Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(3): 1926–1935.
DOI:10.3934/math.2020128
Received: 09 October 2019
Accepted: 14 February 2020
Published: 18 February 2020

Research article

A class of analytic functions related to convexity and functions with bounded turning

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Abstract: In this paper, we define a new subclass k-$\mathcal{Q}(\alpha)$ of analytic functions, which generalizes the class of k-uniformly convex functions. Various interesting relationships between k-$\mathcal{Q}(\alpha)$ and the class $\mathcal{B}(\delta)$ of functions with bounded turning are derived.

Keywords: analytic functions; convex functions; uniformly convex functions; functions with bounded turning; subordination
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$. So each $f \in \mathcal{A}$ has series representation of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For two analytic functions f and g, f is said to be subordinated to g (written as $f \prec g$) if there exists an analytic function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$ such that $f(z) = (g \circ \omega)(z)$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}$ if f is univalent in Δ. A function $f \in \mathcal{S}$ is in class $\mathcal{C}$ of normalized convex functions if $f(\Delta)$ is a convex domain. For $0 \leq \alpha \leq 1$, Mocanu [23] introduced the class $\mathcal{M}_\alpha$ of functions $f \in \mathcal{A}$ such that $\frac{f(z)f'(z)}{z} \neq 0$ for all $z \in \Delta$ and

$$\Re\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right) > 0 \quad (z \in \Delta). \tag{1.2}$$

Geometrically, $f \in \mathcal{M}_\alpha$ maps the circle centred at origin onto α-convex arcs which leads to the condition (1.2). The class $\mathcal{M}_\alpha$ was studied extensively by several researchers, see [1, 10–12, 24–27] and the references cited therein.

A function $f \in \mathcal{S}$ is uniformly starlike if f maps every circular arc Γ contained in Δ with center at $\zeta \in \Delta$ onto a starlike arc with respect to $f(\zeta)$. A function $f \in \mathcal{C}$ is uniformly convex if f maps every circular arc Γ contained in Δ with center $\zeta \in \Delta$ onto a convex arc. We denote the classes of uniformly starlike and uniformly convex functions by $\mathcal{UST}$ and $\mathcal{UCV}$, respectively. For recent study on these function classes, one can refer to [7, 9, 13, 19, 20, 31].

In 1999, Kanas and Wisniowska [15] introduced the class k-$\mathcal{UCV}$ ($k \geq 0$) of k-uniformly convex functions. A function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{UCV}$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf'(z)}{f'(z)}\right| \quad (z \in \Delta). \tag{1.3}$$

In recent years, many researchers investigated interesting properties of this class and its generalizations. For more details, see [2–4, 14–18, 30, 32, 35] and references cited therein.

In 2015, Sokół and Nunokawa [33] introduced the class $\mathcal{MN}$, a function $f \in \mathcal{MN}$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \Delta).$$

In [28], it is proved that if $\Re(f') > 0$ in Δ, then f is univalent in Δ. In 1972, MacGregor [21] studied the class $\mathcal{B}$ of functions with bounded turning, a function $f \in \mathcal{B}$ if it satisfies the condition $\Re(f') > 0$ for $z \in \Delta$. A natural generalization of the class $\mathcal{B}$ is $\mathcal{B}(\delta_1)$ $(0 \leq \delta_1 < 1)$, a function $f \in \mathcal{B}(\delta_1)$ if it satisfies the condition

$$\Re(f'(z)) > \delta_1 \quad (z \in \Delta;\ 0 \leq \delta_1 < 1), \tag{1.4}$$

for details associated with the class $\mathcal{B}(\delta_1)$ (see [5, 6, 34]).

Motivated essentially by the above work, we now introduce the following class k-$\mathcal{Q}(\alpha)$ of analytic functions.

Definition 1. *Let $k \geq 0$ and $0 \leq \alpha \leq 1$. A function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{Q}(\alpha)$ if it satisfies the condition*

$$\Re\left(\frac{(zf'(z))'}{f'(z)}\right) > k\left|(1-\alpha)f'(z) + \alpha\frac{(zf'(z))'}{f'(z)} - 1\right| \quad (z \in \Delta). \tag{1.5}$$

It is worth mentioning that, for special values of parameters, one can obtain a number of well-known function classes, some of them are listed below:

1. k-$\mathcal{Q}(1) = k$-$\mathcal{UCV}$;
2. 0-$\mathcal{Q}(\alpha) = \mathcal{C}$.

In what follows, we give an example for the class k-$\mathcal{Q}(\alpha)$.

Example 1. *The function* $f(z) = \frac{z}{1-Az}$ $(A \neq 0)$ *is in the class* k-$\mathcal{Q}(\alpha)$ *with*

$$k \leq \frac{1-b^2}{b\sqrt{b(1+\alpha)\left[b(1+\alpha)+2\right]+4}} \quad (b = |A|). \tag{1.6}$$

The main purpose of this paper is to establish several interesting relationships between k-$\mathcal{Q}(\alpha)$ and the class $\mathcal{B}(\delta)$ of functions with bounded turning.

2. Preliminaries

To prove our main results, we need the following lemmas.

Lemma 1. ([8]) *Let* h *be analytic in* Δ *with* $h(0) = 1$, $\beta > 0$ *and* $0 \leq \gamma_1 < 1$. *If*

$$h(z) + \beta\frac{zh'(z)}{h(z)} \prec \frac{1+(1-2\gamma_1)z}{1-z},$$

then

$$h(z) \prec \frac{1+(1-2\delta)z}{1-z},$$

where

$$\delta = \frac{(2\gamma_1-\beta)+\sqrt{(2\gamma_1-\beta)^2+8\beta}}{4}. \tag{2.1}$$

Lemma 2. *Let* h *be analytic in* Δ *and of the form*

$$h(z) = 1 + \sum_{n=m}^{\infty} b_n z^n \quad (b_m \neq 0)$$

with $h(z) \neq 0$ *in* Δ. *If there exists a point* z_0 $(|z_0| < 1)$ *such that* $\left|\arg h(z)\right| < \frac{\pi\rho}{2}$ $(|z| < |z_0|)$ *and* $\left|\arg h(z_0)\right| = \frac{\pi\rho}{2}$ *for some* $\rho > 0$, *then* $\frac{z_0 h'(z_0)}{h(z_0)} = i\ell\rho$, *where*

$$\ell : \begin{cases} \ell \geq \frac{n}{2}\left(c+\frac{1}{c}\right) & \left(\arg h(z_0) = \frac{\pi\rho}{2}\right), \\[2ex] \ell \leq -\frac{n}{2}\left(c+\frac{1}{c}\right) & \left(\arg h(z_0) = -\frac{\pi\rho}{2}\right), \end{cases}$$

and $(h(z_0))^{1/\rho} = \pm ic$ $(c > 0)$.

This result is a generalization of the Nunokawa's lemma [29].

Lemma 3. ([37]) *Let* ε *be a positive measure on* $[0,1]$. *Let* F *be a complex-valued function defined on* $\Delta \times [0,1]$ *such that* $F(.,t)$ *is analytic in* Δ *for each* $t \in [0,1]$ *and* $F(z,.)$ *is* ε*-integrable on* $[0,1]$ *for all* $z \in \Delta$. *In addition, suppose that* $\Re(F(z,t)) > 0$, $F(-r,t)$ *is real and* $\Re(1/F(z,t)) \geq 1/F(-r,t)$ *for* $|z| \leq r < 1$ *and* $t \in [0,1]$. *If* $F(z) = \int_0^1 F(z,t)\,d\varepsilon(t)$, *then* $\Re\left(1/F(z)\right) \geq 1/F(-r)$.

Lemma 4. ([22]) *If* $-1 \leq D < C \leq 1,\ \lambda_1 > 0$ *and* $\Re(\gamma_2) \geq -\lambda_1(1-C)/(1-D)$*, then the differential equation*

$$s(z) + \frac{zs'(z)}{\lambda_1 s(z) + \gamma_2} = \frac{1+Cz}{1+Dz} \quad (z \in \Delta)$$

has a univalent solution in Δ *given by*

$$s(z) = \begin{cases} \dfrac{z^{\lambda_1+\gamma_2}(1+Dz)^{\lambda_1(C-D)/D}}{\lambda_1 \int_0^z t^{\lambda_1+\gamma_2-1}(1+Dt)^{\lambda_1(C-D)/D}dt} - \dfrac{\gamma_2}{\lambda_1} & (D \neq 0), \\ \\ \dfrac{z^{\lambda_1+\gamma_2}e^{\lambda_1 Cz}}{\lambda_1 \int_0^z t^{\lambda_1+\gamma_2-1}e^{\lambda_1 Ct}dt} - \dfrac{\gamma_2}{\lambda_1} & (D = 0). \end{cases}$$

If $r(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *satisfies the condition*

$$r(z) + \frac{zr'(z)}{\lambda_1 r(z) + \gamma_2} \prec \frac{1+Cz}{1+Dz} \quad (z \in \Delta),$$

then

$$r(z) \prec s(z) \prec \frac{1+Cz}{1+Dz},$$

and $s(z)$ *is the best dominant.*

Lemma 5. ([36, Chapter 14]) *Let* $w,\ x$ *and* $y \neq 0, -1, -2, \ldots$ *be complex numbers. Then, for* $\Re(y) > \Re(x) > 0$*, one has*

1. ${}_2G_1(w,x,y;z) = \frac{\Gamma(y)}{\Gamma(y-x)\Gamma(x)} \int_0^1 s^{x-1}(1-s)^{y-x-1}(1-sz)^{-w}\,ds;$
2. ${}_2G_1(w,x,y;z) = {}_2G_1(x,w,y;z);$
3. ${}_2G_1(w,x,y;z) = (1-z)^{-w}\ {}_2G_1\left(w, y-x, y; \frac{z}{z-1}\right).$

3. Main results

Firstly, we derive the following result.

Theorem 1. *Let* $0 \leq \alpha < 1$ *and* $k \geq \frac{1}{1-\alpha}$*. If* $f \in k\text{-}\mathcal{Q}(\alpha)$*, then* $f \in \mathcal{B}(\delta)$*, where*

$$\delta = \frac{(2\mu-\lambda) + \sqrt{(2\mu-\lambda)^2 + 8\lambda}}{4} \quad \left(\lambda = \frac{1+\alpha k}{k(1-\alpha)};\ \mu = \frac{k-\alpha k-1}{k(1-\alpha)}\right). \tag{3.1}$$

Proof. Let $f' = \hbar$, where $\hbar$ is analytic in Δ with $\hbar(0) = 1$. From inequality (1.5) which takes the form

$$\Re\left(1 + \frac{z\hbar'(z)}{\hbar(z)}\right) > k\left|(1-\alpha)\hbar(z) + \alpha\left(1 + \frac{z\hbar'(z)}{\hbar(z)}\right) - 1\right| = k\left|1 - \alpha - \hbar(z) + \alpha\hbar(z) - \alpha\frac{z\hbar'(z)}{\hbar(z)}\right|,$$

we find that

$$\Re\left(\hbar(z) + \frac{1+\alpha k}{k(1-\alpha)}\frac{z\hbar(z)}{\hbar(z)}\right) > \frac{k-\alpha k-1}{k(1-\alpha)},$$

which can be rewritten as

$$\Re\left(\hbar(z) + \lambda\frac{z\hbar(z)}{\hbar(z)}\right) > \mu \quad \left(\lambda = \frac{1+\alpha k}{k(1-\alpha)};\ \mu = \frac{k-\alpha k-1}{k(1-\alpha)}\right).$$

The above relationship can be written as the following Briot-Bouquet differential subordination

$$\hbar(z)+\lambda\frac{z\hbar'(z)}{\hbar(z)}\prec\frac{1+(1-2\mu)\,z}{1-z}.$$

Thus, by Lemma 1, we obtain

$$\hbar\prec\frac{1+(1-2\delta)\,z}{1-z},\tag{3.2}$$

where δ is given by (3.1). The relationship (3.2) implies that $f\in\mathcal{B}(\delta)$. We thus complete the proof of Theorem 1. □

Theorem 2. *Let* $0<\alpha\leq 1$, $0<\beta<1$, $c>0$, $k\geq 1$, $n\geq m+1$ $(m\in\mathbb{N})$, $|\ell|\geq\frac{n}{2}\left(c+\frac{1}{c}\right)$ *and*

$$\left|\alpha\beta\ell\pm(1-\alpha)\,c^{\beta}\sin\frac{\beta\pi}{2}\right|\geq 1.\tag{3.3}$$

If

$$f(z)=z+\sum_{n=m+1}^{\infty}a_nz^n\quad(a_{m+1}\neq 0)$$

and $f\in k\text{-}\mathcal{Q}(\alpha)$*, then* $f\in\mathcal{B}(\beta_0)$*, where*

$$\beta_0=\min\{\beta:\beta\in(0,1)\}$$

such that (3.3) *holds.*

Proof. By the assumption, we have

$$f'(z)=\hbar(z)=1+\sum_{n=m}^{\infty}c_nz^n\quad(c_m\neq 0).\tag{3.4}$$

In view of (1.5) and (3.4), we get

$$\Re\left(1+\frac{z\hbar'(z)}{\hbar(z)}\right)>k\left|(1-\alpha)\,\hbar(z)+\alpha\left(1+\frac{z\hbar'(z)}{\hbar(z)}\right)-1\right|.$$

If there exists a point $z_0\in\Delta$ such that

$$\left|\arg\hbar\,(z)\right|<\frac{\beta\pi}{2}\quad(|z|<|z_0|\,;\;0<\beta<1)$$

and

$$\left|\arg\hbar\,(z_0)\right|=\frac{\beta\pi}{2}\quad(0<\beta<1),$$

then from Lemma 2, we know that

$$\frac{z_0\hbar'\,(z_0)}{\hbar\,(z_0)}=i\ell\beta,$$

where

$$(\hbar\,(z_0))^{1/\beta}=\pm ic\quad(c>0)$$

and

$$\ell : \begin{cases} \ell \geq \frac{n}{2}\left(c + \frac{1}{c}\right) & (\arg \hbar\,(z_0) = \frac{\beta\pi}{2}), \\ \\ \ell \leq -\frac{n}{2}\left(c + \frac{1}{c}\right) & (\arg \hbar\,(z_0) = -\frac{\beta\pi}{2}). \end{cases}$$

For the case

$$\arg \hbar\,(z_0) = \frac{\beta\pi}{2},$$

we get

$$\Re\left(1 + \frac{z_0\hbar'(z_0)}{\hbar(z_0)}\right) = \Re\,(1 + i\ell\beta) = 1. \tag{3.5}$$

Moreover, we find from (3.3) that

$$\begin{aligned} & k\left|(1-\alpha)\,\hbar(z_0) + \alpha\left(1 + \frac{z_0\hbar'(z_0)}{\hbar(z_0)}\right) - 1\right| \\ =& k\left|(1-\alpha)\,(\hbar(z_0) - 1) + \alpha\frac{z_0\hbar'(z_0)}{\hbar(z_0)}\right| \\ =& k\left|(1-\alpha)\left[(\pm ic)^{\beta} - 1\right] + i\alpha\beta\ell\right| \\ =& k\sqrt{(1-\alpha)^2\left(c^{\beta}\cos\frac{\beta\pi}{2} - 1\right)^2 + \left[\alpha\beta\ell \pm (1-\alpha)\,c^{\beta}\sin\frac{\beta\pi}{2}\right]^2} \\ \geq& 1. \end{aligned} \tag{3.6}$$

By virtue of (3.5) and (3.6), we have

$$\Re\left(1 + \frac{z\hbar'(z_0)}{\hbar(z_0)}\right) \leq k\left|(1-\alpha)\,\hbar(z_0) + \alpha\left(1 + \frac{z_0\hbar(z_0)}{\hbar(z_0)}\right) - 1\right|,$$

which is a contradiction to the definition of k-$\mathcal{Q}(\alpha)$. Since $\beta_0 = \min\{\beta : \beta \in (0,1)\}$ such that (3.3) holds, we can deduce that $f \in \mathcal{B}(\beta_0)$.

By using the similar method as given above, we can prove the case

$$\arg \hbar(z_0) = -\frac{\beta\pi}{2}$$

is true. The proof of Theorem 2 is thus completed. □

Theorem 3. *If* $0 < \beta < 1$ *and* $0 \leq \nu < 1$. *If* $f \in k$-$\mathcal{Q}(\alpha)$, *then*

$$\Re(f') > \left[{}_2G_1\left(\frac{2}{\beta}\,(1-\nu)\,, 1; \frac{1}{\beta} + 1; \frac{1}{2}\right)\right]^{-1},$$

or equivalently, k-$\mathcal{Q}\,(\alpha) \subset \mathcal{B}\,(\nu_0)$, *where*

$$\nu_0 = \left[{}_2G_1\left(\frac{2}{\beta}\,(1-\mu)\,, 1; \frac{1}{\beta} + 1; \frac{1}{2}\right)\right]^{-1}.$$

Proof. For

$$w = \frac{2}{\beta}(1-\nu),\ x = \frac{1}{\beta},\ y = \frac{1}{\beta}+1,$$

we define

$$F(z) = (1+Dz)^{w}\int_0^1 t^{x-1}(1+Dtz)^{-w}\,dt = \frac{\Gamma(x)}{\Gamma(y)}\,{}_2G_1\left(1, w, y; \frac{z}{z-1}\right). \tag{3.7}$$

To prove $k\text{-}\mathcal{Q}(\alpha) \subset \mathcal{B}(\nu_0)$, it suffices to prove that

$$\inf_{|z|<1}\left\{\Re(q(z))\right\} = q(-1),$$

which need to show that

$$\Re\,(1/F(z)) \geq 1/F(-1).$$

By Lemma 3 and (3.7), it follows that

$$F(z) = \int_0^1 F(z,t)\,d\varepsilon(t),$$

where

$$F(z,t) = \frac{1-z}{1-(1-t)z} \quad (0 \leq t \leq 1),$$

and

$$d\varepsilon(t) = \frac{\Gamma(x)}{\Gamma(w)\Gamma(y-w)}t^{w-1}(1-t)^{y-w-1}\,dt,$$

which is a positive measure on $[0,1]$.

It is clear that $\Re(F(z,t)) > 0$ and $F(-r,t)$ is real for $|z| \leq r < 1$ and $t \in [0,1]$. Also

$$\Re\left(\frac{1}{F(z,t)}\right) = \Re\left(\frac{1-(1-t)z}{1-z}\right) \geq \frac{1+(1-t)r}{1+r} = \frac{1}{F(-r,t)}$$

for $|z| \leq r < 1$. Therefore, by Lemma 3, we get

$$\Re(1/F(z)) \geq 1/F(-r).$$

If we let $r \to 1^-$, it follows that

$$\Re\,(1/F(z)) \geq 1/F(-1).$$

Thus, we deduce that $k\text{-}\mathcal{Q}(\alpha) \subset \mathcal{B}(\nu_0)$. □

Theorem 4. *Let* $0 \leq \alpha < 1$ *and* $k \geq \frac{1}{1-\alpha}$. *If* $f \in k\text{-}\mathcal{Q}(\alpha)$, *then*

$$f'(z) \prec s(z) = \frac{1}{g(z)},$$

where

$$g(z) = {}_2G_1\left(\frac{2}{\lambda}, 1, \frac{1}{\lambda}+1; \frac{z}{z-1}\right) \quad \left(\lambda = \frac{1+\alpha k}{k(1-\alpha)}\right).$$

Proof. Suppose that $f' = \hbar$. From the proof of Theorem 1, we see that

$$\hbar(z) + \frac{z\hbar'(z)}{\frac{1}{\lambda}\hbar(z)} \prec \frac{1 + (1 - 2\mu)\, z}{1 - z} \prec \frac{1 + z}{1 - z} \quad \left(\lambda = \frac{1 + \alpha k}{k\,(1 - \alpha)};\ \mu = \frac{k - \alpha k - 1}{k(1 - \alpha)}\right).$$

If we set $\lambda_1 = \frac{1}{\lambda}$, $\gamma_2 = 0$, $C = 1$ and $D = -1$ in Lemma 4, then

$$\hbar(z) \prec s(z) = \frac{1}{g(z)} = \frac{z^{\frac{1}{\lambda}}\,(1 - z)^{-\frac{2}{\lambda}}}{1/\lambda \int_0^z t^{(1/\lambda)-1}\,(1 - t)^{-2/\lambda}\,dt}.$$

By putting $t = uz$, and using Lemma 5, we obtain

$$\hbar(z) \prec s(z) = \frac{1}{g(z)} = \frac{1}{\frac{1}{\lambda}\,(1 - z)^{\frac{2}{\lambda}} \int_0^1 u^{(1/\lambda)-1}\,(1 - uz)^{-2/\lambda}\,du} = \left[{}_2G_1\left(\frac{2}{\lambda}, 1, \frac{1}{\lambda} + 1; \frac{z}{z - 1}\right)\right]^{-1},$$

which is the desired result of Theorem 4. □

Acknowledgments

The present investigation was supported by the *Key Project of Education Department of Hunan Province* under Grant no. 19A097 of the P. R. China. The authors would like to thank the referees for their valuable comments and suggestions, which was essential to improve the quality of this paper.

Conflict of interest

The authors declare no conflict of interest.

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