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Research article

On generalized k-fractional derivative operator

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Abstract: The principal aim of this paper is to introduce k-fractional derivative operator by using the definition of k-beta function. This paper establishes some results related to the newly defined fractional operator such as the Mellin transform and the relations to k-hypergeometric and k-Appell's functions. Also, we investigate the k-fractional derivative of k-Mittag-Leffler and the Wright hypergeometric functions.

Keywords: beta function; k-beta function; hypergeometric function; k-hypergeometric function; Mellin transform; fractional derivative; Appell's function; k-Mittag-Leffler function
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1. Introduction

The classical beta function

$$\mathbb{B}(\delta_1,\delta_2)=\int_0^\infty t^{\delta_1-1}(1-t)^{\delta_2-1}dt,\ (\Re(\delta_1)>0,\Re(\delta_2)>0) \tag{1.1}$$

and its relation with well known gamma function is given by

$$\mathbb{B}(\delta_1,\delta_2)=\frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_1+\delta_2)},\ \Re(\delta_1)>0,\Re(\delta_2)>0.$$

The Gauss hypergeometric, confluent hypergeometric and Appell's functions which are respectively defined by(see [27])

$${}_2F_1(\delta_1,\delta_2;\delta_3;z)=\sum_{n=0}^{\infty}\frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n}\frac{z^n}{n!},\ (|z|<1), \tag{1.2}$$

$$\Big(\delta_1, \delta_2, \delta_3 \in \mathbb{C} \text{ and } \delta_3 \neq 0, -1, -2, -3, \cdots\Big),$$

and

$$_1\Phi_1(\delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{(\delta_2)_n}{(\delta_3)_n} \frac{z^n}{n!}, (|z| < 1), \tag{1.3}$$

$$\Big(\delta_2, \delta_3 \in \mathbb{C} \text{ and } \delta_3 \neq 0, -1, -2, -3, \cdots\Big).$$

The Appell's series or bivariate hypergeometric series is defined by

$$F_1(\delta_1, \delta_2, \delta_3; \delta_4; x, y) = \sum_{m,n=0}^{\infty} \frac{(\delta_1)_{m+n}(\delta_2)_m(\delta_3)_n x^m y^n}{(\delta_4)_{m+n} m! n!}; \tag{1.4}$$

for all $\delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{C}, \delta_4 \neq 0, -1, -2, -3, \cdots, \quad |x|, |y| < 1 < 1.$

The integral representation of hypergeometric, confluent hypergeometric and Appell's functions are respectively defined by

$$_2F_1(\delta_1, \delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1 - t)^{\delta_3 - \delta_2 - 1}(1 - zt)^{-\delta_1} dt, \tag{1.5}$$

$$\Big(\Re(\delta_3) > \Re(\delta_2) > 0, |\arg(1 - z)| < \pi\Big),$$

and

$$_1\Phi_1(\delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1 - t)^{\delta_3 - \delta_2 - 1} e^{zt} dt, \tag{1.6}$$

$$\Big(\Re(\delta_3) > \Re(\delta_2) > 0\Big).$$

$$\begin{aligned} & F_1\Big(\delta_1, \delta_2, \delta_3; \delta_4; x, y\Big) \\ = \; & \frac{\Gamma(\delta_4)}{\Gamma(\delta_1)\Gamma(\delta_4 - \delta_1)} \int_0^1 t^{\delta_1 - 1}(1 - t)^{\delta_4 - \delta_1 - 1}(1 - xt)^{-\delta_2}(1 - yt)^{-\delta_3} dt. \end{aligned} \tag{1.7}$$

The k-gamma function, k-beta function and the k-Pochhammer symbol introduced and studied by Diaz and Pariguan [5]. The integral representation of k-gamma function and k-beta function respectively given by

$$\Gamma_{\mathrm{k}}(z) = \mathrm{k}^{\frac{z}{\mathrm{k}} - 1}\Gamma\left(\frac{z}{\mathrm{k}}\right) = \int_0^\infty t^{z-1} e^{-\frac{t^{\mathrm{k}}}{\mathrm{k}}} dt, \quad \Re(z) > 0, \mathrm{k} > 0 \tag{1.8}$$

$$\mathbb{B}_{\mathrm{k}}(x, y) = \frac{1}{\mathrm{k}} \int_0^1 t^{\frac{x}{\mathrm{k}} - 1}(1 - t)^{\frac{y}{\mathrm{k}} - 1} dt, \quad \Re(x) > 0, \Re(y) > 0. \tag{1.9}$$

Here, we recall the following relations (see [5]).

$$\mathbb{B}_{\mathrm{k}}(x,y)=\frac{\Gamma_{\mathrm{k}}(x)\Gamma_{\mathrm{k}}(y)}{\Gamma_{\mathrm{k}}(x+y)}, \tag{1.10}$$

$$(z)_{n,\mathrm{k}}=\frac{\Gamma_{\mathrm{k}}(z+n\mathrm{k})}{\Gamma_{\mathrm{k}}(z)}, \tag{1.11}$$

where $(z)_{n,\mathrm{k}}=(z)(z+\mathrm{k})(z+2\mathrm{k})\cdots(z+(n-1)\mathrm{k});\quad (z)_{0,\mathrm{k}}=1$ and $\mathrm{k}>0$
and

$$\sum_{n=0}^{\infty}(\alpha)_{n,\mathrm{k}}\frac{z^n}{n!}=(1-\mathrm{k}z)^{\frac{-\alpha}{\mathrm{k}}}. \tag{1.12}$$

These studies were followed by Mansour [16], Kokologiannaki [13], Krasniqi [14] and Merovci [17]. In 2012, Mubeen and Habibullah [18] defined the k-hypergeometric function as

$${}_2F_{1,\mathrm{k}}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\sum_{n=0}^{\infty}\frac{(\delta_1)_{n,\mathrm{k}}(\delta_2)_{n,\mathrm{k}}}{(\delta_3)_{n,\mathrm{k}}}\frac{z^n}{n!}, \tag{1.13}$$

where $\delta_1,\delta_2,\delta_3\in\mathbb{C}$ and $\delta_3\neq 0,-1,-2,\cdots$ and its integral representation is given by

$$\begin{aligned}{}_2F_{1,\mathrm{k}}\Big(\delta_1,\delta_2;\delta_3;z\Big)=&\frac{1}{\mathrm{k}\mathbb{B}_{\mathrm{k}}(\delta_2,\delta_3-\delta_2)}\\&\times\int_0^1 t^{\frac{\delta_2}{\mathrm{k}}-1}(1-t)^{\frac{\delta_3-\delta_2}{\mathrm{k}}-1}(1-\mathrm{k}tz)^{-\frac{\delta_1}{\mathrm{k}}}dt.\end{aligned} \tag{1.14}$$

The k-Riemann-Liouville (R-L) fractional integral using k-gamma function introduced in [19]:

$$\Big(I_{\mathrm{k}}^{\alpha}f(t)\Big)(x)=\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\alpha)}\int_0^x f(t)(x-t)^{\frac{\alpha}{\mathrm{k}}-1}dt,\mathrm{k},\alpha\in\mathbb{R}^+. \tag{1.15}$$

Later on Mubeen and Iqbal [11] established the improved version of Gruss type inequalities by utilizing k-fractional integrals. In [1], Agarwal et al. presented certain Hermite-Hadamard type inequalities for generalized k-fractional integrals. Set et al. [29] presented an integral identity and generalized Hermite–Hadamard type inequalities for Riemann–Liouville fractional integral. Mubeen et al. [24] established integral inequalities of Ostrowski type for k-fractional Riemann–Liouville integrals. Recently, many researchers have introduced generalized version of k-fractional integrals and investigated a large bulk of various inequalities via the said fractional integrals. The interesting readers are referred to see the work of [9, 10, 26, 30]. Farid et al. [7] introduced Hadamard k-fractional integrals. In [8] introduced Hadamard-type inequalities for k-fractional Riemann-Liouville integrals. In [12, 31], the authors established certain inequalities by utilizing Hadamard-type inequalities for k-fractional Riemann-Liouville integrals. In [25], Nisar et al. established certain Gronwall type inequalities associated with Riemann-Liouville k- and Hadamard k-fractional derivatives and their applications. In [25], they presented dependence solutions of certain k-fractional differential equations of arbitrary real order with initial conditions. Recently, Samraiz et al. [28] defined an extension of Hadamard k-fractional derivative and proved its various properties.

The solution of some integral equations involving confluent k-hypergeometric functions and k-analogue of Kummer's first formula are given in [22, 23]. While the k-hypergeometric and confluent k-hypergeometric differential equations are introduced in [20]. In 2015, Mubeen et al. [21] introduced k-Appell hypergeometric function as

$$F_{1,\mathrm{k}}(\delta_1,\delta_2,\delta_3;\delta_4;z_1,z_2)=\sum_{m,n=0}^{\infty}\frac{(\delta_1)_{m+n,\mathrm{k}}(\delta_2)_{m,\mathrm{k}}(\delta_3)_{m,\mathrm{k}}}{(\delta_4)_{m+n,\mathrm{k}}}\frac{z_1^m z_2^n}{m!n!} \tag{1.16}$$

for all $\delta_1,\delta_2,\delta_3,\delta_4\in\mathbb{C},\delta_4\neq 0,-1,-2,-3,\cdots,\quad \max\{|z_1|,|z_2|\}<\frac{1}{\mathrm{k}}$ and $\mathrm{k}>0$. Also, Mubeen et al. defined its integral representation as

$$\begin{aligned}&F_{1,\mathrm{k}}\Big(\delta_1,\delta_2,\delta_3;\delta_4;z_1,z_2\Big)\\&=\frac{1}{\mathrm{k}\mathbb{B}_{\mathrm{k}}(\delta_1,\delta_4-\delta_1)}\int\limits_0^1 t^{\frac{\delta_1}{\mathrm{k}}-1}(1-t)^{\frac{\delta_4-\delta_1}{\mathrm{k}}-1}(1-\mathrm{k}z_1t)^{-\frac{\delta_2}{\mathrm{k}}}(1-\mathrm{k}z_2t)^{-\frac{\delta_3}{\mathrm{k}}}dt,\\&(\Re(\delta_4)>\Re(\delta_1)>0).\end{aligned} \tag{1.17}$$

2. Extension of fractional derivative operator

In this section, we recall the following definition of fractional derivatives from and give a new extension called Riemann-Liouville k-fractional derivative.

Definition 2.1. *The well-known R-L fractional derivative of order μ is defined by*

$$\mathfrak{D}_x^{\mu}\{f(x)\}=\frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}dt,\Re(\mu)<0. \tag{2.1}$$

For the case $m-1<\Re(\mu)<m$ where $m=1,2,\cdots$, it follows

$$\begin{aligned}\mathfrak{D}_x^{\mu}\{f(x)\}&=\frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\Big\{f(x)\Big\}\\&=\frac{d^m}{dx^m}\Big\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}dt\Big\}.\end{aligned} \tag{2.2}$$

For further study and applications, we refer the readers to the work of [2–4, 15, 32]. In the following, we define Riemann-Liouville k-fractional derivative of order μ as

Definition 2.2.

$${}_{\mathrm{k}}\mathfrak{D}_x^{\mu}\{f(x)\}=\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^x f(t)(x-t)^{-\frac{\mu}{\mathrm{k}}-1}dt,\Re(\mu)<0,\mathrm{k}\in\mathbb{R}^+. \tag{2.3}$$

For the case $m-1<\Re(\mu)<m$ where $m=1,2,\cdots$, it follows

$$\begin{aligned}{}_{\mathrm{k}}\mathfrak{D}_x^{\mu}\{f(x)\}&=\frac{d^m}{dx^m}{}_{\mathrm{k}}\mathfrak{D}_x^{\mu-m\mathrm{k}}\Big\{f(x)\Big\}\\&=\frac{d^m}{dx^m}\Big\{\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu+m\mathrm{k})}\int_0^x f(t)(x-t)^{-\frac{\mu}{\mathrm{k}}+m-1}dt\Big\}.\end{aligned} \tag{2.4}$$

Note that for $\mathrm{k} = 1$, definition 2.2 reduces to the classical R-L fractional derivative operator given in definition 2.1.

Now, we are ready to prove some theorems by using the new definition 2.2.

Theorem 1. *The following formula holds true,*

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\{z^{\frac{\eta}{\mathrm{k}}}\} = \frac{z^{\frac{\eta-\mu}{\mathrm{k}}}}{\Gamma_{\mathrm{k}}(-\mu)}\mathbb{B}_{\mathrm{k}}(\eta+\mathrm{k},-\mu), \Re(\mu) < 0. \tag{2.5} $$

Proof. From (2.3), we have

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\{z^{\frac{\eta}{\mathrm{k}}}\} = \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^z t^{\frac{\eta}{\mathrm{k}}}(z-t)^{-\frac{\mu}{\mathrm{k}}-1}\,dt. \tag{2.6} $$

Substituting $t = uz$ in (2.6), we get

$$ \begin{aligned} {}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\{z^{\frac{\eta}{\mathrm{k}}}\} &= \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^1 (uz)^{\frac{\eta}{\mathrm{k}}}(z-uz)^{-\frac{\mu}{k}-1}\,zdu \\ &= \frac{z^{\frac{\eta-\mu}{\mathrm{k}}}}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^1 u^{\frac{\eta}{\mathrm{k}}}(1-u)^{-\frac{\mu}{\mathrm{k}}-1}\,du. \end{aligned} $$

Applying definition (1.9) to the above equation, we get the desired result. □

Theorem 2. *Let* $\Re(\mu) > 0$ *and suppose that the function* $f(z)$ *is analytic at the origin with its Maclaurin expansion given by* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *where* $|z| < \rho$ *for some* $\rho \in \mathbb{R}^+$*. Then*

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\{f(z)\} = \sum_{n=0}^{\infty} a_n\, {}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\{z^n\}. \tag{2.7} $$

Proof. Using the series expansion of the function $f(z)$ in (2.3) gives

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\{f(z)\} = \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^z \sum_{n=0}^{\infty} a_n t^n (z-t)^{-\frac{\mu}{\mathrm{k}}-1}\,dt. $$

As the series is uniformly convergent on any closed disk centered at the origin with its radius smaller then ρ, therefore the series so does on the line segment from 0 to a fixed z for $|z| < \rho$. Thus it guarantee terms by terms integration as follows

$$ \begin{aligned} {}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\{f(z)\} &= \sum_{n=0}^{\infty} a_n \Big\{\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^z t^n (z-t)^{-\frac{\mu}{\mathrm{k}}-1}\,dt \\ &= \sum_{n=0}^{\infty} a_n\, {}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\{z^n\}, \end{aligned} $$

which is the required proof. □

Theorem 3. *The following result holds true:*

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}z)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{\Gamma_{\mathrm{k}}(\eta)}{\Gamma_{\mathrm{k}}(\mu)} z^{\frac{\mu}{\mathrm{k}}-1}\, {}_2F_{1,\mathrm{k}}\Big(\beta,\eta;\mu;z\Big), \tag{2.8} $$

where $\Re(\mu) > \Re(\eta) > 0$ *and* $|z| < 1$.

Proof. By direct calculation, we have

$$
\begin{aligned}
{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}z)^{-\frac{\beta}{\mathrm{k}}}\} &= \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\mu-\eta)}\int_0^z t^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}t)^{-\frac{\beta}{\mathrm{k}}}(z-t)^{\frac{\mu-\eta}{\mathrm{k}}-1}dt \\
&= \frac{z^{\frac{\mu-\eta}{\mathrm{k}}-1}}{\mathrm{k}\Gamma_{\mathrm{k}}(\mu-\eta)}\int_0^z t^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}t)^{-\frac{\beta}{\mathrm{k}}}(1-\frac{t}{z})^{\frac{\mu-\eta}{\mathrm{k}}-1}dt.
\end{aligned}
$$

Substituting $t = zu$ in the above equation, we get

$$
{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}z)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{z^{\frac{\mu}{\mathrm{k}}-1}}{\mathrm{k}\Gamma_{\mathrm{k}}(\mu-\eta)}\int_0^1 u^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}uz)^{-\frac{\beta}{\mathrm{k}}}(1-u)^{\frac{\mu-\eta}{\mathrm{k}}-1}zdu.
$$

Applying (1.14) and after simplification we get the required proof. □

Theorem 4. *The following result holds true:*

$$
{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}az)^{-\frac{\alpha}{\mathrm{k}}}(1-\mathrm{k}bz)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{\Gamma_{\mathrm{k}}(\eta)}{\Gamma_{\mathrm{k}}(\mu)}z^{\frac{\mu}{\mathrm{k}}-1}F_{1,\mathrm{k}}\Big(\eta,\alpha,\beta;\mu;az,bz\Big), \tag{2.9}
$$

where $\Re(\mu) > \Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\max\{|az|,|bz|\} < \frac{1}{\mathrm{k}}$.

Proof. To prove (2.9), we use the power series expansion

$$
(1-\mathrm{k}az)^{-\frac{\alpha}{\mathrm{k}}}(1-\mathrm{k}bz)^{-\frac{\beta}{\mathrm{k}}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m,\mathrm{k}}(\beta)_{n,\mathrm{k}}\frac{(az)^m}{m!}\frac{(bz)^n}{n!}.
$$

Now, applying Theorem 1, we obtain

$$
\begin{aligned}
&{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}az)^{-\frac{\alpha}{\mathrm{k}}}(1-\mathrm{k}bz)^{-\frac{\beta}{\mathrm{k}}}\} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m,\mathrm{k}}(\beta)_{n,\mathrm{k}}\frac{(a)^m}{m!}\frac{(b)^n}{n!}{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}+m+n-1}\} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m,\mathrm{k}}(\beta)_{n,\mathrm{k}}\frac{(a)^m}{m!}\frac{(b)^n}{n!}\frac{\beta_{\mathrm{k}}(\eta+m\mathrm{k}+n\mathrm{k},\mu-\eta)}{\Gamma_{\mathrm{k}}(\mu-\eta)}z^{\frac{\mu}{\mathrm{k}}+m+n-1} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m,k}(\beta)_{n,k}\frac{(a)^m}{m!}\frac{(b)^n}{n!}\frac{\Gamma_k(\eta+mk+nk)}{\Gamma_k(\mu+mk+nk)}z^{\frac{\mu}{k}+m+n-1}.
\end{aligned}
$$

In view of (1.16), we get

$$
{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}az)^{-\frac{\alpha}{\mathrm{k}}}(1-\mathrm{k}bz)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{\Gamma_{\mathrm{k}}(\eta)}{\Gamma_{\mathrm{k}}(\mu)}z^{\frac{\mu}{\mathrm{k}}-1}F_{1,\mathrm{k}}\Big(\eta,\alpha,\beta;\mu;az,bz\Big).
$$

□

Theorem 5. *The following Mellin transform formula holds true:*

$$
M\Big\{e^{-x}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}(z^{\frac{\eta}{\mathrm{k}}});s\Big\} = \frac{\Gamma(s)}{\Gamma_{\mathrm{k}}(-\mu)}\mathbb{B}_{\mathrm{k}}(\eta+\mathrm{k},-\mu)z^{\frac{\eta-\mu}{\mathrm{k}}}, \tag{2.10}
$$

where $\Re(\eta) > -1$, $\Re(\mu) < 0$, $\Re(s) > 0$.

Proof. Applying the Mellin transform on definition (2.3), we have

$$M\Big\{e^{-x}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}(z^{\frac{\eta}{\mathrm{k}}});s\Big\}=\int_0^\infty x^{s-1}e^{-x}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}(z^{\eta});s\Big\}dx$$
$$=\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^\infty x^{s-1}e^{-x}\Big\{\int_0^z t^{\frac{\eta}{\mathrm{k}}}(z-t)^{-\frac{\mu}{\mathrm{k}}-1}\,dt\Big\}dx$$
$$=\frac{z^{-\frac{\mu}{\mathrm{k}}-1}}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^\infty x^{s-1}e^{-x}\Big\{\int_0^z t^{\frac{\eta}{\mathrm{k}}}(1-\frac{t}{z})^{-\frac{\mu}{\mathrm{k}}-1}\,dt\Big\}dx$$
$$=\frac{z^{\frac{\eta-\mu}{\mathrm{k}}}}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^\infty x^{s-1}e^{-x}\Big\{\int_0^1 u^{\frac{\eta}{\mathrm{k}}}(1-u)^{-\frac{\mu}{\mathrm{k}}-1}\,du\Big\}dx$$

Interchanging the order of integrations in above equation, we get

$$M\Big\{e^{-x}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}(z^{\frac{\eta}{\mathrm{k}}});s\Big\}=\frac{z^{\frac{\eta-\mu}{\mathrm{k}}}}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\int_0^1 u^{\frac{\eta}{\mathrm{k}}}(1-u)^{-\frac{\mu}{\mathrm{k}}-1}\Big(\int_0^\infty x^{s-1}e^{-x}\,dx\Big)du.$$
$$=\frac{z^{\frac{\eta-\mu}{\mathrm{k}}}}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\Gamma(s)\int_0^1 u^{\frac{\eta}{\mathrm{k}}}(1-u)^{-\frac{\mu}{\mathrm{k}}-1}du$$
$$=\frac{\Gamma(s)}{\Gamma_{\mathrm{k}}(-\mu)}\mathbb{B}_{\mathrm{k}}(\eta+\mathrm{k},-\mu)z^{\frac{\eta-\mu}{\mathrm{k}}},$$

which completes the proof. □

Theorem 6. *The following Mellin transform formula holds true:*

$$M\Big\{e^{-x}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}((1-\mathrm{k}z)^{-\frac{\alpha}{\mathrm{k}}});s\Big\}=\frac{z^{-\frac{\mu}{\mathrm{k}}}\Gamma(s)}{\Gamma_{\mathrm{k}}(-\mu)}\mathbb{B}_{\mathrm{k}}(\mathrm{k},-\mu)\,{}_2F_{1,\mathrm{k}}\Big(\alpha,\mathrm{k};-\mu+\mathrm{k};z\Big), \tag{2.11}$$

where $\Re(\alpha)>0$, $\Re(\mu)<0$, $\Re(s)>0$, *and* $|z|<1$.

Proof. Using the power series for $(1-\mathrm{k}z)^{-\frac{\alpha}{\mathrm{k}}}$ and applying Theorem 5 with $\eta=n\mathrm{k}$, we can write

$$M\Big\{e^{-x}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}((1-\mathrm{k}z)^{-\frac{\alpha}{\mathrm{k}}});s\Big\}=\sum_{n=0}^{\infty}\frac{(\alpha)_{n,\mathrm{k}}}{n!}M\Big\{e^{-x}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}(z^{n});s\Big\}$$
$$=\frac{\Gamma(s)}{\mathrm{k}\Gamma_{\mathrm{k}}(-\mu)}\sum_{n=0}^{\infty}\frac{(\alpha)_{n,\mathrm{k}}}{n!}\mathbb{B}_{\mathrm{k}}(n\mathrm{k}+\mathrm{k},-\mu)z^{n-\frac{\mu}{\mathrm{k}}}$$
$$=\frac{\Gamma(s)z^{-\frac{\mu}{\mathrm{k}}}}{\Gamma_{\mathrm{k}}(-\mu)}\sum_{n=0}^{\infty}\mathbb{B}_{\mathrm{k}}(n\mathrm{k}+\mathrm{k},-\mu)\frac{(\alpha)_{n,\mathrm{k}}z^n}{n!}$$
$$=\Gamma(s)z^{-\frac{\mu}{k}}\sum_{n=0}^{\infty}\frac{\Gamma_k(k+nk)}{\Gamma_k(-\mu+k+nk)}\frac{(\alpha)_{n,k}z^n}{n!}$$
$$=\frac{\Gamma(s)}{\Gamma_k(-\mu+k)}z^{-\frac{\mu}{k}}\sum_{n=0}^{\infty}\frac{(k)_{n,k}}{(-\mu+k)_{n,k}}\frac{(\alpha)_{n,k}z^n}{n!}$$
$$=\frac{\Gamma(s)z^{-\frac{\mu}{\mathrm{k}}}}{\Gamma_{\mathrm{k}}(-\mu)}\mathbb{B}_{\mathrm{k}}(\mathrm{k},-\mu)\,{}_2F_{1,\mathrm{k}}\Big(\alpha,\mathrm{k};-\mu+\mathrm{k};z\Big),$$

which is the required proof. □

Theorem 7. *The following result holds true:*

$$
{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\Big[z^{\frac{\eta}{\mathrm{k}}-1}E^{\mu}_{\mathrm{k},\gamma,\delta}(z)\Big] = \frac{z^{\frac{\mu}{\mathrm{k}}-1}}{\mathrm{k}\Gamma_{\mathrm{k}}(\mu-\eta)}\sum_{n=0}^{\infty}\frac{(\mu)_{n,\mathrm{k}}}{\Gamma_{\mathrm{k}}(\gamma n+\delta)}\mathbb{B}_{\mathrm{k}}(\eta+n\mathrm{k},\mu-\eta)\frac{z^n}{n!}, \tag{2.12}
$$

where $\gamma, \delta, \mu \in \mathbb{C}$, $\Re(p) > 0$, $\Re(q) > 0$, $\Re(\mu) > \Re(\eta) > 0$ *and* $E^{\mu}_{\mathrm{k},\gamma,\delta}(z)$ *is* k*-Mittag-Leffler function (see [6]) defined as:*

$$
E^{\mu}_{\mathrm{k},\gamma,\delta}(z) = \sum_{n=0}^{\infty}\frac{(\mu)_{n,\mathrm{k}}}{\Gamma_{\mathrm{k}}(\gamma n+\delta)}\frac{z^n}{n!}. \tag{2.13}
$$

Proof. Using (2.13), the left-hand side of (2.12) can be written as

$$
{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\Big[z^{\frac{\eta}{\mathrm{k}}-1}E^{\mu}_{\mathrm{k},\gamma,\delta}(z)\Big] = {}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\Big[z^{\frac{\eta}{\mathrm{k}}-1}\Big\{\sum_{n=0}^{\infty}\frac{(\mu)_{n,\mathrm{k}}}{\Gamma_{\mathrm{k}}(\gamma n+\delta)}\frac{z^n}{n!}\Big\}\Big].
$$

By Theorem 2, we have

$$
{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\Big[z^{\frac{\eta}{\mathrm{k}}-1}E^{\mu}_{\mathrm{k},\gamma,\delta}(z)\Big] = \sum_{n=0}^{\infty}\frac{(\mu)_{n,\mathrm{k}}}{\Gamma_{\mathrm{k}}(\gamma n+\delta)}\Big\{{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}\Big[z^{\frac{\eta}{\mathrm{k}}+n-1}\Big]\Big\}.
$$

In view of Theorem 1, we get the required proof. □

Theorem 8. *The following result holds true:*

$$
{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\Big\{z^{\frac{\eta}{\mathrm{k}}-1}\,{}_m\Psi_n\left[\begin{array}{cc}(\alpha_i, A_i)_{1,m}; & \\ & |z \\ (\beta_j, B_j)_{1,n}; & \end{array}\right]\Big\} = \frac{z^{\frac{\mu}{\mathrm{k}}-1}}{\mathrm{k}\Gamma_{\mathrm{k}}(\mu-\eta)}
$$

$$
\times\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(\alpha_i+A_i n)}{\prod_{j=1}^{n}\Gamma(\beta_j+B_j n}\mathbb{B}_{\mathrm{k}}(\eta+n\mathrm{k},\mu-\eta)\frac{z^n}{n!}, \tag{2.14}
$$

where $\Re(p) > 0$, $\Re(q) > 0$, $\Re(\mu) > \Re(\eta) > 0$ *and* ${}_m\Psi_n(z)$ *is the Fox-Wright function defined by (see [15], pages 56–58)*

$$
{}_m\Psi_n(z) = {}_m\Psi_n\left[\begin{array}{cc}(\alpha_i, A_i)_{1,m}; & \\ & |z \\ (\beta_j, B_j)_{1,n}; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(\alpha_i+A_i n)}{\prod_{j=1}^{n}\Gamma(\beta_j+B_j n}\frac{z^n}{n!}. \tag{2.15}
$$

Proof. Applying Theorem 1 and followed the same procedure used in Theorem 7, we get the desired result. □

3. Concluding remarks

Recently, many researchers have introduced various generalizations of fractional integrals and derivatives. In this line, we have established a k-fractional derivative and its various properties. If we letting $\mathrm{k} \to 1$ then all the results established in this paper will reduce to the results related to the classical Reimann-Liouville fractional derivative operator.

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Conflict of interest

The authors declare no conflict of interest.

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