



Research article

Fractional order SIR model with generalized incidence rate

Muhammad Altaf Khan^{1,*}, Muhammad Ismail², Saif Ullah³ and Muhammad Farhan⁴

¹ Faculty of Natural and Agricultural Sciences, University of the Free State, South Africa

² Department of mathematics, City university of Science and Information Technology, Peshawar, KP, Pakistan

³ Department of mathematics university of Peshawar, KP, Pakistan

⁴ Department of mathematics Abdul Wali Khan university, Mardan, KP, Pakistan

* **Correspondence:** Email: altafdir@gmail.com.

Abstract: The purpose of this work is to present the dynamics of a fractional SIR model with generalized incidence rate using two differential derivatives, that are the Caputo and the Atangana-Baleanu. Firstly, we formulate the proposed model in Caputo sense and carried out the basic mathematical analysis such as positivity, basic reproduction number, local and global dynamics of the disease free and endemic case. The disease free equilibrium is locally and globally asymptotically stable when the basic reproduction number is less than 1. We also show that the SIR fractional model is locally and globally asymptotically stable at endemic state when the basic reproduction number greater than unity. Further, to analyze the dynamics of Caputo model we provide some numerical results by considering various incidence rates and with different fractional order parameters. Then, we reformulate the same model using the Atangana-Baleanu operator having nonsingular and non-local kernel. We prove the existence and uniqueness of the Atangana-Baleanu SIR model using Picard-Lindelof approach. Furthermore, we present an iterative scheme of the model and provide graphical results for various values of the fractional order α . From the graphical interpretation we conclude that the Atangana-Baleanu derivative is more prominent and provides biologically more feasible results than Caputo operator.

Keywords: SIR model; Caputo operator; Atangana-Baleanu operator; stability results; Lyapunov function; Picard-Lindelof theorem; numerical solution

Mathematics Subject Classification: 34A08, 37N30

1. Introduction

Infectious diseases are massive threat for humans and also economy of the state. Proper understanding of a disease dynamics plays an important role in curtailing of the infection in a community. Implementation of a suitable strategy against the disease transmission is another challenge. Mathematical modeling approach is one of the key tools in order to handle these challenges. A number of disease models have been developed in existing literature which enables us to explore and control the spread of infectious diseases in a better way [1–3].

Most of these models are based on integer-order differential equations (IDEs). However, in the last few years, it is noticed that fractional-order differential equations (FDEs) can be used to model universal phenomena with greater degree of accuracy and its applications can be found in various fields for example engineering, economics, control theory, finance and in epidemiology [4, 5]. Fractional calculus is the generalization of classical integer-order calculus. The increasing interest of using FDEs in modeling of real world complex problems is due to its various properties which are not found in IDEs. In contrast of IDEs which local in nature, the FDEs are non-local and possess the memory effects which make it more superior because in many situations the future state of the model depends not only upon the current state but also on the previous history. These features enable FDEs to model the phenomena having not only the non-Gaussian but also for non-Markovian behavior. Further, the classical IDEs are unable to provide the information in between two different integer values. Various types of fractional-order operators were introduced in existing literature to overcome such limitations of integer-order derivative [6–8]. The applications of these fractional operators can be found in various fields [9–11]. In epidemiology, the previous experiences and history of the past states of an epidemic have a great impact to explore its dynamics in a better way. Additionally, due to the non-Markovian behavior of any real epidemic process, models with FDEs are more realistic and therefore, play an essential role in the understanding the spread and possible control of diseases. Some recent and interesting work regarding fractional analysis to the real life problem have been studied in [12–15]. In these works, the authors studied mathematical models with delay in fractional derivatives and presented the results for stability and bifurcation.

In Mathematical models, the transmission of a disease i.e., the progression at which susceptible become infected is known as incidence rate function. This function plays a key role in epidemiological models and ensure that the model provides an acceptable qualitative analysis of the disease under consideration. A variety of incidence rate functions have been implemented in the literature [16–18]. Most commonly, in disease models the bilinear incidence rate of the type bSI has been adopted, where b indicates the per unit contact rate. In 1927 Kermack and McKendrick [19] introduced a simple epidemic model using bilinear incidence rate and the assumptions that there are no births or deaths from the infection and each person of the population is initially susceptible to the disease. In [20] the authors introduced the saturated incidence after the cholera epidemic in Bari. Later on, Korobeinikov and Maini [20] studied the global analysis of the SIR and SIRS models with nonlinear incidence rate. Recently, a fractional order SIR epidemic model with nonlinear incidence rate has been developed in [21].

Keeping the above discussion in view and applicability of fractional order derivatives, in the present investigation, we develop a fractional SIR epidemic model with generalized incidence rate function of the form $f(I)S$. The fractional derivative is considered in the Caputo and Atangana-Baleanu sense. We

simulate the proposed SIR model for different values of relevant parameters and $\alpha \in (0, 1]$. Further, we provide a detail stability analysis of both disease free and endemic equilibriums of the model. The reaming sections of the manuscript are organized as: The basic definitions and relevant results are provided in section 2. Model description in Caputo case and its basic properties are presented in section 3. The stability result of the Caputo model is discussed in section in 4 and its numerical simulations and brief discussion are given in section 5. Model formulation of the ABC case and its existence and uniqueness are presented in section 6. In section 7 the iterative scheme is developed for ABC model and simulation results are presented. finally, concluding remarks are given in section 8.

2. Preliminaries

Here some background material for the Caputo and Atangana-Baleanu fractional derivatives are presented [7, 22, 23].

Definition 2.1. Consider $h \in C^n$ be a function then the fractional Caputo derivative of order α is defined as:

$${}^c D_t^\alpha(h(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^n(\mathfrak{Y})}{(t-\mathfrak{Y})^{\alpha-n+1}} d\mathfrak{Y},$$

where $\alpha \in (n-1, n)$ and $n \in \mathbb{N}$. Clearly, ${}^c D_t^\alpha(h(t))$ tends to $h'(t)$ as $\alpha \rightarrow 1$.

Definition 2.2. The corresponding fractional integral is described as:

$$I_t^\alpha(h(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\mathfrak{Y})^{\alpha-1} h(\mathfrak{Y}) d\mathfrak{Y}.$$

In the above integral Γ represents the Gamma function.

Definition 2.3. A fixed point $x = x^*$ is said to be the equilibrium point (or solution) of the Caputo fractional system defined by:

$${}^c D_t^\alpha x(t) = \mathcal{F}(t, x(t)), \quad \alpha \in (0, 1), \quad (2.1)$$

if and only if $\mathcal{F}(t, x^*) = 0$.

To present the stability analysis of non linear fractional system in the caputo sense via Lyapunov method we first recall the following necessary results from [24, 25].

Theorem 2.4. Suppose x^* be an equilibrium point for the system (2.1) and $\Upsilon \in \mathbb{R}^n$ be a domain containing x^* . Let $L : [0, \infty) \times \Upsilon$ tend to \mathbb{R} be a continuously differentiable function such that

$$M_1(x) \leq L(t, x(t)) \leq M_2(x), \quad (2.2)$$

and

$${}^c D_t^\alpha L(t, x(t)) \leq -M_3(x), \quad (2.3)$$

$\forall 0 < \alpha < 1$ and $x \in \Upsilon$. Where $M_1(x)$, $M_2(x)$ and $M_3(x)$ are continuously positive definite functions on Υ , then x^* is uniformly asymptotically stable equilibrium point for the system (2.1).

Definition 2.5. Let $\mathcal{F} \in H^1(b_1, b_2)$, $b_2 > b_1$, $\alpha \in (0, 1]$, then Atangana-Beleanu-Caputo (ABC) derivatives is defined as [7]:

$${}_{b_1}^{ABC} D_t^\alpha \mathcal{F}(t) = \frac{B(\alpha)}{1-\alpha} \int_{b_1}^t \mathcal{F}'(\zeta) E_\alpha \left[-\alpha \frac{(t-\zeta)^\alpha}{1-\alpha} \right] d\zeta. \quad (2.4)$$

The term $B(\alpha)$ represents a normalization function such that $B(0) = B(1) = 1$.

The corresponding AB integral is given as:

$${}_{b_1}^{ABC} I_t^\alpha \mathcal{F}(t) = \frac{1-\alpha}{B(\alpha)} \mathcal{F}(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{b_1}^t \mathcal{F}(\zeta) (t-\zeta)^{\alpha-1} d\zeta. \quad (2.5)$$

3. Mathematical model and discussion

To formulate the model, the total population is represented with $N(t) = S(t) + I(t) + R(t)$, where $S(t)$ represents susceptible, $I(t)$ represents number of infected and $R(t)$ represents the number of recovered individuals. The complete model that describes a system of fractional differential equation is presented as follows

$$\begin{cases} {}^C D_t^\alpha S(t) = \Lambda - dS - S f(I) + \mu_1 I + \theta R, \\ {}^C D_t^\alpha I(t) = S f(I) - (d + \mu_1 + \mu_2 + \delta) I, \\ {}^C D_t^\alpha R(t) = \mu_2 I - (d + \theta) R. \end{cases} \quad (3.1)$$

The initial condition is given by

$$S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad R(0) = R_0 \geq 0.$$

In the model (3.1), Λ is used for the recruitment rate, d denotes the natural mortality rate in all classes and δ is disease induced death rate. Parameter μ_1 shows the rate at which the infected individuals move to the susceptible class and μ_2 is the transfer rate from the infected class to the recovered class. The parameter θ is the susceptibility of recovered individuals.

To start the analysis of the system (3.1), it is necessary to present the existence, uniqueness, and non-negativity of the system solution.

To make our study more effective we have assumed that $f(I)$ is to be non-negative and continuously differentiable in the interior of \mathbb{R}_+ and further, the same hypotheses is taken in account as in [26] that is, $f(I)$ is a real locally Lipschitz function on $\mathbb{R}_+ = [0, \infty)$ satisfying

- (i) $f(0) = 0$ and $f(I) > 0$ for $I > 0$;
- (ii) $f(I)/I$ is continuous and monotonously non-increasing for $I > 0$ and $\lim_{I \rightarrow 0^+} \frac{f(I)}{I}$ exists.

It is easy to analyze from condition (ii) that

$$f(I) \leq \beta I \quad \text{for } I \in \mathbb{R}_+.$$

Then $\int_{0+}^1 1/f(u)du = 1$ and hence assumption (iii) in [26] is redundant. The incidence rate $f(I)$ is dependent on the concentration of infection. Furthermore, its most famous generalized forms satisfying the above hypothesis are given in the Table 1.

Table 1. Some famous incidence functions $f(I)$, for $\beta, a_i \geq 0, i = 0, \dots, 3$.

Incidence functions	$f(I)$	Source
Bi-linear	βI	[27]
Saturated	$\frac{\beta I}{1+a_1 I}$	[28–30]
Beddington-DeAngelis	$\frac{\beta I}{1+a_1 S+a_2 I}$	[31, 32]
Specific nonlinear	$\frac{\beta I}{1+a_1 S+a_2 I+a_3 S I}$	[33]

3.1. Existence and positivity of the solution

This section investigates the positivity of the solution of the fractional SIR model system (3.1). Let us consider

$$\mathbb{R}_+^3 = \{y \in \mathbb{R}^3 \mid y \geq 0\} \text{ and } y(t) = (S(t), I(t), R(t))^T.$$

To proceed further first we recall the generalized mean values theorem [34].

Lemma 3.1. Consider that $h(y) \in \mathbb{C}[a_1, a_2]$ and ${}^C D_t^\alpha h(y) \in (a_1, a_2]$, then

$$h(t) = h(a_1) + \frac{1}{\Gamma(\alpha)} (h)(\mathfrak{J})(t - a_1)^\alpha,$$

with $a_1 \leq \mathfrak{J} \leq t, \forall t \in (a_1, a_2]$.

Corollary 3.2. Consider that $h(y) \in \mathbb{C}[a_1, a_2]$ and ${}^C D_t^\alpha h(y) \in (a_1, a_2]$, where $\alpha \in (0, 1]$, then if

(i). ${}^C D_t^\alpha h(y) \geq 0, \forall y \in (a_1, a_2)$, then $h(y)$ is non – decreasing

(ii). ${}^C D_t^\alpha h(y) \leq 0, \forall y \in (a_1, a_2)$, then $h(y)$ is non – increasing

The following main result is presented:

Theorem 3.3. A unique solution $y(t)$ of the system (3.1) will exist and remain in \mathbb{R}_+^3 . Furthermore, the solution is positive.

Proof. The existence of the Caputo fractional SIR model can be shown with the help of theorem (3.1) in [35], while the uniqueness of the solution can be easily obtained by making use of the remark 3.2, [35], $\forall t > 0$. In order to explore the solution non-negativity, it is necessary to show that on each hyperplane bounding the positive orthant, the vector field point \mathbb{R}_+^3 . Form the system (3.1) we obtained

$${}^C D_t^\alpha S|_{S=0} = \Lambda \geq 0, \quad {}^C D_t^\alpha I|_{I=0} = S f(I) = 0, \quad {}^C D_t^\alpha R|_{R=0} = \mu_2 I \geq 0.$$

□

Therefore, by using the above theorem, we conclude that the solution will stay in \mathbb{R}_+^3 and hence biologically feasible region is constructed as:

$$\Phi = \{(S, I, R) \in \mathbb{R}_+^3 : S, I, R \geq 0.\}$$

3.2. Equilibria and reproduction number

To obtain the equilibria of the fractional SIR model we solve the system (3.1) at the steady state we have

$${}^C D_t^\alpha S = {}^C D_t^\alpha I = {}^C D_t^\alpha R = 0.$$

then the following is established:

Theorem 3.4. *The proposed model (3.1) have at most two equilibria obtained as below:*

1. The disease free equilibrium (DFE) expressed by $\mathcal{E}^0 = (S^0, I^0, R^0) = (\frac{\Lambda}{d}, 0, 0)$,
2. and the endemic equilibrium (EE) expressed by $\mathcal{E}_1 = (S^{**}, I^{**}, R^{**})$,

such that,

$$S^{**} = \frac{(d + \mu_1 + \mu_2 + \delta)I^{**}}{f(I^{**})}, \quad R^{**} = \frac{\mu_2 I^{**}}{(d + \theta)}, \quad (3.2)$$

where I^{**} is a positive solution of the function H ,

$$H(I) = d(d + \mu_1 + \mu_2 + \delta) \frac{I}{f(I)} + \left(d + \delta + \frac{d\theta}{(d + \theta)}\right)I - \Lambda. \quad (3.3)$$

By condition (ii) H is a strictly increasing function on \mathbb{R}_+ . Furthermore,

$\lim_{I \rightarrow 0^+} H(I) = \frac{d(d + \mu_1 + \mu_2 + \delta)}{\beta} - \Lambda$ and $H(\frac{\Lambda}{d}) > 0$. Therefore, H has a positive zero (and only one in this case) iff $\frac{d(d + \mu_1 + \mu_2 + \delta)}{\beta} - \Lambda < 0$ or equivalently

$$\frac{\Lambda\beta}{d(d + \mu_1 + \mu_2 + \delta)} > 1. \quad (3.4)$$

The expression for the most biologically important threshold parameter known as the basic reproduction number \mathcal{R}_0 obtained by the next generation approach which is given as below:

$$\mathcal{R}_0 = \frac{\Lambda}{d(d + \mu_1 + \mu_2 + \delta)} \frac{\partial f(0)}{\partial I}, \quad (3.5)$$

where $\frac{\partial f(0)}{\partial I}$ denote partial derivative of incidence function with respect to I at disease free state.

4. Stability of DFE

In this section we proceed to confirm the stability results in both local and global case. The corresponding Jacobian of (3.1) around the DFE is given as

$$J_{E^0} = \begin{pmatrix} -d & \mu_1 - S^0 \frac{\partial f(0)}{\partial I} & \theta \\ 0 & S^0 \frac{\partial f(0)}{\partial I} - (d + \mu_1 + \mu_2 + \delta) & 0 \\ 0 & \mu_2 & -(d + \theta) \end{pmatrix}. \quad (4.1)$$

Theorem 4.1. For any $p_1, p_2 \in \mathbb{Z}$, such that $\gcd(p_1, p_2) = 1$. Let $\alpha = (\frac{p_1}{p_2})$ and consider $M = p_2$, then the DFE of the system (3.1) is locally asymptotically stable (LAS) if $|\arg(\Pi)| > \frac{\pi}{2M}$, for all roots Π of the characteristic equation (4.2) of the matrix J_{E^0} .

$$\det(\text{diag}[\Pi^{p_1} \Pi^{p_1} \Pi^{p_1}] - J_{E^0}) = 0. \quad (4.2)$$

Proof. Evaluating the determinant (4.2), we derived the below equation in term of Π .

$$(\Pi^{p_1} + d)(\Pi^{2p_1} + a_1 \Pi^{p_1} + a_2) = 0, \quad (4.3)$$

where the coefficients are given below:

$$a_1 = (d + \theta) + (d + \delta + \mu_1 + \mu_2)(1 - \mathcal{R}_0),$$

$$a_2 = (d + \theta)(d + \delta + \mu_1 + \mu_2)(1 - \mathcal{R}_0).$$

The arguments of the roots of the equation $\Pi^{p_1} + d = 0$, are as follow:

$$\arg(\Pi_k) = \frac{\pi}{p_1} + k \frac{2\pi}{p_1} > \frac{\pi}{M} > \frac{\pi}{2M}, \quad \text{where } k = 0, 1, \dots, (p_1 - 1). \quad (4.4)$$

In similar pattern, if $\mathcal{R}_0 < 1$, then the necessary condition fulfil for all the roots of polynomial (4.3) i.e., $|\arg(\Pi)| > \frac{\pi}{2M}$. Thus the DFE is LAS for $\mathcal{R}_0 < 1$. \square

Theorem 4.2. For $\alpha \in (0, 1)$ and $\mathcal{R}_0 < 1$ then the DFE, of the fractional SIR model (3.1) is globally asymptotically stable (GAS) and unstable for $\mathcal{R}_0 > 1$.

Proof. To confirm the GAS around the DFE, we formulate the suitable Lyapunov function as:

$$\mathcal{L}(t) = I(t),$$

Calculative the time fractional derivative of the above function $\mathcal{L}(t)$ we obtain

$${}^C D_t^\alpha \mathcal{L}(t) = {}^C D_t^\alpha I.$$

Utilizing system (3.1) we get

$$\begin{aligned} {}^C D_t^\alpha \mathcal{L}(t) &= S f(I) - (d + \mu_1 + \mu_2 + \delta)I, \\ &\leq \{\beta S^0 - (d + \mu_1 + \mu_2 + \delta)\}I, \text{ as } f(I) \leq \beta I \end{aligned}$$

$$\begin{aligned} &\leq \left\{ \frac{\Lambda\beta}{d} - (d + \mu_1 + \mu_2 + \delta) \right\} I \\ &= (d + \mu_1 + \mu_2 + \delta) I \{ \mathcal{R}_0 - 1 \}. \end{aligned}$$

Clearly once $\mathcal{R}_0 < 1$ then ${}^C D_t^\alpha \mathcal{L}(t)$ is $-ve$, and hence, by result (2.4), we ensure that the DFE, \mathcal{E}^0 is GAS in Φ . \square

4.1. Global stability of endemic case

Here, we investigate the global stability of the endemic equilibrium of the fractional SIR model (3.1). At the steady state of the model (3.1) we obtained

$$\begin{cases} \Lambda = dS^{**} + S^{**}f(I^{**}) - \mu_1 I^{**} - \theta R^{**}, \\ (d + \mu_1 + \mu_2 + \delta) = \frac{S^{**}f(I^{**})}{I^{**}}, \\ (d + \theta) = \frac{\mu_2 I^{**}}{R^{**}}. \end{cases} \quad (4.5)$$

Theorem 4.3. *If $\mathcal{R}_0 > 1$ and with condition (4.10) satisfy, then the endemic equilibrium of the system (3.1) is globally asymptotically stable.*

Proof. We define the following Lyapunov function

$$\begin{aligned} \mathcal{V}(t) &= \left(S - S^{**} - S^{**} \ln \frac{S}{S^{**}} \right) + \left(I - I^{**} - I^{**} \ln \frac{I}{I^{**}} \right) \\ &\quad + \left(R - R^{**} - R^{**} \ln \frac{R}{R^{**}} \right). \end{aligned}$$

The differentiation of $\mathcal{V}(t)$ with the solution of (3.1) is

$$\begin{aligned} {}^C D_t^\alpha \mathcal{V}(t) &= \left(1 - \frac{S^{**}}{S} \right) {}^C D_t^\alpha S(t) + \left(1 - \frac{I^{**}}{I} \right) {}^C D_t^\alpha I(t) \\ &\quad + \left(1 - \frac{R^{**}}{R} \right) {}^C D_t^\alpha R(t). \end{aligned}$$

By direct calculations, we have

$$\begin{aligned} \left(1 - \frac{S^{**}}{S} \right) {}^C D_t^\alpha S(t) &= \left(1 - \frac{S^{**}}{S} \right) \left[\Lambda - dS - S f(I) + \mu_1 I + \theta R \right] \\ &= \left(1 - \frac{S^{**}}{S} \right) \left[dS^{**} + S^{**}f(I^{**}) - \mu_1 I^{**} - \theta R^{**} - dS - S f(I) + \mu_1 I + \theta R \right] \\ &= dS^{**} \left(2 - \frac{S}{S^{**}} - \frac{S^{**}}{S} \right) + S^{**}f(I^{**}) \left(1 - \frac{S f(I)}{S^{**}f(I^{**})} - \frac{S^{**}}{S} + \frac{f(I)}{f(I^{**})} \right) \\ &\quad + \theta R^{**} \left(\frac{R}{R^{**}} - 1 - \frac{S^{**}R}{S R^{**}} + \frac{S^{**}}{S} \right) + \mu_1 I^{**} \left(\frac{I}{I^{**}} - 1 - \frac{S^{**}I}{S I^{**}} + \frac{S^{**}}{S} \right), \end{aligned} \quad (4.6)$$

$$\begin{aligned}
\left(1 - \frac{I^{**}}{I}\right)^c D_t^\alpha I(t) &= \left(1 - \frac{I^{**}}{I}\right) [Sf(I) - (d + \mu_1 + \mu_2 + \delta)I] \\
&= \left(1 - \frac{I^{**}}{I}\right) \left[Sf(I) - S^{**}f(I^{**})\frac{I}{I^{**}} \right] \\
&= Sf(I) - S^{**}f(I^{**})\frac{I}{I^{**}} - Sf(I)\frac{I^{**}}{I} + S^{**}f(I^{**}), \tag{4.7}
\end{aligned}$$

$$\begin{aligned}
\left(1 - \frac{R^{**}}{R}\right)^c D_t^\alpha R(t) &= \left(1 - \frac{R^{**}}{R}\right) (\mu_2 I - (d + \theta)R) \\
&= \left(1 - \frac{R^{**}}{R}\right) \left(\mu_2 I - \frac{\mu_2 I^{**}}{R^{**}} R \right) \\
&= \mu_2 I^{**} \left(\frac{I}{I^{**}} - \frac{R}{R^{**}} - \frac{R^{**}I}{RI^{**}} + 1 \right). \tag{4.8}
\end{aligned}$$

After some arrangement we have

$$\begin{aligned}
{}^c D_t^\alpha \mathcal{V}(t) &= dS^{**} \left(2 - \frac{S}{S^{**}} - \frac{S^{**}}{S} \right) + \mu_1 I^{**} \left(\frac{I}{I^{**}} - 1 - \frac{S^{**}}{S} \left(\frac{I}{I^{**}} - 1 \right) \right) + \theta R^{**} \left(\frac{R}{R^{**}} - 1 \right. \\
&\quad \left. - \frac{S^{**}}{S} \left(\frac{R}{R^{**}} - 1 \right) \right) + \mu_2 I^{**} \left(1 - \frac{R}{R^{**}} - \frac{I}{I^{**}} \left(\frac{R^{**}}{R} - 1 \right) \right) \\
&\quad + S^{**} f(I^{**}) \left(2 - \frac{S^{**}}{S} - \frac{I}{I^{**}} - \frac{f(I)}{f(I^{**})} \left(\frac{SI^{**}}{S^{**}I} - 1 \right) \right). \tag{4.9}
\end{aligned}$$

It follows from the property arithmetic mean and we have

$$\left(2 - \frac{S}{S^*} - \frac{S^*}{S} \right) \leq 0,$$

and if

$$\begin{aligned}
\left(\frac{I}{I^{**}} - 1 - \frac{S^{**}}{S} \left(\frac{I}{I^{**}} - 1 \right) \right) &\leq 0, \\
\left(\frac{R}{R^{**}} - 1 - \frac{S^{**}}{S} \left(\frac{R}{R^{**}} - 1 \right) \right) &\leq 0, \\
\left(1 - \frac{R}{R^*} - \frac{I}{I^*} \left(\frac{R^*}{R} - 1 \right) \right) &\leq 0, \\
\left(2 - \frac{S^{**}}{S} - \frac{I}{I^{**}} - \frac{f(I)}{f(I^{**})} \left(\frac{SI^{**}}{S^{**}I} - 1 \right) \right) &\leq 0, \tag{4.10}
\end{aligned}$$

then it ensures that the model given by (3.1) at the EE case is GAS when $\mathcal{R}_0 > 1$ and with the condition (4.10) satisfied. \square

5. Numerical results for the Caputo model

In the current section, we investigate the numerical results of the Caputo SIRS model given by system (3.1) with general incidence rate. The model in each case is solved numerically using

predictor corrector of molten type technique. In order to perform the numerical experiment and analyze the importance of incidence function on disease dynamics we assign different incidence rate to the generalized incidence function f and simulate the model with various fractional values of α . The parameter values considered in the numerical solution for the fractional SIRS model are depicted in each Figure caption. Initially, we consider the incidence rate f in the form as a bilinear incidence rate i.e., $f(S, I) = \beta SI$ and obtain the graphical results in sub-plots (a-c) of Figure 1, while in sub-plot (d), we depicted the phase portrait analysis of the model variables. We can observed that the by decreasing the fractional order α the population in susceptible class is increasing whereas the population of infected and recovered classes is decreasing sharply. The graphical dynamics of the model (3.1) for various values of α with non-linear incidence function is shown in sub-plots (a-c) of Figure 2. The corresponding phase portrait analysis in this case is depicted in sub-plot (d) of Figure 2. The influence of parameter m (the half saturation constant) for different values of $\alpha = 1, 0.8, 0.5$, is shown in Figures 3, 4 and 5 respectively. We can see that population in the infected class is decreasing sharply with the increasing values of m . The influence of m is more significant biologically for fractional order of ABC operator rather than integer case. Finally, the dynamics of the Caputo model (3.1) with Beddington-DeAngelis incidence function with various fractional order α of the derivative are depicted in Figure 6. The corresponding phase portrait analysis in this case is presented in sub-plot (d) of Figure 6. These graphical interpretations explore the importance of the fractional derivative to the proposed model where one can see the behavior of the model for any non-integer case.

6. Model formulation in Atangana-Baleanu framework

In this part of the paper we reconstruct the proposed model using the generalized fractional operator known as Atangana-Baleanu-Caputo (ABC) derivative in order to explore the dynamics of the model in more better way. The model in ABC sense is given by the following system

$$\begin{cases} {}_0^{ABC}D_t^\alpha S(t) = \Lambda - dS - S f(I) + \mu_1 I + \theta R, \\ {}_0^{ABC}D_t^\alpha I(t) = S f(I) - (d + \mu_1 + \mu_2 + \delta)I, \\ {}_0^{ABC}D_t^\alpha R(t) = \mu_2 I - (d + \theta)R, \end{cases} \quad (6.1)$$

with some appropriate nonnegative inial conditions. Before to analyze the dynamics of fractional model (6.1), first we study the existence and uniqueness od the model in the following theorem.

6.1. Existence and uniqueness of the solution

We apply the fixed point theory in order to explore the existence and uniqueness of the solution of fractional model (6.1). To do this we firstly re-write the system (6.1) in the following form:

$$\begin{cases} {}_0^{ABC}D_t^\alpha u(t) = G(t, u(t)), \\ u(0) = u_0, \quad 0 < t < \mathcal{T} < \infty. \end{cases} \quad (6.2)$$

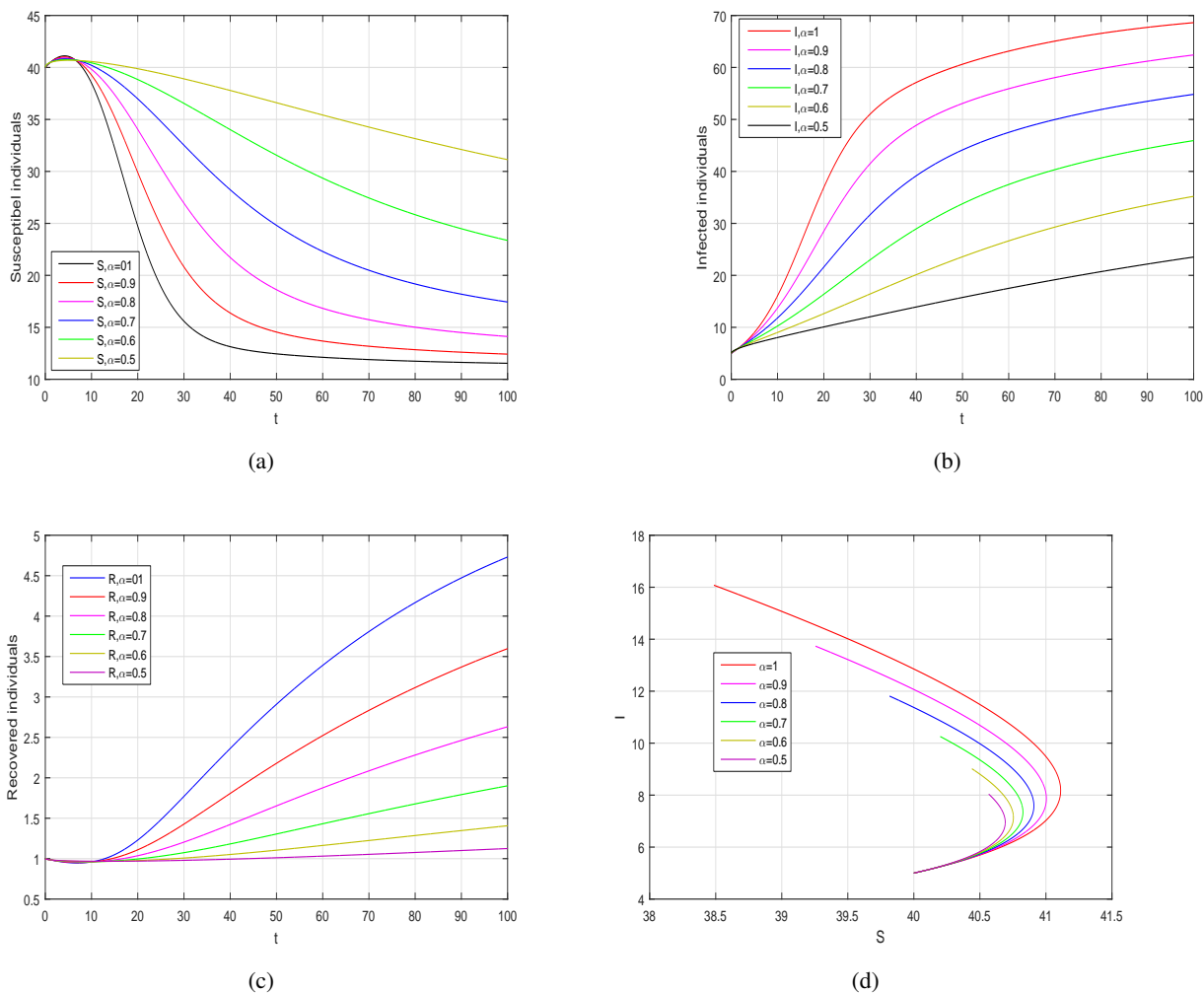


Figure 1. Simulation of the Caputo fractional SIR model (3.1) for $f(S, I) = \beta SI$ with $d = 0.02, \beta = 0.004, \mu_1 = 0.02, \mu_2 = 0.002, \delta = 0.003, \theta = 0.004, \Lambda = 02$.

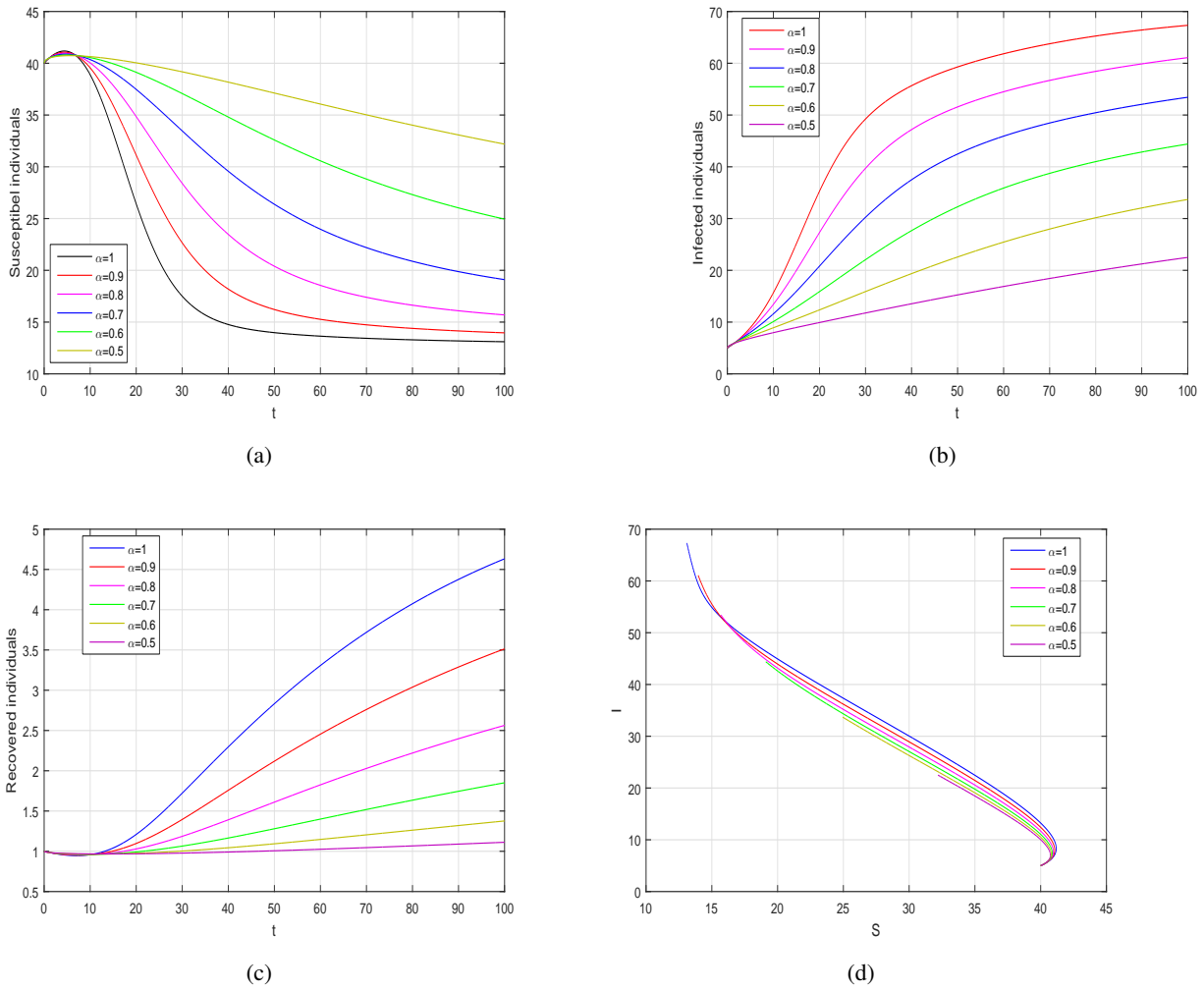


Figure 2. Simulation of the Caputo fractional SIR model (3.1) for $f(I) = \beta I / (1 + mI)$ with $d = 0.02, \beta = 0.004, \mu_1 = 0.02, \mu_2 = 0.002, \delta = 0.003, \theta = 0.004, \Lambda = 02, m = 0.002$.

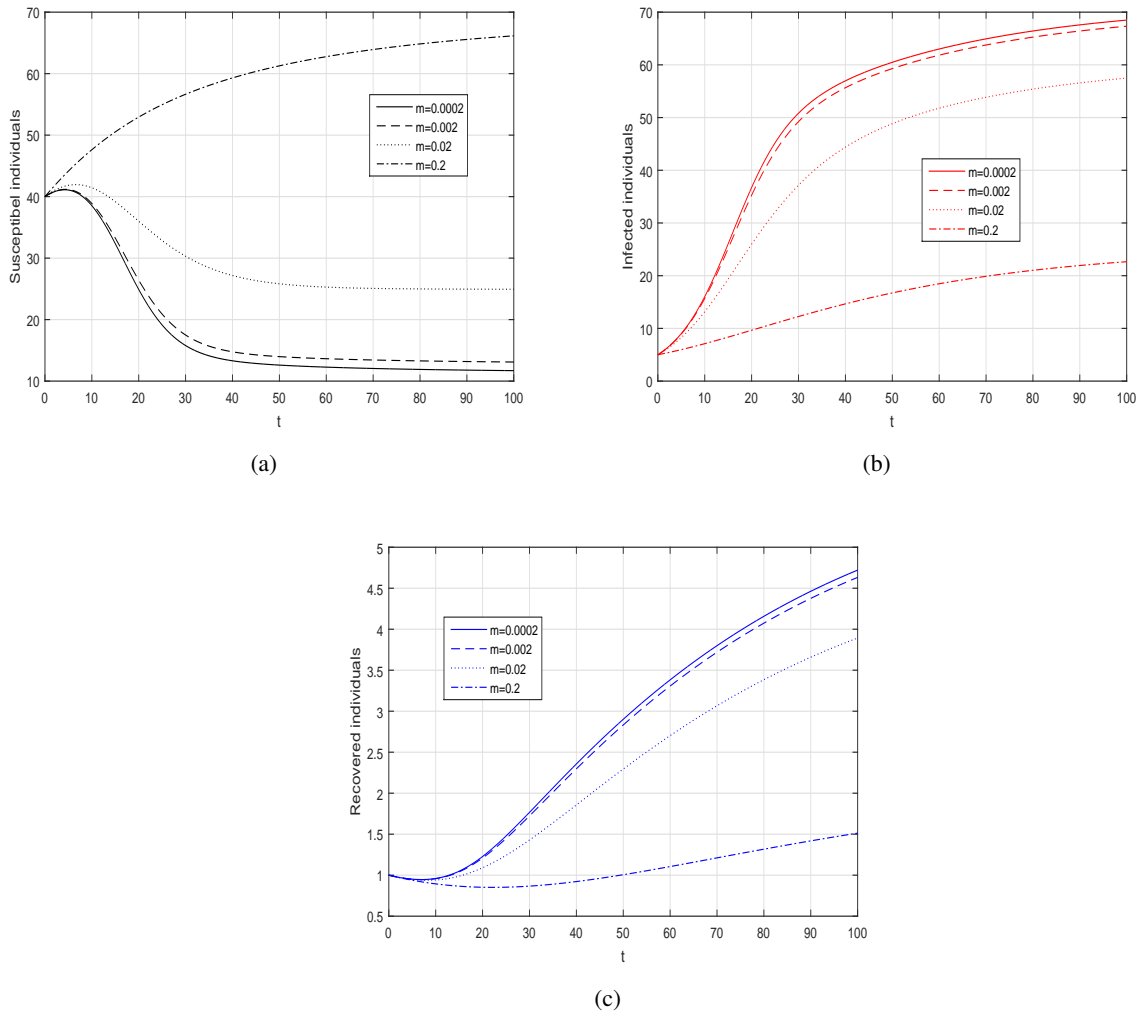


Figure 3. Simulation of the Caputo fractional SIR model (3.1) for $f(I) = \beta I / (1 + mI)$ with $d = 0.02$, $\beta = 0.004$, $\mu_1 = 0.02$, $\mu_2 = 0.002$, $\delta = 0.003$, $\theta = 0.004$, $\Lambda = 02$, $m = 0.0002, 0.002, 0.02, 0.2$.

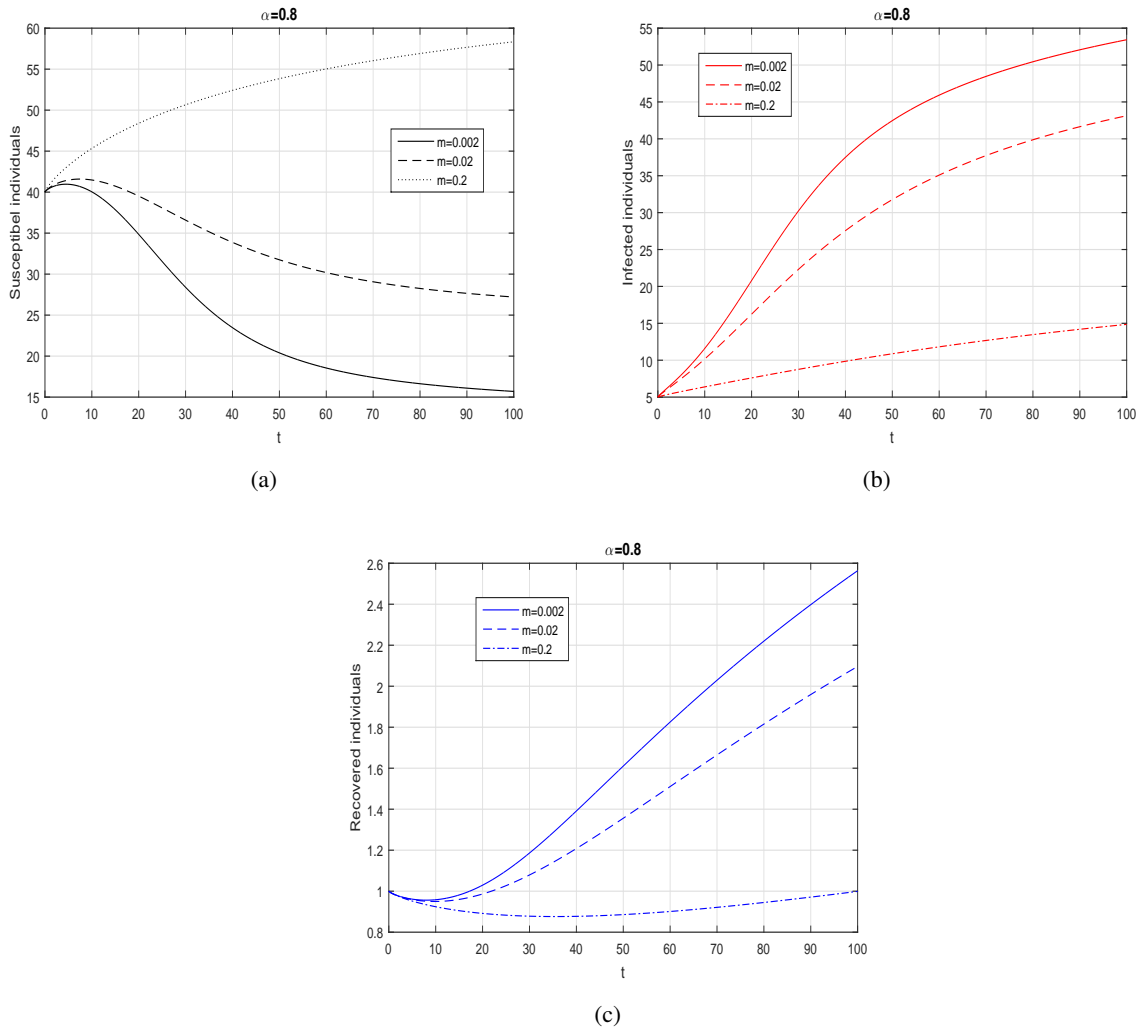


Figure 4. Simulation of the Caputo fractional SIR model (3.1) for $f(I) = \beta I / (1 + mI)$ with $d = 0.02$, $\beta = 0.004$, $\mu_1 = 0.02$, $\mu_2 = 0.002$, $\delta = 0.003$, $\theta = 0.004$, $\Lambda = 02$, $m = 0.0002, 0.002, 0.02, 0.2$.

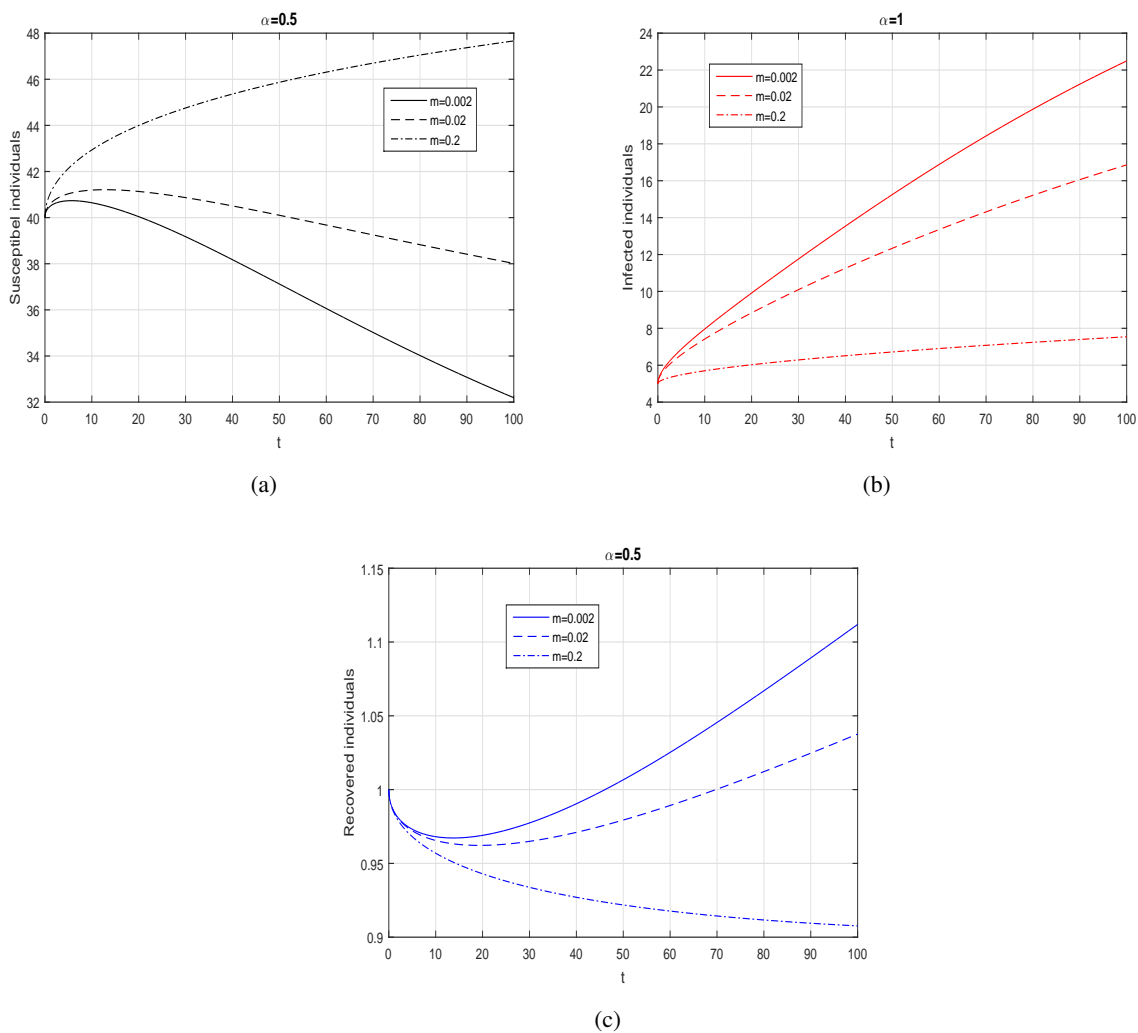


Figure 5. Simulation of the Caputo fractional SIR model (3.1) for $f(I) = \beta I / (1 + mI)$ with $d = 0.02$, $\beta = 0.004$, $\mu_1 = 0.02$, $\mu_2 = 0.002$, $\delta = 0.003$, $\theta = 0.004$, $\Lambda = 02$, $m = 0.0002, 0.002, 0.02, 0.2$.

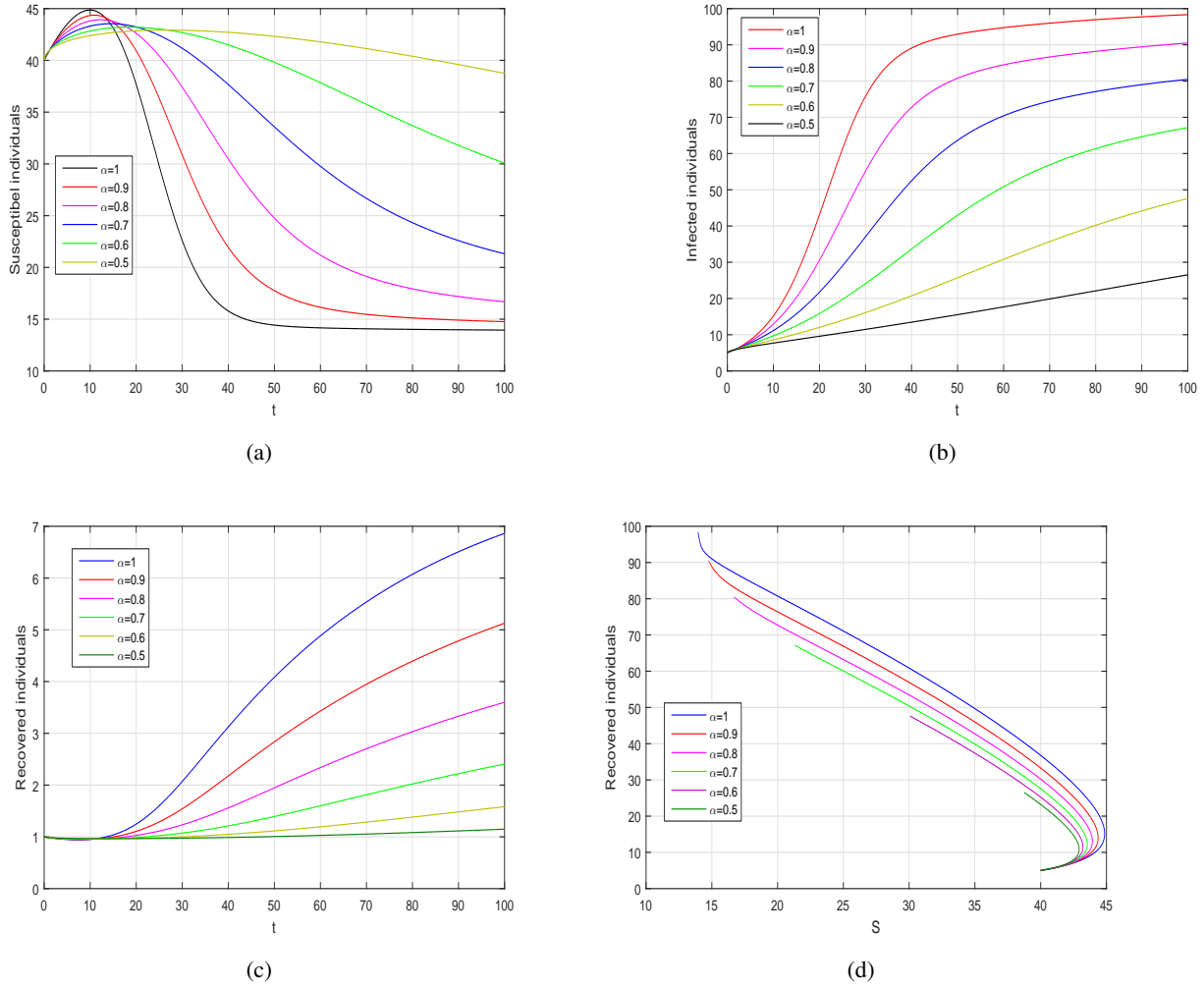


Figure 6. Simulation of the Caputo fractional SIR model (3.1) for $f(I) = \beta I / (1 + m_1 S + m_2 I)$ with $d = 0.02$, $\beta = 0.004$, $\mu_1 = 0.02$, $\mu_2 = 0.002$, $\delta = 0.003$, $\theta = 0.004$, $\Lambda = 02$, $m_1 = 0.002$, $m_2 = 0.002$.

The vector $u(t) = (S, I, R)$ and G in the system (6.2) represent the state variables and a continuous vector function respectively defined as follows:

$$G = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} \Lambda - dS - Sf(I) + \mu_1 I + \theta R, \\ Sf(I) - (d + \mu_1 + \mu_2 + \delta)I, \\ \mu_2 I - (d + \theta)R \end{pmatrix},$$

with the initial condition by $u_0(t) = (S(0), I(0), R(0))$. Furthermore, the G satisfies the Lipschitz condition given as:

$$\|G(t, u_1(t)) - G(t, u_2(t))\| \leq \mathcal{M}\|u_1(t) - u_2(t)\|. \quad (6.3)$$

To confirm the existence and uniqueness for the solution of system (6.1) with fractional order, we proceed as in the following theorem.

Theorem 6.1. *The exists a unique solution of the SIR epidemic model (6.1) if the below condition holds:*

$$\frac{(1 - \alpha)}{ABC(\alpha)}\mathcal{M} + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\mathcal{M}\mathcal{T}_{max}^\alpha < 1. \quad (6.4)$$

Proof. On applying the ABC fractional integral to both sides of the model (6.2), we get the non-linear Volterra integral equation:

$$u(t) = u_0 + \frac{1 - \alpha}{ABC(\alpha)}G(t, u(t)) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} G(\zeta, u(\zeta)) d\zeta. \quad (6.5)$$

Consider $J = (0, \mathcal{T})$, and $\Xi : C(J, \mathbb{R}^3) \rightarrow C(J, \mathbb{R}^3)$ stated as

$$\Xi[u(t)] = u_0 + \frac{1 - \alpha}{ABC(\alpha)}G(t, u(t)) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} G(\zeta, u(\zeta)) d\zeta. \quad (6.6)$$

Eq (6.5) becomes:

$$u(t) = \Xi[u(t)]. \quad (6.7)$$

The supremum norm on u denoted by $\|\cdot\|_J$ and is defined as:

$$\|u(t)\|_J = \sup_{t \in J} \|u(t)\|, \quad u(t) \in C. \quad (6.8)$$

Clearly, $C(J, \mathbb{R}^3)$ with the norm $\|\cdot\|_J$ construct a Banach space. Further, the following inequality fulfil

$$\left\| \int_0^t \mathcal{K}(t, \zeta) u(\zeta) d\zeta \right\| \leq T \|\mathcal{K}(t, \zeta)\|_J \|u(t)\|_J, \quad (6.9)$$

with $w(t) \in C(J, \mathbb{R}^3)$, $\mathcal{H}(t, \zeta) \in C(J^2, \mathbb{R})$ such that

$$\|\mathcal{K}(t, \zeta)\|_J = \sup_{t, \zeta \in J} |\mathcal{K}(t, \zeta)|. \quad (6.10)$$

Applying result in (6.7), we have

$$\begin{aligned} \|\Xi[u_1(t)] - \Xi[u_2(t)]\|_J &\leq \left\| \frac{(1-\alpha)}{ABC(\alpha)}(G(t, u_1(t)) - G(t, u_2(t))) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \times \right. \\ &\quad \left. \int_0^t (t-\zeta)^{\alpha-1}(G(\zeta, u_1(\zeta)) - G(\zeta, u_2(\zeta)))d\zeta \right\|_J. \end{aligned} \quad (6.11)$$

Furthermore, taking in account equations (6.3), (6.9) and triangular inequality, we have:

$$\|\Xi[u_1(t)] - \Xi[u_2(t)]\|_J \leq \left(\frac{(1-\alpha)\mathcal{M}}{ABC(\alpha)} + \frac{\alpha}{\Gamma(\alpha)ABC(\alpha)}\mathcal{M}\mathcal{T}_{max}^\alpha \right) \|u_1(t) - u_2(t)\|_J. \quad (6.12)$$

Finally, we derived the following expression

$$\|\Xi[u_1(t)] - \Xi[u_2(t)]\|_J \leq F \|u_1(t) - u_2(t)\|_J, \quad (6.13)$$

where,

$$F = \frac{(1-\alpha)\mathcal{K}}{ABC(\alpha)} + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\mathcal{K}\mathcal{T}_{max}^\alpha.$$

The operator Ξ is known as a contraction if it follows the condition in equation (6.4). Hence, the solution of the model (6.2) will also has a unique solution. \square

7. Numerical scheme of the ABC model

Here firstly, we obtain the iterative scheme for the system (6.1). The approach developed in [36] is applied in order to approximate the ABC integral. To proceed further, using the fundamental theorem on the system (6.2), we obtained

$$u(t) - u(0) = \frac{(1-\alpha)}{ABC(\alpha)}G(t, u(t)) + \frac{\alpha}{ABC(\alpha) \times \Gamma(\alpha)} \int_0^t G(\zeta, u(\zeta))(t-\zeta)^{\alpha-1}d\zeta. \quad (7.1)$$

At $t = t_{m+1}$, $m = 0, 1, 2, \dots$, we have

$$\begin{aligned} u(t_{m+1}) - u(0) &= \frac{1-\alpha}{ABC(\alpha)}G(t_m, u(t_m)) + \\ &\quad \frac{\alpha}{\Gamma(\alpha)} \times ABC(\alpha) \int_0^{t_{m+1}} G(\zeta, u(\zeta))(t_{m+1} - \zeta)^{\alpha-1}d\zeta, \\ &= \frac{1-\alpha}{ABC(\alpha)}G(t_m, u(t_m)) + \\ &\quad \frac{\alpha}{ABC(\alpha) \times \Gamma(\alpha)} \sum_{j=0}^m \int_{t_j}^{t_{j+1}} G(\zeta, u(\zeta))(t_{m+1} - \zeta)^{\alpha-1}d\zeta. \end{aligned} \quad (7.2)$$

Utilizing the interpolation polynomial approach, the function $G(\zeta, u(\zeta))$ can be approximated over $[t_j, t_{j+1}]$:

$$G(\zeta, u(\zeta)) \cong P_k(\zeta) = \frac{G(t_j, u(t_j))}{h}(\zeta - t_{j-1}) - \frac{G(t_{j-1}, u(t_{j-1}))}{h}(\zeta - t_j). \quad (7.3)$$

Equation (7.2) becomes:

$$\begin{aligned}
 u(t_{m+1}) = & u(0) + \frac{1-\alpha}{ABC(\alpha)} G(t_m, u(t_m)) + \\
 & \frac{\alpha}{ABC(\alpha) \times \Gamma(\alpha)} \sum_{j=0}^m \left(\frac{G(t_j, u(t_j))}{h} \int_{t_j}^{t_{j+1}} (\zeta - t_{j-1})(t_{m+1} - \zeta)^{\alpha-1} d\zeta \right. \\
 & \left. - \frac{G(t_{j-1}, u(t_{j-1}))}{h} \int_{t_j}^{t_{j+1}} (\zeta - t_j)(t_{m+1} - \zeta)^{\alpha-1} d\zeta \right). \tag{7.4}
 \end{aligned}$$

On further simplifications we finally arrived the approximate solution as below:

$$\begin{aligned}
 u(t_{m+1}) = & u(t_0) + \frac{1-\alpha}{ABC(\alpha)} G(t_m, u(t_m)) + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^m \\
 & \left(\frac{h^\alpha G(t_j, u(t_j))}{\Gamma(\alpha+2)} ((m-j+1)^\alpha (m+\alpha-j+2) - (m-j)^\alpha (m+2\alpha-j+2)) \right. \\
 & \left. - \frac{h^\alpha G(t_{j-1}, u(t_{j-1}))}{\Gamma(\alpha+2)} ((m-j+1)^{\alpha+1} - (m-j)^\alpha (m+\alpha-j+1)) \right). \tag{7.5}
 \end{aligned}$$

Hence, for the presented fractional system, we have the following recursive formulae:

$$\begin{aligned}
 S(t_{m+1}) = & S(t_0) + \frac{1-\alpha}{ABC(\alpha)} G_1(t_m, u(t_m)) + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^m \\
 & \left(\frac{h^\alpha G_1(t_j, u(t_j))}{\Gamma(\alpha+2)} ((m-j+1)^\alpha (m+\alpha-j+2) - (m-j)^\alpha (m+2\alpha-j+2)) \right. \\
 & \left. - \frac{h^\alpha G_1(t_{j-1}, u(t_{j-1}))}{\alpha(\alpha+2)} ((m-j+1)^{\alpha+1} - (m-j)^\alpha (m+\alpha-j+1)) \right), \\
 I(t_{m+1}) = & I(t_0) + \frac{1-\alpha}{ABC(\alpha)} G_2(t_m, u(t_m)) + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^m \\
 & \left(\frac{h^\alpha G_2(t_j, u(t_j))}{\alpha(\alpha+2)} ((m+1-j)^\alpha (m+\alpha-j+2) - (m-j)^\alpha (m+2\alpha-j+2)) \right. \\
 & \left. - \frac{h^\alpha G_2(t_{j-1}, u(t_{j-1}))}{\alpha(\alpha+2)} ((m-j+1)^{\alpha+1} - (m-j)^\alpha (m+\alpha-j+1)) \right), \\
 R(t_{m+1}) = & R(t_0) + \frac{1-\alpha}{ABC(\alpha)} G_3(t_m, u(t_m)) + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^m \\
 & \left(\frac{h^\alpha G_3(t_j, u(t_j))}{\alpha(\alpha+2)} ((m-j+1)^\alpha (m-j+2+\alpha) - (m-j)^\alpha (m+2\alpha-j+2)) \right. \\
 & \left. - \frac{h^\alpha G_3(t_{j-1}, u(t_{j-1}))}{\alpha(\alpha+2)} ((m-j+1)^{\alpha+1} - (m-j)^\alpha (m+\alpha-j+1)) \right). \tag{7.6}
 \end{aligned}$$

7.1. Simulation

Based on numerical scheme derived in (7.6), we present the graphical results of the proposed ABC fractional model (6.1) in Figures 7–9. To analyze the influence of fractional order and incidence

function on the disease dynamics and control we simulate the ABC SIRS model (6.1) with different incidence rates as in Caputo case. In Figure 7, we have shown the dynamics of the model by taking bilinear incidence rate in account. The graphical behavior of the model with nonlinear saturated incidence rate and for different values of $\alpha \in (0, 1]$ is analyzed in Figure 8. Finally, the graphical results of fractional SIRS model by incorporating Beddington-DeAngelis type incidence function for various values of α are depicted in Figure 9.

From the graphical analysis, we observed that the population in infected class decrease significantly by decreasing the fractional order and the disease endemic state move to disease free state in each case. Further, from the graphical results of ABC model reveal that this operator is more helpful to explore the model dynamics and provide biologically more feasible results in compression with Caputo case.

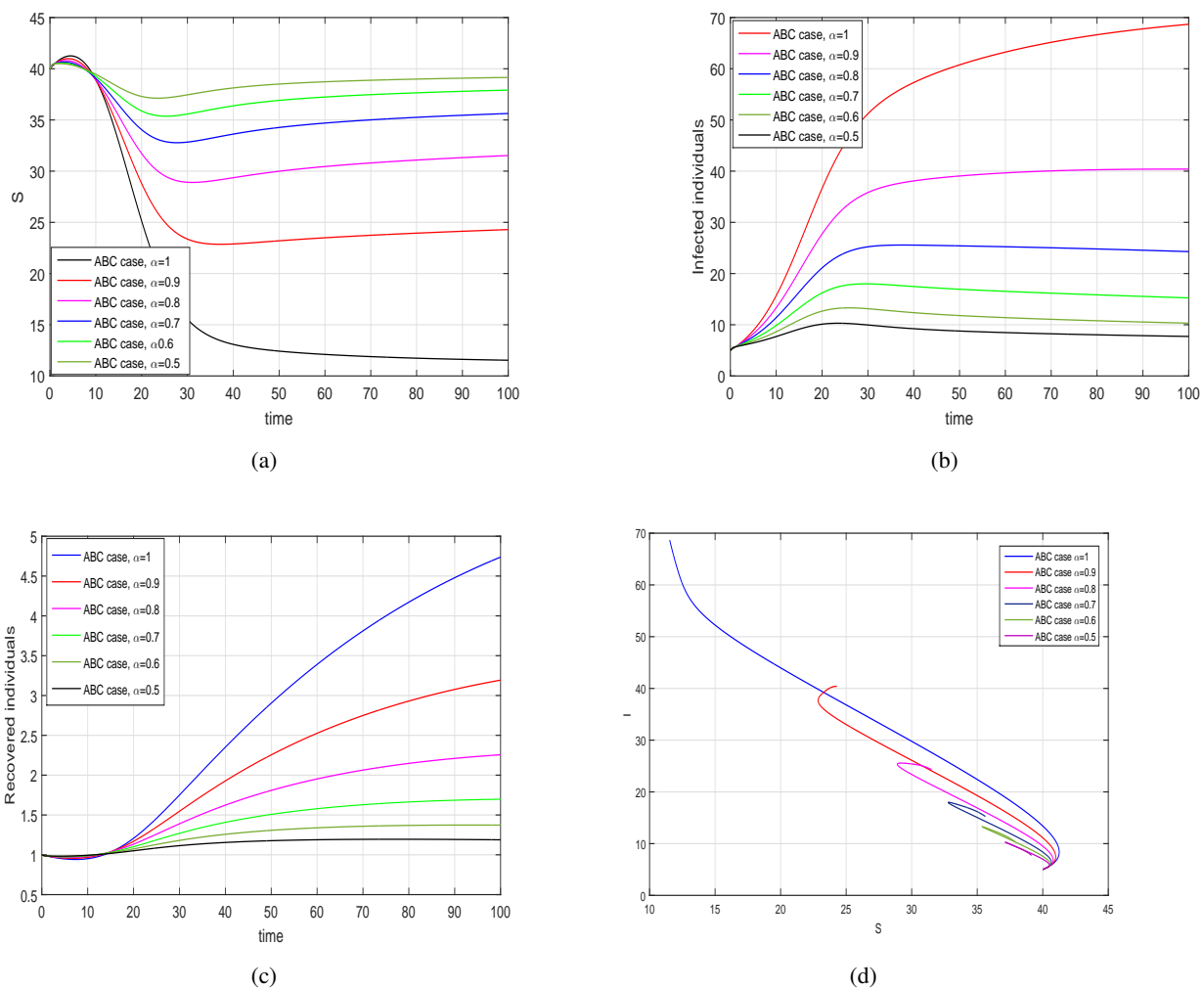


Figure 7. Simulation of the ABC fractional SIR model (6.1) for $f(S, I) = \beta SI$ with $d = 0.02$, $\beta = 0.004$, $\mu_1 = 0.02$, $\mu_2 = 0.002$, $\delta = 0.003$, $\theta = 0.004$, $\Lambda = 02$.

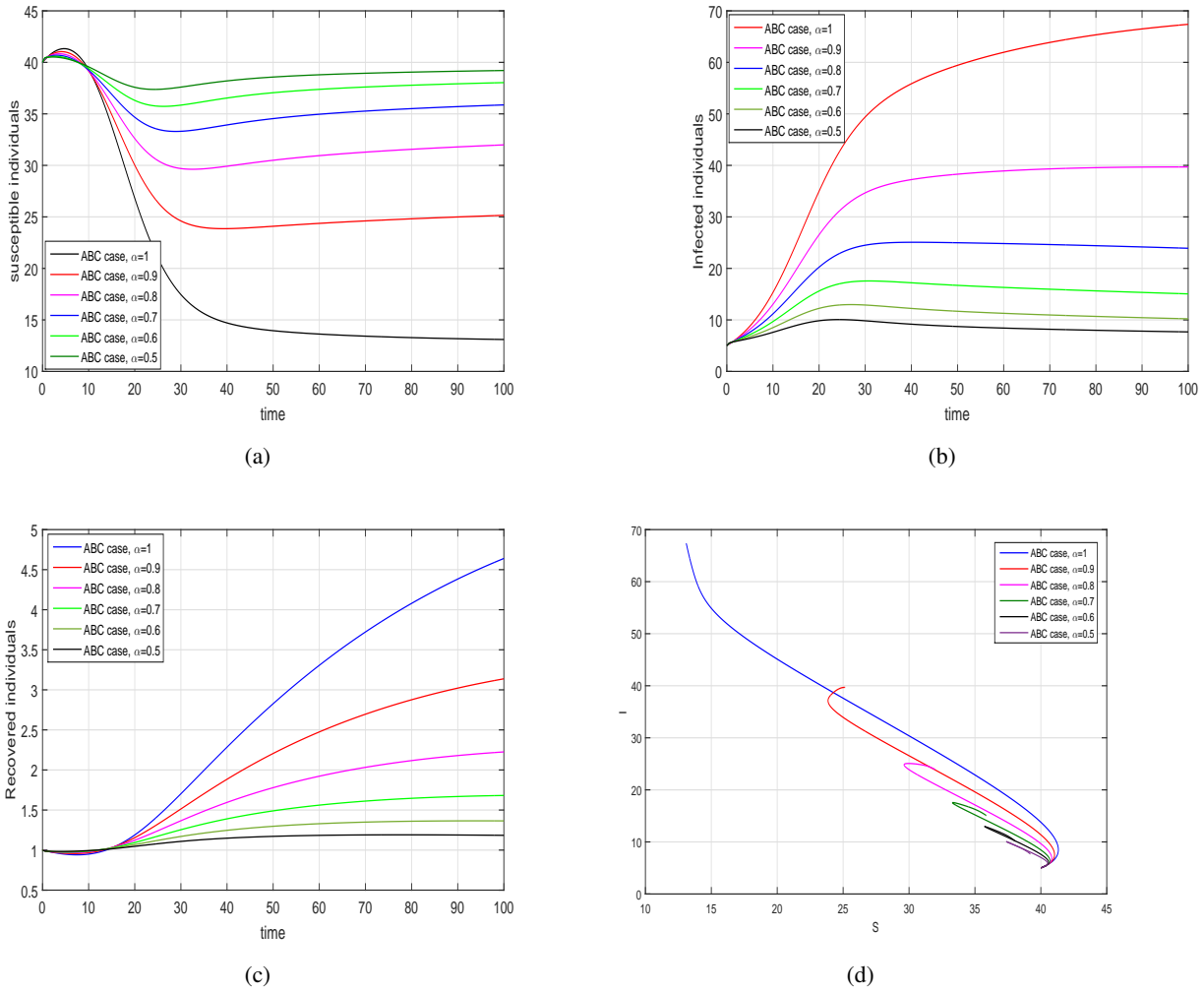


Figure 8. Simulation of the fractional ABC SIR model (6.1) for $f(I) = \beta I / (1 + mI)$ with $d = 0.02, \beta = 0.004, \mu_1 = 0.02, \mu_2 = 0.002, \delta = 0.003, \theta = 0.004, \Lambda = 02, m = 0.002$.

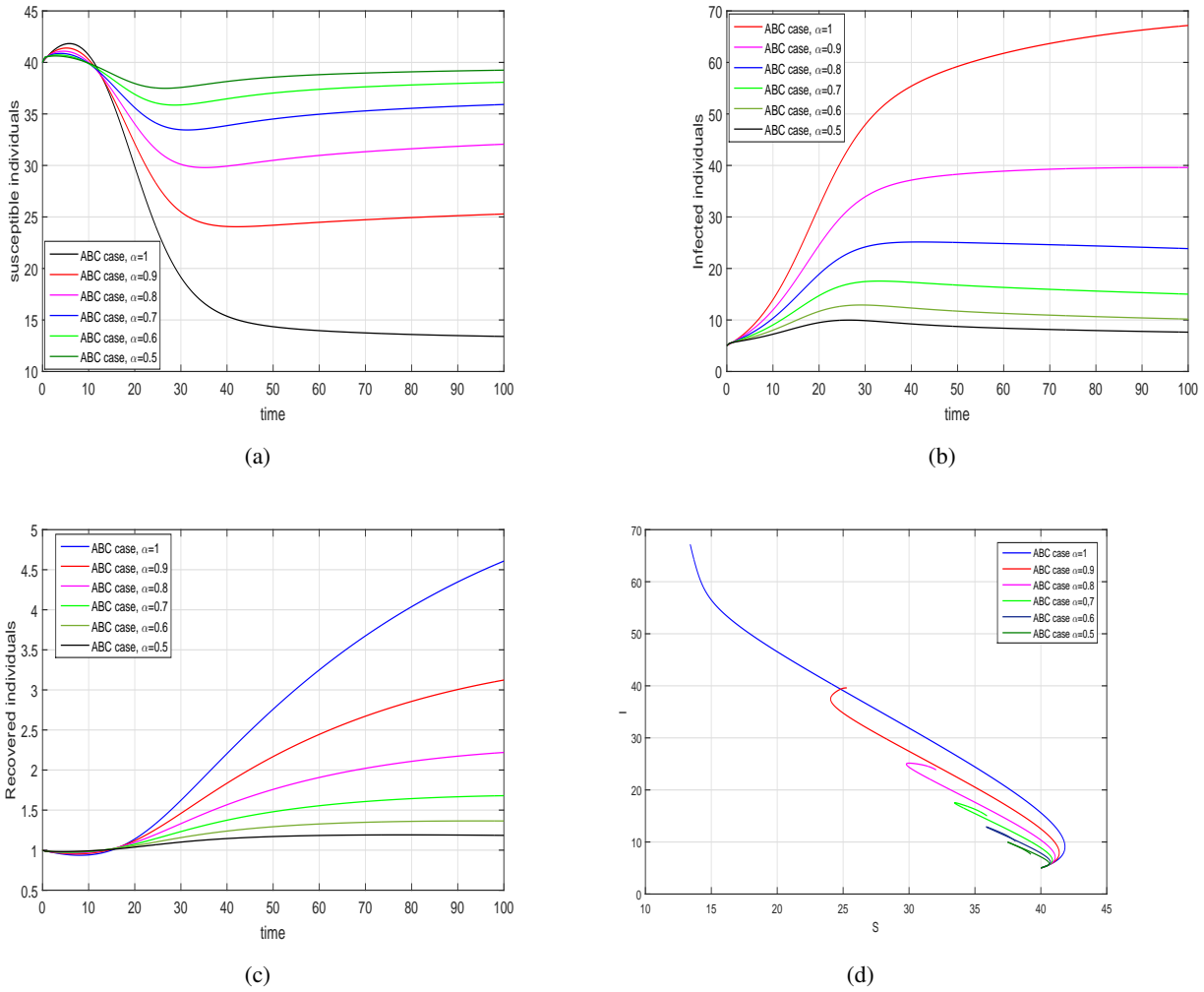


Figure 9. Simulation of the ABC fractional SIR model (6.1) for $f(I) = \beta I / (1 + m_1 S + m_2 I)$ with $d = 0.02$, $\beta = 0.004$, $\mu_1 = 0.02$, $\mu_2 = 0.002$, $\delta = 0.003$, $\theta = 0.004$, $\Lambda = 02$, $m_1 = 0.002$, $m_2 = 0.002$.

8. Conclusion

In this paper, we analyzed the dynamics of a fractional SIR model with nonlinear generalized incidence rate. Initially, we formulated a general SIR model by apply the Caputo derivative. The local and global stability for the fractional SIR model are obtained and discussed in detailed. It is shown that the model at the disease free case is locally asymptotically stable when the basic reproduction number less than 1. Further, we obtain that the fractional SIR model at the disease free case is globally asymptotically stable for the case of basic reproduction number less than 1. We used fractional Lyapunov function for the proof of the global stability of the disease free case and also we proved by defining the fractional Lyapunov function for the proof of the global stability of the endemic equilibrium when the basic reproduction number greater than 1. Numerical results with various values of the fractional order parameter are obtained and discussed in detailed by choosing some different type of incidence functions. Since the Caputo fractional operator has singular kernel and can not explore the crossover behavior of some biological phenomena. Therefore, we further reformulated the proposed model using ABC derivative having nonsingular and nonlocal kernel. The existence and uniqueness of the ABC model is proved via Picard-Landelof approach. An efficient numerical scheme is used to carried out the iterative solution of the model and perform various simulation results r to analyzed the role of different incidence function in disease dynamic and control. These results indicates that the fractional order model in ABC case is more beneficial and important than that of the ordinary integer order model.

Conflict of interest

All authors declare no conflict of interest in this paper.

References

1. S. Ullah, M. A. Khan, J. F. Gez-Aguilar, *Mathematical formulation of hepatitis B virus with optimal control analysis*, Optim. Contr. Appl. Meth., **40** (2019), 529–544.
2. E. Bonyah, M. A. Khan, K. O. Okosun, et al. *A theoretical model for Zika virus transmission*, PLoS ONE, **12** (2017), 1–26.
3. S. Ullah, M. A. Khan, M. Farooq, et al. *Modeling and analysis of Tuberculosis (TB) in Khyber Pakhtunkhwa, Pakistan*, Math. Comput. Simulat., **165** (2019), 181–199.
4. D. P. Ahokposi, A. Atangana and D. P. Vermeulen, *Modelling groundwater fractal flow with fractional differentiation via Mittag effler law*, Eur. Phy. J. Plus, **132** (2017), 165.
5. M. A. Khan, Y. Khan and S. Islam, *Complex dynamics of an SEIR epidemic model with saturated incidence rate and treatment*, Phy. A., **493** (2018), 210–227.
6. M. Caputo and M. Fabrizio, *A New Definition of Fractional Derivative without Singular Kernel*, Progr. Fract. Differ. Appl., **1** (2015), 73–85.
7. A. Atangana and D. Baleanu, *New Fractional Derivatives with Nonlocal and Non-Singular Kernel: Theory and Application to Heat Transfer Model*, Therm. Sci., **20** (2016), 763–769.

8. M. A. Khan, S. Ullah, M. Farhan, *The dynamics of Zika virus with Caputo fractional derivative*, AIMS Math., **4** (2019), 134–146.
9. S. Ullah, M. A. Khan, M. Farooq, *A new fractional model for tuberculosis with relapse via Atangana-Baleanu derivative*, Chaos, Solitons and Fractals, **116** (2018), 227–238.
10. S. Qureshi, A. Yusuf, *Modeling chickenpox disease with fractional derivatives: From caputo to atangana-baleanu*, Chaos, Solitons and Fractals, **122** (2019), 111–118.
11. S. Qureshi, E. Bonyah, A. A. Shaikh, *Classical and contemporary fractional operators for modeling diarrhea transmission dynamics under real statistical data*, Phy. A, **535** (2019), 1–22.
12. Z. Wang, Y. K. Xie, J. Lu, et al. *Stability and bifurcation of a delayed generalized fractional-order prey redator model with interspecific competition*, Appl. Math. Comput., **347** (2019), 360–369.
13. X. Wang, Z. Wang, X. Huang, et al. *Dynamic Analysis of a Delayed Fractional-Order SIR Model with Saturated Incidence and Treatment Functions*, Int. J. Bifurcat. Chaos, **28** (2018), 1850180.
14. X. Wang, Z. Wang, H. Shen, *Dynamical analysis of a discrete-time SIS epidemic model on complex networks*, Appl. Math. Lett., **94** (2019), 292–299.
15. X. Wang, Z. Wang, J. Xia, *Stability and bifurcation control of a delayed fractional-order eco-epidemiological model with incommensurate orders*, J. Franklin I., **356** (2019), 8278–8295.
16. M. A. Khan, K. Shah, Y. Khan, et al. *Mathematical modeling approach to the transmission dynamics of pine wilt disease with saturated incidence rate*, Int. J. Biomath., **11** (2018), 1850035.
17. F. Rao, P. S. Mandal and Y. Kang, *Complicated endemics of an SIRS model with a generalized incidence under preventive vaccination and treatment controls*, App. Math. Mod., **67** (2019), 38–61.
18. M. A. Khan, M. Farhan, S. Islam, et al. *Modeling the transmission dynamics of avian influenza with saturation and psychological effect*, Discrete and Continuous Dynamical Systems-S, **12** (2019), 455–474.
19. W. O. Kermack, A. G. MCKendrick, *A contribution to the mathematical theory of epidemics*, Proc. Roy. Soc. Lond. A, **115** (1927), 700–721.
20. V. Capasso and G. Serio, *A generalization of the Kermack-Mckendrick deterministic epidemic model*, Math. Biosci., **42** (1978), 41–61.
21. A. Korobeinikov and P. K Maini, *A Lyapunov function and global properties for SIR and SEIR epidemiological models with nonlinear incidence*, Math. Biosci. Eng., **1** (2004), 57–60.
22. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1999.
23. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, 1993.
24. H. Delavari, D. Baleanu, J. Sadati, *Stability analysis of Caputo fractional-order nonlinear systems revisited*, Nonlinear Dyn., **67** (2012), 2433–2439.
25. C. Vargas-De-Leon, *Volterra-type Lyapunov functions for fractional order epidemic systems*, Commun. Nonlinear Sci. Numer. Simul., **24** (2015), 75–85.

26. J. Li, Y. Yang, Y. Xiao, et al. *A class of Lyapunov functions and the global stability of some epidemic models with nonlinear incidence*, *J. Appl. Anal. Comput.*, **6** (2016), 38–46.
27. J. J. Wang, J. Z. Zhang and Z. Jin, *Analysis of an SIR model with bilinear incidence rate*, *Nonlinear Analysis: Real World Applications*, **11** (2010), 2390–2402.
28. M. A. Khan, Q. Badshah, S. Islam, et al. *Global dynamics of SEIRS epidemic model with non-linear generalized incidences and preventive vaccination*, *Adv. Diff. Equ.*, **2015** (2015), 88.
29. M. A. Khan, Y. Khan and S. Islam, *Complex dynamics of an SEIR epidemic model with saturated incidence rate and treatment*, *Phy. A.*, **493** (2018), 210–227.
30. X. Liu and L. Yang, *Stability analysis of an SEIQV epidemic model with saturated incidence rate*, *Nonlinear Analysis: Real World Applications*, **13** (2012), 2671–2679.
31. J. R. Beddington, *Mutual interference between parasites or predators and its effect on searching efficiency*, *The Journal of Animal Ecology*, (1975), 331–341.
32. D. L. DeAngelis, R. A. Goldstein and R. V. Oeill, *A model for tropic interaction*, *Ecology*, **56** (1975), 881–892.
33. K. Hattaf, M. Mahrouf, J. Adnani, et al. *Qualitative analysis of a stochastic epidemic model with specific functional response and temporary immunity*, *Phys. A*, **490** (2018), 591–600.
34. Z. M. Odibat and N. T. Shawagfeh, *Generalized Taylor's formula*, *Appl. Math. Comput.*, **186** (2007), 286–293.
35. W. Lin, *Global existence theory and chaos control of fractional differential equations*, *J. Math. Anal. Appl.*, **332** (2007), 709–726.
36. M. Toufik, A. Atangana, *New numerical approximation of fractional derivative with non-local and non-singular kernel: application to chaotic models*, *Eur. Phys. J. Plus*, **132** (2017), 444.



AIMS Press

© 2020 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)