b]$ hold in the ring $M_n(\mathbf{Z})$ if and only if both $\frac{x^a y^b - 1}{x^a y^a - 1}$ and $\frac{x^b y^a - 1}{x^a y^a - 1}$ are integers, where $S = S_1 \cup S_2$ with S_1 and S_2 being divisor chains and $x = \min(S_1)$ and $y = \min(S_2)$. From this one can read that even though $a|b$ and S consists of two coprime divisor chains, but if $1 \notin S$, then the divisibility theorems among power GCD matrices and among power LCM matrices need not always hold. Meanwhile, Tan, Lin and Liu found surprisingly that the divisibility theorems among determinants of power GCD matrices and among determinants of power LCM matrices should always hold. That is, they showed that if $a|b$ and S consists of two coprime divisor chains as well $1 \notin S$, then $\det(S^a)|\det(S^b)$, $\det[S^a]|\det[S^b]$ and $\det(S^a)|\det[S^b]$.

The main aim of this paper is to generalize this interesting result to the matrices of the forms $\det(f^a(S))$ and $\det(f^a[S])$, where the arithmetic function f^a is defined for any positive integer x by $f^a(x) = (f(x))^a$. We will study the divisibility among $\det(f^a(S))$ and $\det(f^b(S))$ and among $\det(f^a[S])$ and $\det(f^b[S])$ when $a|b$. We also investigate the divisibility among $\det(f(S^a))$ and $\det(f(S^b))$ and among $\det(f[S^a])$ and $\det(f[S^b])$ when $a|b$, where $S^a := \{x^a | x \in S\}$ is the ath *power set* of S. In particular, we show that if S consists of two coprime divisor chains with $1 \notin S$ and f is an integer-valued *multiplicative function* (see, for instance, [2]), then for any positive integer a, we have $\det(f(S^a)) | \det(f[S^a])$. But it is unclear whether or not the $n \times n$ matrix $(f[S^a])$ is divisible by the $n \times n$ matrix $(f(S^a))$ in the ring $M_n(\mathbf{Z})$ when S consists of two coprime divisor chains with $1 \notin S$ and f is integer-valued and multiplicative. This problem remains open. We guess that the answer to this question is affirmative.

This paper is organized as follows. First of all, we recall in Section 2 Hong's formulas for $\det(f(S))$ and $\det(f[S])$ when S is gcd closed, and then use them to give formulae for the determinants of matrices associated with arithmetic functions on divisor chains. Consequently, in Section 3, we use these results to derive the formulae for the determinants of matrices associated with arithmetic functions on two coprime divisor chains. The final section is to present the main results and their proofs. Our results extend Hong's results [20, 22] and the Tan-Lin-Liu results [36].

In the close future, we will explore the divisibility among the power matrices associated with integer-valued arithmetic functions.

2. Determinants of matrices associated with arithmetic functions on divisor chains

In the present section, we provide formulas for the determinants of matrices associated with arithmetic functions on divisor chains. For this purpose, we need the concept of greatest-type divisor introduced by Hong in 1996 (see, for example, [16] and [17]). Notice that the concept of greatest-type divisor played central roles in Hong's solution [16, 17] to the Bourque-Ligh conjecture [5], in Cao's partial answer [9] to Hong's conjecture [18] as well as in Li's partial answer [28] to Hong's conjecture [21]. We begin with the following formulas due to Hong.

Lemma 2.1. *([21]) Let f be an arithmetic function and S be a gcd-closed set. Then*

$$\det(f(S)) = \prod_{x\in S} \sum_{J\subset G_S(x)} (-1)^{|J|} f(\gcd(J \cup \{x\}))$$

and if f is multiplicative, then

$$\det(f[S]) = \prod_{x\in S} f(x)^2 \sum_{J\subset G_S(x)} \frac{(-1)^{|J|}}{f(\gcd(J \cup \{x\}))}.$$

We can now use Hong's formulae to deduce the formulae for $\det(S^a)$ and $\det[S^a]$ when S is a divisor chain.

Theorem 2.2. *Let f be an arithmetic function and $S = \{x_1, ..., x_n\}$ be a divisor chain such that $x_1|...|x_n$ and $n \geq 2$. Then*

$$\det(f(S)) = f(x_1) \prod_{i=2}^{n} (f(x_i) - f(x_{i-1}))$$

and if f is multiplicative, then

$$\det(f[S]) = (-1)^{n-1} f(x_n) \prod_{i=2}^{n} (f(x_i) - f(x_{i-1})).$$

Proof. Since $x_1|x_2|...|x_n$, we have $G_S(x_1) = \phi$ and $G_S(x_i) = \{x_{i-1}\}$ for $2 \le i \le n$. Then Theorem 2.2 follows immediately from Lemma 2.1.

This completes the proof of Theorem 2.2. □

3. Determinants of matrices associated with arithmetic functions on two coprime divisor chains

In this section, we give the formulae calculating the determinants of matrices associated with arithmetic functions on two coprime divisor chains.

Theorem 3.1. *Let f be an arithmetic function and $S = \{x_1, ..., x_n, y_1, ..., y_m\}$, where $x_1|...|x_n$, $y_1|...|y_m$ and $\gcd(x_n, y_m) = 1$. Then*

$$\det(f(S)) = (f(x_1)f(y_1) - f(1)^2)\Big(\prod_{i=1}^{n-1}(f(x_{i+1}) - f(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f(y_{j+1}) - f(y_j))\Big)$$

and if f is multiplicative, then

$$\det(f[S]) = (-1)^{m+n-1}f(x_n)f(y_m)(f(x_1)f(y_1) - 1)\Big(\prod_{i=1}^{n-1}(f(x_{i+1}) - f(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f(y_{j+1}) - f(y_j))\Big).$$

Proof. Write $S_i := \{x_1, ..., x_i\}$ and $T_j := \{y_1, ..., y_j\}$ for all integers i and j with $1 \le i \le n$ and $1 \le j \le m$. Then $S = S_n \cup T_m$.

First let $n = 1$. Then

$$\begin{aligned}\det(f(S)) &= \det(f(S_1 \cup T_m))\\ &= \det\begin{pmatrix} f(x_1) & f(1) & f(1) & \cdots & f(1)\\ f(1) & f(y_1) & f(y_1) & \cdots & f(y_1)\\ f(1) & f(y_1) & f(y_2) & \cdots & f(y_2)\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ f(1) & f(y_1) & f(y_2) & \cdots & f(y_m)\end{pmatrix}.\end{aligned}$$

Let $f(y_1) = 0$. If $m = 1$, then it is clear that

$$\det(f(S)) = f(x_1)f(y_1) - f(1)^2$$

as expected. If $m \ge 2$, then we can calculate that

$$\det(f(S)) = -f(1)^2\det(f(\tilde{T}_{m-1})),$$

where $\tilde{T}_{m-1} := T_m \setminus \{y_1\}$. If $m = 2$, then $\det(f(S)) = -f(1)^2 f(y_2)$ since $\det(f(\tilde{T}_1)) = f(y_2)$. If $m \ge 3$, then it follows from Theorem 2.2 that

$$\det(f(S)) = -f(1)^2 f(y_2)\prod_{j=2}^{m-1}(f(y_{j+1}) - f(y_j))$$

as desired.

Now let $f(y_1) \neq 0$. Then replacing the first row by the sum of itself and $-\frac{f(1)}{f(y_1)}$ multiple of the second row and using Theorem 2.2, one arrives at

$$\begin{aligned}
\det(f(S)) &= \det\begin{pmatrix} f(x_1)-\frac{f(1)^2}{f(y_1)} & 0 & 0 & \cdots & 0 \\ f(1) & f(y_1) & f(y_1) & \cdots & f(y_1) \\ f(1) & f(y_1) & f(y_2) & \cdots & f(y_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f(1) & f(y_1) & f(y_2) & \cdots & f(y_m) \end{pmatrix} \\
&= \Big(f(x_1) - \frac{f(1)^2}{f(y_1)}\Big)\det(f(T_m)) \\
&= \Big(f(x_1) - \frac{f(1)^2}{f(y_1)}\Big) f(y_1)\prod_{j=1}^{m-1}(f(y_{j+1}) - f(y_j)) \\
&= (f(x_1)f(y_1) - f(1)^2)\prod_{j=1}^{m-1}(f(y_{j+1}) - f(y_j))
\end{aligned}$$

as required. Thus the first formula of Theorem 3.1 is true when $n = 1$.

Consequently, let $n > 1$. Then we deduce that

$$\begin{aligned}
\det(f(S)) &= \det(f(S_n \cup T_m)) \\
&= \det\begin{pmatrix}
f(x_1) & f(x_1) & \cdots & f(x_1) & f(x_1) & f(1) & f(1) & \cdots & f(1) \\
f(x_1) & f(x_2) & \cdots & f(x_2) & f(x_2) & f(1) & f(1) & \cdots & f(1) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
f(x_1) & f(x_2) & \cdots & f(x_{n-1}) & f(x_{n-1}) & f(1) & f(1) & \cdots & f(1) \\
f(x_1) & f(x_2) & \cdots & f(x_{n-1}) & f(x_n) & f(1) & f(1) & \cdots & f(1) \\
f(1) & f(1) & \cdots & f(1) & f(1) & f(y_1) & f(y_1) & \cdots & f(y_1) \\
f(1) & f(1) & \cdots & f(1) & f(1) & f(y_1) & f(y_2) & \cdots & f(y_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
f(1) & f(1) & \cdots & f(1) & f(1) & f(y_1) & f(y_2) & \cdots & f(y_m)
\end{pmatrix}.
\end{aligned}$$

Replacing nth row by the sum of itself and (-1) multiple of $(n-1)$th row gives us that

$$\begin{aligned}
&\det(f(S)) \\
&= \det\begin{pmatrix}
f(x_1) & f(x_1) & \cdots & f(x_1) & f(x_1) & f(1) & f(1) & \cdots & f(1) \\
f(x_1) & f(x_2) & \cdots & f(x_2) & f(x_2) & f(1) & f(1) & \cdots & f(1) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
f(x_1) & f(x_2) & \cdots & f(x_{n-1}) & f(x_{n-1}) & f(1) & f(1) & \cdots & f(1) \\
0 & 0 & \cdots & 0 & f(x_n) - f(x_{n-1}) & 0 & 0 & \cdots & 0 \\
f(1) & f(1) & \cdots & f(1) & f(1) & f(y_1) & f(y_1) & \cdots & f(y_1) \\
f(1) & f(1) & \cdots & f(1) & f(1) & f(y_1) & f(y_2) & \cdots & f(y_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
f(1) & f(1) & \cdots & f(1) & f(1) & f(y_1) & f(y_2) & \cdots & f(y_m)
\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&=(f(x_n)-f(x_{n-1}))\det(f(S_{n-1}\cup T_m))\\
&=(f(x_n)-f(x_{n-1}))(f(x_{n-1})-f(x_{n-2}))\dots(f(x_2)-f(x_1))\det(f(S_1\cup T_m))\\
&=(f(x_1)f(y_1)-f(1)^2)\Big(\prod_{i=1}^{n-1}\big(f(x_{i+1})-f(x_i)\big)\Big)\Big(\prod_{j=1}^{m-1}\big(f(y_{j+1})-f(y_j)\big)\Big)
\end{aligned}
$$

as desired. This concludes the proof of the first part of Theorem 3.1.

We are now in the position to show the second part of Theorem 3.1. Since f is multiplicative, one has $f(1)=1$ and

$$f(\gcd(x_i,x_j))f(\operatorname{lcm}(x_i,x_j))=f(x_i)f(x_j).$$

It then follows that

$$(f[S])=\Lambda\cdot\Big(\frac{1}{f}(S)\Big)\cdot\Lambda,$$

where

$$\Lambda:=\operatorname{diag}(f(x_1),...,f(x_n),f(y_1),...,f(y_m))$$

is the $(n+m)\times(n+m)$ diagonal matrix with $f(x_1),...,f(x_n),f(y_1),...,f(y_m)$ as its diagonal elements. Therefore

$$\det(f[S])=\Big(\prod_{i=1}^{n}f^2(x_i)\Big)\Big(\prod_{j=1}^{m}f^2(y_j)\Big)\det\Big(\frac{1}{f}(S)\Big).$$

By the first part of Theorem 3.1, one derives that

$$
\begin{aligned}
\det\Big(\frac{1}{f}(S)\Big)&=\Big(\frac{1}{f(x_1)f(y_1)}-\frac{1}{f^2(1)}\Big)\cdot\prod_{i=1}^{n-1}\Big(\frac{1}{f(x_{i+1})}-\frac{1}{f(x_i)}\Big)\cdot\prod_{j=1}^{m-1}\Big(\frac{1}{f(y_{j+1})}-\frac{1}{f(y_j)}\Big)\\
&=\frac{1-f(x_1)f(y_1)}{f(x_1)f(y_1)}\cdot\frac{\prod_{i=1}^{n-1}\big(f(x_i)-f(x_{i+1})\big)}{f(x_1)f^2(x_2)\cdots f^2(x_{n-1})f(x_n)}\cdot\frac{\prod_{j=1}^{m-1}\big(f(y_j)-f(y_{j+1})\big)}{f(y_1)f^2(y_2)\cdots f^2(y_{m-1})f(y_m)}.
\end{aligned}
$$

So we obtain that

$$
\begin{aligned}
\det(f[S])&=\Big(\prod_{i=1}^{n}f^2(x_i)\Big)\Big(\prod_{j=1}^{m}f^2(y_j)\Big)\\
&\quad\times\frac{1-f(x_1)f(y_1)}{f(x_1)f(y_1)}\cdot\frac{\prod_{i=1}^{n-1}\big(f(x_i)-f(x_{i+1})\big)}{f(x_1)f^2(x_2)\cdots f^2(x_{n-1})f(x_n)}\cdot\frac{\prod_{j=1}^{m-1}\big(f(y_j)-f(y_{j+1})\big)}{f(y_j)f^2(y_2)\cdots f^2(y_{m-1})f(y_m)}\\
&=(-1)^{m+n-1}f(x_n)f(y_m)(f(x_1)f(y_1)-1)\Big(\prod_{i=1}^{n-1}(f(x_{i+1})-f(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f(y_{j+1})-f(y_j))\Big)
\end{aligned}
$$

as desired.

This ends the proof of Theorem 3.1. □

4. Divisibility among determinants of power matrices associated with integer-valued arithmetic functions

In this last section, we first study the divisibility among determinants of power matrices associated with integer-valued arithmetic functions. We begin with the following result that is the first main result of this section.

Theorem 4.1. *Let f be an integer-valued arithmetic function and let a and b be positive integers such that $a|b$. Let S consist of two coprime divisor chains with $1 \notin S$. Then* $\det(f^a(S))|\det(f^b(S))$. *Furthermore, if f is multiplicative, then* $\det(f^a[S])|\det(f^b[S])$ *and* $\det(f^a(S))|\det(f^b[S])$.

Proof. Since S consists of two coprime divisor chains and $1 \notin S$, without loss of any generality, we may let $S = \{x_1, ..., x_n, y_1, ..., y_m\}$, where $x_1|...|x_n$, $y_1|...|y_m$ and $\gcd(x_n, y_m) = 1$. Then with f replaced by f^a and f^b, Theorem 3.1 tells us that

$$\det(f^a(S)) = (f^a(x_1)f^a(y_1) - f(1)^{2a})\Big(\prod_{i=1}^{n-1}(f^a(x_{i+1}) - f^a(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^a(y_{j+1}) - f^a(y_j))\Big),$$

$$\det(f^b(S)) = (f^b(x_1)f^b(y_1) - f(1)^{2b})\Big(\prod_{i=1}^{n-1}(f^b(x_{i+1}) - f^b(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^b(y_{j+1}) - f^b(y_j))\Big),$$

$$\begin{aligned}\det(f^a[S]) =&(-1)^{m+n-1}f^a(x_n)f^a(y_m)(f^a(x_1)f^a(y_1) - 1)\\ &\times\Big(\prod_{i=1}^{n-1}(f^a(x_{i+1}) - f^a(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^a(y_{j+1}) - f^a(y_j))\Big)\end{aligned}$$

and

$$\det(f^b[S]) =(-1)^{m+n-1}f^b(x_n)f^b(y_m)(f^b(x_1)f^b(y_1) - 1)\Big(\prod_{i=1}^{n-1}(f^b(x_{i+1}) - f^b(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^b(y_{j+1}) - f^b(y_j))\Big).$$

Now let $\det(f^a(S)) = 0$. Then $f^a(x_1)f^a(y_1) - f(1)^{2a} = 0$, or $f^a(x_{i+1}) - f^a(x_i) = 0$ for some integer i with $1 \le i \le n-1$, or $f^a(y_{j+1}) - f^a(y_j)) = 0$ for some integer j with $1 \le j \le m-1$. Since $a|b$, one then deduces that $f^b(x_1)f^b(y_1) - f(1)^{2b} = 0$, or $f^b(x_{i+1}) - f^b(x_i) = 0$ for some integer i with $1 \le i \le n-1$, or $f^b(y_{j+1}) - f^b(y_j)) = 0$ for some integer j with $1 \le j \le m-1$. Thus $\det(f^b(S)) = \det(f^b[S]) = 0$ which infers that $\det(f^a(S))|\det(f^b(S))$, $\det(f^a[S])|\det(f^b[S])$ and $\det(f^a(S))|\det(f^b[S])$ as desired. Likewise, if $\det(f^a[S]) = 0$, then we can deduce that $\det(f^b[S]) = 0$. Hence $\det(f^a[S])|\det(f^b[S])$ as expected. So Theorem 4.1 holds in this case.

In what follows, we let $\det(f^a(S)) \neq 0$ and $\det(f^a[S]) \neq 0$. Since $a|b$, we may let $b = ka$ for some integer k. Therefore

$$\frac{\det(f^b(S))}{\det(f^a(S))}$$

$$
\begin{aligned}
&=\frac{(f^b(x_1)f^b(y_1)-f(1)^{2b})\Big(\prod_{i=1}^{n-1}(f^b(x_{i+1})-f^b(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^b(y_{j+1})-f^b(y_j))\Big)}{(f^a(x_1)f^a(y_1)-f(1)^{2a})\Big(\prod_{i=1}^{n-1}(f^a(x_{i+1})-f^a(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^a(y_{j+1})-f^a(y_j))\Big)}\\
&=\frac{(f^{ka}(x_1)f^{ka}(y_1)-f(1)^{2ka})\Big(\prod_{i=1}^{n-1}(f^{ka}(x_{i+1})-f^{ka}(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^{ka}(y_{j+1})-f^{ka}(y_j))\Big)}{(f^a(x_1)f^a(y_1)-f(1)^{2a})\Big(\prod_{i=1}^{n-1}(f^a(x_{i+1})-f^a(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^a(y_{j+1})-f^a(y_j))\Big)}\\
&=\Big(\sum_{t=1}^{k}(f(x_1)f(y_1))^{(k-t)a}f^{2(t-1)a}(1)\Big)\Big(\prod_{i=1}^{n-1}\sum_{t=1}^{k}(f(x_{i+1}))^{(k-t)a}f^{(t-1)a}(x_i)\Big)\\
&\quad\times\Big(\prod_{j=1}^{m-1}\sum_{t=1}^{k}(f(y_{j+1}))^{(k-t)a}f^{(t-1)a}(y_j)\Big)\in\mathbf{Z}.
\end{aligned}
$$

This implies that $\det(f^a(S))|\det(f^b(S))$.

Similarly, if f is multiplicative and integer-valued, then one deduces that $f(1)=1$,

$$
\begin{aligned}
&\frac{\det(f^b[S])}{\det(f^a[S])}\\
&=\frac{f^b(x_n)f^b(y_m)(f^b(x_1)f^b(y_1)-1)\Big(\prod_{i=1}^{n-1}(f^b(x_{i+1})-f^b(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^b(y_{j+1})-f^b(y_j))\Big)}{f^a(x_n)f^a(y_m)(f^a(x_1)f^a(y_1)-1)\Big(\prod_{i=1}^{n-1}(f^a(x_{i+1})-f^a(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^a(y_{j+1})-f^a(y_j))\Big)}\\
&=(f(x_n)f(y_m))^{(k-1)a}\Big(\sum_{t=1}^{k}(f(x_1)f(y_1))^{(k-t)a}\Big)\Big(\prod_{i=1}^{n-1}\sum_{t=1}^{k}(f(x_{i+1}))^{(k-t)a}f^{(t-1)a}(x_i)\Big)\\
&\quad\times\Big(\prod_{j=1}^{m-1}\sum_{t=1}^{k}(f(y_{j+1}))^{(k-t)a}f^{(t-1)a}(y_j)\Big)\in\mathbf{Z}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\det(f^b[S])}{\det(f^a(S))}\\
&=(-1)^{m+n-1}\times\frac{f^b(x_n)f^b(y_m)(f^b(x_1)f^b(y_1)-1)\Big(\prod_{i=1}^{n-1}(f^b(x_{i+1})-f^b(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^b(y_{j+1})-f^b(y_j))\Big)}{(f^a(x_1)f^a(y_1)-1)\Big(\prod_{i=1}^{n-1}(f^a(x_{i+1})-f^a(x_i))\Big)\Big(\prod_{j=1}^{m-1}(f^a(y_{j+1})-f^a(y_j))\Big)}\\
&=(-1)^{m+n-1}f^b(x_n)f^b(y_m)\Big(\sum_{t=1}^{k}(f(x_1)f(y_1))^{(k-t)a}\Big)\\
&\quad\times\Big(\prod_{i=1}^{n-1}\sum_{t=1}^{k}(f(x_{i+1}))^{(k-t)a}f^{(t-1)a}(x_i)\Big)\Big(\prod_{j=1}^{m-1}\sum_{t=1}^{k}(f(y_{j+1}))^{(k-t)a}f^{(t-1)a}(y_j)\Big)\in\mathbf{Z}
\end{aligned}
$$

as one requires. Thus Theorem 4.1 is true if $\det(f^a(S)) \neq 0$ and $\det(f^a[S]) \neq 0$.

This finishes the proof of Theorem 4.1. □

We point out that the condition $a|b$ in Theorem 4.1 is not necessary as the following example shows.

Example 4.1. (i). Let $f(x) = x + 1$, $a = 2, b = 5$ and $S = \{2, 4, 3\}$. Then $a \nmid b$. Clearly, one has

$$(f^2(S)) = \begin{pmatrix} 9 & 9 & 4 \\ 9 & 25 & 4 \\ 4 & 4 & 16 \end{pmatrix} \text{ and } (f^5(S)) = \begin{pmatrix} 243 & 243 & 32 \\ 243 & 3125 & 32 \\ 32 & 32 & 1024 \end{pmatrix}.$$

So we can compute and get that

$$\det(f^2(S)) = 2048 \text{ and } \det(f^5(S)) = 714182656.$$

Hence

$$\frac{\det(f^5(S))}{\det(f^2(S))} = 348722 \in \mathbf{Z}.$$

That is, one has $\det(f^2(S))|\det(f^5(S))$.

(ii). Let $f(x) = \varphi(x)$, $a = 2, b = 3$ and $S = \{2, 4, 7\}$. Then $a \nmid b$ and

$$(\varphi^2(S)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 36 \end{pmatrix}, (\varphi^2[S]) = \begin{pmatrix} 1 & 4 & 36 \\ 4 & 4 & 144 \\ 36 & 144 & 36 \end{pmatrix}$$

and

$$(\varphi^3[S]) = \begin{pmatrix} 1 & 8 & 216 \\ 8 & 8 & 1728 \\ 216 & 1728 & 216 \end{pmatrix}.$$

One can easily calculate and obtain that

$$\det(\varphi^2(S)) = 105, \det(\varphi^2[S]) = 15120 \text{ and } \det(\varphi^3[S]) = 2600640.$$

Thus

$$\frac{\det(\varphi^3[S])}{\det(\varphi^2(S))} = 24768 \in \mathbf{Z} \text{ and } \frac{\det(\varphi^3[S])}{\det(\varphi^2[S])} = 172 \in \mathbf{Z}.$$

In other words, we have $\det(\varphi^2[S])|\det(\varphi^3[S])$ and $\det(\varphi^2(S))|\det(\varphi^3[S])$. □

It is also remarked that the condition that f is multiplicative in Theorem 4.1 is necessary as the following example shows.

Example 4.2. Letting $f(x) := x + 1$, $a := 1, b := 3$ and $S := \{2, 4, 3\}$ gives us that

$$(f(S)) = \begin{pmatrix} 3 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix}, (f[S]) = \begin{pmatrix} 3 & 5 & 7 \\ 5 & 5 & 13 \\ 7 & 13 & 4 \end{pmatrix}$$

and

$$(f^3(S)) = \begin{pmatrix} 27 & 27 & 8 \\ 27 & 125 & 8 \\ 8 & 8 & 64 \end{pmatrix}, (f^3[S]) = \begin{pmatrix} 27 & 125 & 343 \\ 125 & 125 & 2197 \\ 343 & 2197 & 64 \end{pmatrix}.$$

So we obtain that $\det(f(S)) = 16, \det(f[S]) = 118, \det(f^3(S)) = 163072$ and $\det(f^3[S]) = 42578782$. Thus

$$\frac{\det(f^3(S))}{\det(f(S))} = 10192 \in \mathbf{Z}, \frac{\det(f^3[S])}{\det(f(S))} = \frac{21289391}{8} \notin \mathbf{Z} \text{ and } \frac{\det(f^3[S])}{\det(f[S])} = \frac{21289391}{59} \notin \mathbf{Z}.$$

So $\det(f(S)) | \det(f^3(S))$, $\det(f(S)) \nmid \det(f^3[S])$ and $\det(f[S]) \nmid \det(f^3[S])$. □

Subsequently, we explore the divisibility of determinants of the matrices associated to the integer-valued multiplicative function on the power set S^a. We present the second main result of this section as follows.

Theorem 4.2. *Let f be an integer-valued arithmetic function and let a and b be positive integers such that $a|b$. Let S consist of two coprime divisor chains with $1 \notin S$. Then each of the following is true:*

(i). *If f is multiplicative, then* $\det(f(S^a)) | \det(f[S^a])$.

(ii). *If f is completely multiplicative, then we have* $\det(f(S^a)) | \det(f(S^b))$, $\det(f[S^a]) | \det(f[S^b])$ *and* $\det(f(S^a)) | \det(f[S^b])$.

Moreover, there exist multiplicative functions f, positive integers a and b with $a|b$ and $b > a$, and a set S consisting of two coprime divisor chains with $1 \notin S$, such that $\det(f(S^a)) \nmid \det(f(S^b))$, $\det(f[S^a]) \nmid \det(f[S^b])$ *and* $\det(f(S^a)) \nmid \det(f[S^b])$.

Proof. We begin with the proof of the first part of Theorem 4.2.

(i). Since S consists of two coprime divisor chains with $1 \notin S$, the power set S^a consists of two coprime divisor chains with $\gcd(S^a) = 1 \notin S^a$. Furthermore, since f is multiplicative, one has either $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$, then f is the zero function and so one has $\det(f(S^a)) = \det(f[S^a]) = 0$. Thus $\det(f(S^a)) | \det(f[S^a])$ as desired. Now let $f(1) = 1$. Then by Lemma 3.1, we have

$$\begin{aligned}
&(-1)^{m+n-1} f(x_n^a) f(y_m^a) \det(f(S^a)) \\
=&(-1)^{m+n-1} f(x_n^a) f(y_m^a) (f(x_1^a) f(y_1^a) - 1) \Big(\prod_{i=1}^{n-1} (f(x_{i+1}^a) - f(x_i^a)) \Big) \Big(\prod_{j=1}^{m-1} (f(y_{j+1}^a) - f(y_j^a)) \Big) \\
=& \det(f[S^a])
\end{aligned}$$

However, since f is integer valued, one has $f(x_n^a) f(y_m^a) \in \mathbf{Z}$. Therefore the desired result $\det(f(S^a)) | \det(f[S^a])$ follows. Part (i) is proved.

(ii). If f is complete multiplicative, then it is clear that $f(x^a) = f^a(x)$ for any positive integers a and x. So one has

$$(f(S^a)) = (f^a(S)), (f(S^b)) = (f^b(S)), (f[S^a]) = (f^a[S]) \text{ and } (f[S^b]) = (f^b[S]).$$

Since $a|b$ and S consists of two coprime divisor chains with $1 \notin S$, it then follows from Theorem 4.1 that $\det(f^a(S)) | \det(f^b(S))$, $\det(f^a[S]) | \det(f^b[S])$ and $\det(f^a(S)) | \det(f^b[S])$. Thus the desired results

$\det(f(S^a))|\det(f(S^b))$, $\det(f[S^a])|\det(f[S^b])$ and $\det(f(S^a))|\det(f[S^b])$ follow immediately. Part (ii) is proved.

Finally, we turn our attention to the proof of the second part of Theorem 4.2. Letting $S := \{2, 4, 3\}$ and $a := 2, b := 4$ gives us that

$$(S) = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix}, (S^2) = \begin{pmatrix} 4 & 4 & 1 \\ 4 & 16 & 1 \\ 1 & 1 & 9 \end{pmatrix}, (S^4) = \begin{pmatrix} 16 & 16 & 1 \\ 16 & 256 & 1 \\ 1 & 1 & 81 \end{pmatrix}$$

and

$$[S] = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 4 & 12 \\ 6 & 12 & 3 \end{pmatrix}, [S^2] = \begin{pmatrix} 4 & 16 & 36 \\ 16 & 16 & 144 \\ 36 & 144 & 9 \end{pmatrix}, [S^4] = \begin{pmatrix} 16 & 256 & 1296 \\ 256 & 256 & 20736 \\ 1296 & 20736 & 81 \end{pmatrix}.$$

Therefore picking $f = \varphi$ to be the Euler's totient function tells us that

$$(f(S^2)) = (\varphi(S^2)) = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 8 & 1 \\ 1 & 1 & 6 \end{pmatrix}, (f(S^4)) = (\varphi(S^4)) = \begin{pmatrix} 8 & 8 & 1 \\ 8 & 128 & 1 \\ 1 & 1 & 54 \end{pmatrix}$$

and

$$(f[S^2]) = (\varphi[S^2]) = \begin{pmatrix} 2 & 8 & 12 \\ 8 & 8 & 48 \\ 12 & 48 & 6 \end{pmatrix}, (f[S^4]) = (\varphi[S^4]) = \begin{pmatrix} 8 & 128 & 432 \\ 128 & 128 & 6912 \\ 432 & 6912 & 54 \end{pmatrix}.$$

So one deduces that

$$\frac{\det(f(S^b))}{\det(f(S^a))} = \frac{\det(\varphi(S^4))}{\det(\varphi(S^2))} = \frac{51720}{66} = \frac{8620}{11} \notin \mathbf{Z},$$

$$\frac{\det(f[S^b])}{\det(f[S^a])} = \frac{\det(\varphi[S^4])}{\det(\varphi[S^2])} = \frac{357488640}{3168} = \frac{1241280}{11} \notin \mathbf{Z}$$

and

$$\frac{\det(f[S^b])}{\det(f(S^a))} = \frac{\det(\varphi[S^4])}{\det(\varphi(S^2))} = \frac{357488640}{66} = \frac{59581440}{11} \notin \mathbf{Z}.$$

So $\det(f(S^a)) \nmid \det(f(S^b))$, $\det(f[S^a]) \nmid \det(f[S^b])$ and $\det(f(S^a)) \nmid \det(f[S^b])$ as desired.

This concludes the proof of Theorem 4.2. □

Remark 4.3. (i). *If S consists of at least three coprime divisor chains, then the divisibility result in Theorem 4.2 (i) may be false. For instance, letting $S := \{2, 4, 3, 5\}$ and $a := 2$ gives us that*

$$(\varphi(S^2)) = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 8 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 20 \end{pmatrix} \text{ and } (\varphi[S^2]) = \begin{pmatrix} 2 & 8 & 12 & 40 \\ 8 & 8 & 48 & 160 \\ 12 & 48 & 6 & 120 \\ 40 & 160 & 120 & 20 \end{pmatrix}.$$

Hence

$$\frac{\det(\varphi[S^2])}{\det(\varphi(S^2))} = -\frac{148320}{107} \notin \mathbf{Z}.$$

That is, $\det(\varphi(S^2)) \nmid \det(\varphi[S^2])$.

On the other hand, the condition that f *is multiplicative in Theorem 4.2 (i) is necessary. For example, let* $f(x) := x + 1$ *and* $S := \{2, 4, 3\}, a := 1$. *Then* f *is not multiplicative and*

$$\frac{\det(f[S])}{\det(f(S))} = \frac{59}{8} \notin \mathbf{Z}.$$

Hence $\det(f(S)) \nmid \det(f[S])$.

(ii). *We remark that the condition* $a|b$ *is not necessary for the divisibility result in Theorem 4.2 (ii). For example, letting* $f(x) := x, S := \{2, 6, 5\}, a := 2$ *and* $b := 5$ *gives us that*

$$(S^2) = \begin{pmatrix} 4 & 4 & 1 \\ 4 & 36 & 1 \\ 1 & 1 & 25 \end{pmatrix}, (S^5) = \begin{pmatrix} 32 & 32 & 1 \\ 32 & 7776 & 1 \\ 1 & 1 & 3125 \end{pmatrix}$$

and

$$[S^2] = \begin{pmatrix} 4 & 36 & 100 \\ 36 & 36 & 900 \\ 100 & 900 & 25 \end{pmatrix}, [S^5] = \begin{pmatrix} 32 & 7776 & 100000 \\ 7776 & 7776 & 24300000 \\ 100000 & 24300000 & 3125 \end{pmatrix}.$$

Then we compute and get that

$$\frac{\det((S^5))}{\det((S^2))} = 244442 \in \mathbf{Z}, \frac{\det([S^5])}{\det([S^2])} = 659993400 \in \mathbf{Z} \text{ and } \frac{\det([S^5])}{\det((S^2))} = 5939940600000 \in \mathbf{Z}.$$

It follows immediately that $\det((S^2))|\det((S^5)), \det([S^2])|\det([S^5])$ *and* $\det((S^2))|\det([S^5])$.

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Conflict of interest

We declare that we have no conflict of interest.

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