Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(3): 1843–1855.
DOI:10.3934/math.2020123
Received: 01 December 2019
Accepted: 07 February 2020
Published: 17 February 2020

Research article

Determinantal and permanental representations of convolved (u, v)-Lucas first kind p-polynomials

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Abstract: The convolved (u, v)-Lucas first kind p-polynomials are defined using the generating function of the (u, v)-Lucas first kind p-polynomials. The determinantal and permanental representations of the convolved (u, v)-Lucas first kind p-polynomials are used to derive some identities of these polynomials.

Keywords: Lucas first kind p-numbers; Lucas first kind p-polynomials; convolved (u, v)-Lucas first kind p-polynomials; determinantal; permanental; Hessenberg matrix
Mathematics Subject Classification: 11B37, 11B39, 15A15

1. Introduction

The integer sequence $(u_n)_{n\geq 0}$ is said to be a Lucas sequence of first kind if there exist non zero integers A and B such that $u_{n+2} = Au_{n+1} + Bu_n,\ n \geq 0$ with initials $u_0 = 0$ and $u_1 = 1$. Since the last few decades, researchers keep a constant interest in this sequence and have been placed their results to many modern sciences. Lucas sequence of the first kind comprises many sequences, like Fibonacci numbers, Pell numbers, balancing numbers, Jacobsthal numbers etc. that always make a constant attraction to the recent researchers. In one of the communicated papers of the authors, the Lucas first kind p-numbers $L_p(j)$ is defined by the recurrence relation

$$L_p(j) = aL_p(j-1) + bL_p(j-p-1),\ \ j \geq (p+1) \tag{1.1}$$

with initials $L_p(j) = a^{j-1}$, for $j = 1, 2, \ldots, p$ and $L_p(0) = 0$, where p is taken as non-negative integer and the coefficients a and b are non zero integers. For $p = 1$, (1.1) reduces to the recurrence relation of the Lucas first kind numbers.

A new generalization of the Fibonacci sequence based on its generating function is the convolved

Fibonacci numbers $F_j^{(r)}$ that have been studied in several manner (for e.g. [3, 5, 7]) and are defined by

$$(1-t-t^2)^{-r} = \sum_{j=0}^{\infty} F_{j+1}^{(r)} t^j,\ r \in \mathbb{Z}^+.$$

Nowadays it is the most challenging task for the authors to investigate several properties of number and polynomial sequences in a matrix way. Determinantal and permanental representations of numbers, polynomials and functions play a crucial role in many areas of mathematics. Şahin and Ramírez [6] introduced the convolved generalized Lucas polynomials $F_{p,q,j}^{(r)}(x)$ and are defined by

$$g_{p,q}^{(r)}(t) = \left(1-p(x)t-q(x)t^2\right)^{-r} = \sum_{j=0}^{\infty} F_{p,q,j+1}^{(r)}(x)t^j,\ r \in \mathbb{Z}^+,$$

where $p(x)$ and $q(x)$ are polynomials coefficient. They derived several identities using the matrix representations of $F_{p,q,j}^{(r)}(x)$ with real and imaginary entries.

In this article, we generalize Şahin and Ramírez paper by introducing convolved (u, v)-Lucas first kind p-polynomials. Based on determinantal and permanental representations, some similar type identities of [6] are studied for these polynomials using different proof methods.

2. Convolved (u, v)-Lucas first kind p-polynomials

In this section the (u, v)-Lucas first kind p-polynomials and convolved (u, v)-Lucas first kind p-polynomials are defined. Using some results of convolved (u, v)-Lucas first kind p-polynomials the recurrence relation of these polynomials is also established.

Definition 2.1. *Let p be any non negative integer and $u(x)$ and $v(x)$ are polynomials with real coefficients. The (u, v)-Lucas first kind p-polynomials $\{L_{u,v,j}^p(x)\}_{j\in\mathbb{N}}$ are defined by the recurrence relation*

$$L_{u,v,j}^p(x) = u(x)L_{u,v,j-1}^p(x) + v(x)L_{u,v,j-p-1}^p(x) \tag{2.1}$$

with initials $L_{u,v,0}^p(x) = 0$ and $L_{u,v,j}^p(x) = u^{j-1}(x)$ for $j = 1, \ldots, p$.

It is noticed that, when we consider $u(x) = ax$ and $v(x) = b$, equation (2.1) reduced to Lucas first kind p-polynomials $\{L_{p,j}(x)\}$ with initial values $L_{p,0}(x) = 0$ and $L_{p,j}(x) = (ax)^{j-1}$ for $j = 1, 2, \ldots, p$.

If $g_{u,v}^p(t)$ is the generating function of $L_{u,v,j+1}^p(x)$, then it can be easily seen that

$$g_{u,v}^p(t) = \sum_{j=0}^{\infty} L_{u,v,j+1}^p(x)t^j = \frac{1}{1-u(x)t-v(x)t^{p+1}}.$$

By virtue of the gnerating function $g_{u,v}^p(t)$, the convolved (u, v)-Lucas first kind p-polynomials can be defined as follows.

Definition 2.2. *The convolved (u, v)-Lucas first kind p-polynomials $\{L_{u,v,j}^{(p,r)}(x)\}_{j\in\mathbb{N}}$ for $p \geq 1$ are defined by*

$$g_{u,v}^{(p,r)}(t) = \sum_{j=0}^{\infty} L_{u,v,j+1}^{(p,r)}(x)t^j = (1-u(x)t-v(x)t^{p+1})^{-r},\ r \in \mathbb{Z}^+ \tag{2.2}$$

where $u(x)$ and $v(x)$ are polynomials with real coefficients.

From equation (2.2), we have

$$\begin{aligned}\sum_{j=0}^{\infty} L_{u,v,j+1}^{(p,r)}(x)t^j &= \sum_{j=0}^{\infty}\binom{-r}{j}(-t)^j(u(x)+v(x)t^p)^j \\ &= \sum_{j=0}^{\infty}\frac{(r)_j}{j!}t^j\sum_{k=0}^{j}\binom{j}{k}u^{j-k}(x)v^k(x)t^{pk} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\lfloor\frac{j}{p+1}\rfloor}\frac{(r)_{j-pk}}{(j-(p+1)k)!k!}u^{j-(p+1)k}(x)v^k(x)t^j.\end{aligned}$$

Here we conclude that

$$L_{u,v,j+1}^{(p,r)}(x) = \sum_{k=0}^{\lfloor\frac{j}{p+1}\rfloor}\frac{(r)_{j-pk}}{(j-(p+1)k)!k!}u^{j-(p+1)k}(x)v^k(x). \tag{2.3}$$

Using (2.3), we yield convolved (u, v)-Lucas first kind p-polynomials $L_{u,v,j}^{(p,r)}(x)$ for $j = 0, 1, 2, 3, 4, 5,$ and 6 with different (p, r) values, which are listed in both Table 1 and Table 2.

Table 1. Convolved (u, v)-Lucas first kind p-polynomials.

j	$(p,r) = (1,3)$	$(p,r) = (2,3)$	$(p,r) = (3,3)$	$(p,r) = (4,3)$
0	1	1	1	1
1	$3u(x)$	$3u(x)$	$3u(x)$	$3u(x)$
2	$6u^2(x) + 3v(x)$	$6u^2(x)$	$6u^2(x)$	$6u^2(x)$
3	$10u^3(x) + 12u(x)v(x)$	$10u^3(x) + 3v(x)$	$10u^3(x)$	$10u^3(x)$
4	$15u^4(x) + 30u^2(x)v(x) + 6v^2(x)$	$15u^4(x) + 12u(x)v(x)$	$15u^4(x) + 3v(x)$	$15u^4(x)$
5	$21u^5(x) + 60u^3(x)v(x) + 30u(x)v^2(x)$	$21u^5(x) + 30u^2(x)v(x)$	$21u^5(x) + 12u(x)v(x)$	$21u^5(x) + 3v(x)$
6	$28u^6(x) + 105u^4(x)v(x) + 90u^2(x)v^2(x) + 10v^3(x)$	$28u^6(x) + 60u^3(x)v(x) + 6v^2(x)$	$28u^6(x) + 30u^2(x)v(x)$	$28u^6(x) + 12u(x)v(x)$

Table 2. Convolved (u, v)-Lucas first kind p-polynomials.

j	$(p, r) = (1, 4)$	$(p, r) = (2, 4)$	$(p, r) = (3, 4)$	$(p, r) = (4, 4)$
0	1	1	1	1
1	$4u(x)$	$4u(x)$	$4u(x)$	$4u(x)$
2	$10u^2(x) + 4v(x)$	$10u^2(x)$	$10u^2(x)$	$10u^2(x)$
3	$20u^3(x) + 20u(x)v(x)$	$20u^3(x) + 4v(x)$	$20u^3(x)$	$20u^3(x)$
4	$35u^4(x) + 60u^2(x)v(x) + 10v^2(x)$	$35u^4(x) + 20u(x)v(x)$	$35u^4(x) + 4v(x)$	$35u^4(x)$
5	$56u^5(x) + 140u^3(x)v(x) + 60u(x)v^2(x)$	$56u^5(x) + 60u^2(x)v(x)$	$56u^5(x) + 20u(x)v(x)$	$56u^5(x) + 4v(x)$
6	$84u^6(x) + 280u^4(x)v(x) + 210u^2(x)v^2(x) + 20v^3(x)$	$84u^6(x) + 140u^3(x)v(x) + 10v^2(x)$	$84u^6(x) + 60u^2(x)v(x)$	$84u^6(x) + 20u(x)v(x)$

Using the definition of convolved (u, v)-Lucas first kind p-polynomials the following results can be easily verified.

Lemma 2.3. *The following relations holds for convolved (u, v)-Lucas first kind p-polynomials*

(i) $L^{(p,r)}_{u,v,2}(x) = ru(x)$;

(ii) $L^{(p,r)}_{u,v,j+1}(x) = u(x)L^{(p,r)}_{u,v,j}(x) + v(x)L^{(p,r)}_{u,v,j-p}(x) + L^{(p,r-1)}_{u,v,j+1}(x),\ j \geq 2$;

(iii) $jL^{(p,r)}_{u,v,j+1}(x) = r[u(x)L^{(p,r+1)}_{u,v,j}(x) + (p + 1)v(x)L^{(p,r+1)}_{u,v,j-p}(x)],\ j \geq 1.$

Now we are in a position to find the recurrence relation of the convolved (u, v)-Lucas first kind p-polynomials.

Theorem 2.4. *The recurrence relation of the convolved (u, v)-Lucas first kind p-polynomials $\{L^{(p,r)}_{u,v,j}(x)\}_{j\in\mathbb{N}}$ obey the second order recurrence relation*

$$L^{(p,r)}_{u,v,j+1}(x) = \frac{r + j - 1}{j}u(x)L^{(p,r)}_{u,v,j}(x) + \frac{(p + 1)r + j - p - 1}{j}v(x)L^{(p,r)}_{u,v,j-p}(x), \tag{2.4}$$

with initials $L^{(p,r)}_{u,v,1}(x) = 1$ and $L^{(p,r)}_{u,v,k+1}(x) = \prod_{j=1}^{k}\left(\frac{r+j-1}{j}u(x)\right)$ for $k = 1, 2, \ldots, p - 1$.

Proof. From relation (*iii*) of Lemma 2.3, we have

$$jL^{(p,r)}_{u,v,j+1}(x) = (ru(x)t + (p + 1)rv(x)t^{p+1})L^{(p,r+1)}_{u,v,j+1}(x).$$

Multiplying $(1 - u(x)t - v(x)t^{p+1})$ on both the sides yields

$$\begin{aligned} jL^{(p,r)}_{u,v,j+1}(x) - u(x)(j - 1)L^{(p,r)}_{u,v,j}(x) \quad - \quad & v(x)(j - p - 1)L^{(p,r)}_{u,v,j-p}(x) \\ & = ru(x)L^{(p,r)}_{u,v,j}(x) + (p + 1)rv(x)L^{(p,r)}_{u,v,j-p}(x). \end{aligned}$$

Further simplification gives

$$jL^{(p,r)}_{u,v,j+1}(x) = (r + j - 1)u(x)L^{(p,r)}_{u,v,j}(x) + ((p + 1)r + j - p - 1)v(x)L^{(p,r)}_{u,v,j-p}(x),$$

and the result follows. □

3. Determinantal representations of convolved (u, v)-Lucas first kind p-polynomials

In this section we consider various Hessenberg matrices with some adjustable real or imaginary entries. Based upon these matrices we establish some results involving determinantal representations of convolved (u, v)-Lucas first kind p-polynomials.

The following result is useful while proving the subsequent theorems.

Lemma 3.1. [1] Let $A_j = (a_{il})_{j\times j}$ with $1 \le i, l \le j$ be the lower Hessenberg matrix for all $j \ge 1$ and define $det(A_0) = 1$. Then, $det(A_1) = a_{11}$ and for $j \ge 2$

$$det(A_j) = a_{jj}det(A_{j-1}) + \sum_{l=1}^{j-1} [(-1)^{j-r} a_{j,l}(\prod_{i=l}^{j-1} a_{i,i+1})det(A_{l-1})].$$

Theorem 3.2. *Let $F^{(p,r)}_{u,v,j} = (f_{st})$ be $j \times j$ Hessenberg matrix defined as*

$$f_{st} = \begin{cases} \frac{r+s-1}{s}u(x), & \text{if } t = s; \\ \frac{(p+1)r+s-p-1}{s}v(x)(i)^p, & \text{if } s - t = p; \\ i, & \text{if } t - s = 1; \\ 0, & \text{otherwise,} \end{cases}$$

that is, $F^{(p,r)}_{u,v,j} =$

$$\begin{bmatrix} ru(x) & i & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{r+1}{2}u(x) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{r+p-1}{p}u(x) & i & \dots & 0 \\ rv(x)(i)^p & 0 & \dots & 0 & \frac{r+p}{p+1}u(x) & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots \frac{(p+1)r-p+j-1}{j}v(x)(i)^p \dots & 0 & 0 & \dots & \frac{r+j-1}{j}u(x) \end{bmatrix},$$

where $i = \sqrt{-1}$. Then

$$det(F^{(p,r)}_{u,v,j}) = L^{(p,r)}_{u,v,j+1}(x). \tag{3.1}$$

Proof. Using induction on j, the result is clearly holds for $j = 1$ by (2.4). Assume that the result is true for all positive integers less than or equal to $j - 1$, i.e. $det(F^{(p,r)}_{u,v,j}) = L^{(p,r)}_{u,v,j+1}(x)$. By virtue of Lemma 3.1 and the relation (2.4), we have

$$\begin{aligned} det(F^{(p,r)}_{u,v,j+1}) &= f_{j+1,j+1}det(F^{(p,r)}_{u,v,j}) + \sum_{t=1}^{j} [(-1)^{j+1-t} f_{j+1,t}(\prod_{s=t}^{j} f_{s,s+1})det(F^{(p,r)}_{u,v,t-1})] \\ &= \frac{r+j}{j+1}u(x)det(F^{(p,r)}_{u,v,j}) + \sum_{t=1}^{j-p} [(-1)^{j+1-t} f_{j+1,t}(\prod_{s=t}^{j} f_{s,s+1})det(F^{(p,r)}_{u,v,t-1})] \end{aligned}$$

$$
\begin{aligned}
& +(-1)^p f_{j+1,j-p+1}(\prod_{s=j-p+1}^{j} f_{s,s+1})det(F^{(p,r)}_{u,v,j-p}) \\
& + \sum_{t=j-p+2}^{j}[(-1)^{j+1-t} f_{j+1,t}(\prod_{s=t}^{j} f_{s,s+1})det(F^{(p,r)}_{u,v,t-1})] \\
= & \frac{r+j}{j+1}u(x)det(F^{(p,r)}_{u,v,j}) + (-1)^p\frac{(p+1)r-p+j}{j+1}v(x)(i)^p(\prod_{s=j-p+1}^{j} i)det(F^{(p,r)}_{u,v,j-p}) \\
= & \frac{r+j}{j+1}u(x)det(F^{(p,r)}_{u,v,j}) + (-1)^p\frac{(p+1)r-p+j}{j+1}v(x)(i)^p(i)^p det(F^{(p,r)}_{u,v,j-p}) \\
= & \frac{r+j}{j+1}u(x)L^{(p,r)}_{u,v,j+1}(x) + \frac{(p+1)r-p+j}{j+1}v(x)L^{(p,r)}_{u,v,j-p+1}(x) \\
= & L^{(p,r)}_{u,v,j+2}(x).
\end{aligned}
$$

This completes the proof. □

Theorem 3.3. *Let $D^{(p,r)}_{u,v,j} = (d_{st})$ be $j \times j$ Hessenberg matrix defined as*

$$
d_{st} = \begin{cases} \frac{r+s-1}{s}u(x), & \text{if } t = s; \\ \frac{(p+1)r+s-p-1}{s}v(x), & \text{if } s-t = p; \\ -1, & \text{if } t-s = 1; \\ 0, & \text{otherwise,} \end{cases}
$$

that is,

$$
D^{(p,r)}_{u,v,j} = \begin{bmatrix}
ru(x) & -1 & \dots & 0 & 0 & \dots & 0 \\
0 & \frac{r+1}{2}u(x) & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & \frac{r+p-1}{p}u(x) & -1 & \dots & 0 \\
rv(x) & 0 & \dots & 0 & \frac{r+p}{p+1}u(x) & \dots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \dots \frac{(p+1)r-p+j-1}{j}v(x) \dots & 0 & 0 & \dots & \frac{r+j-1}{j}u(x)
\end{bmatrix}.
$$

Then

$$
det(D^{(p,r)}_{u,v,j}) = L^{(p,r)}_{u,v,j+1}(x). \tag{3.2}
$$

Proof. The proof is analogous to the proof of Theorem 3.2. □

To better understand the above theorems, let us consider the following examples.

Example 3.4. *We calculate the polynomial $L^{(p,r)}_{u,v,j+1}(x)$ with $(p, r) = (2, 4)$ and $j = 6$ by using Theorem 3.2.*

$$L^{(2,4)}_{u,v,7}(x) = det\begin{bmatrix} 4u(x) & i & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{2}u(x) & i & 0 & 0 & 0 \\ -4v(x) & 0 & 2u(x) & i & 0 & 0 \\ 0 & \frac{-13}{4}v(x) & 0 & \frac{7}{4}u(x) & i & 0 \\ 0 & 0 & \frac{-14}{5}v(x) & 0 & \frac{8}{5}u(x) & i \\ 0 & 0 & 0 & \frac{-5}{2}v(x) & 0 & \frac{3}{2}u(x) \end{bmatrix}_{6\times 6}$$

$$= 84u^6(x) + 140u^3(x)v(x) + 10v^2(x).$$

Example 3.5. *We calculate the polynomial* $L^{(p,r)}_{u,v,j+1}(x)$ *with* $(p, r) = (3, 4)$ *and* $j = 5$ *by using Theorem 3.3.*

$$L^{(3,4)}_{u,v,6}(x) = det\begin{bmatrix} 4u(x) & -1 & 0 & 0 & 0 \\ 0 & \frac{5}{2}u(x) & -1 & 0 & 0 \\ 0 & 0 & 2u(x) & -1 & 0 \\ 4v(x) & 0 & 0 & \frac{7}{4}u(x) & -1 \\ 0 & \frac{17}{5}v(x) & 0 & 0 & \frac{8}{5}u(x) \end{bmatrix}_{5\times 5}$$

$$= 56u^5(x) + 20u(x)v(x).$$

4. Permanental representations of convolved (u, v)-Lucas first kind p-polynomials

In this section we consider various Hessenberg matrices and upon these matrices we establish some results involving permanental representations of convolved (u, v)-Lucas first kind p-polynomials. Moreover, we consider some non-singular matrices and establish the first column of inverse of these matrices is written in convolved (u, v)-Lucas first kind p-polynomials.

The following result is useful while proving the subsequent theorems.

Lemma 4.1. [4] Let $A_j = (a_{il})_{j\times j}$ with $1 \le i, l \le j$ be the lower Hessenberg matrix for all $j \ge 1$, and define $per(A_0) = 1$. Then $per(A_1) = a_{11}$, and for $j \ge 2$,

$$per(A_j) = a_{j,j}per(A_{j-1}) + \sum_{l=1}^{j-1}(a_{j,l}\prod_{i=l}^{j-1} a_{i,i+1}per(A_{l-1})).$$

Theorem 4.2. *Let* $G^{(p,r)}_{u,v,j} = (g_{st})$ *be* $j \times j$ *Hessenberg matrix, given by*

$$g_{st} = \begin{cases} \frac{r+s-1}{s}u(x), & \textit{if } t = s; \\ \frac{(p+1)r+s-p-1}{s}v(x)(i)^p, & \textit{if } s - t = p; \\ -i, & \textit{if } t - s = 1; \\ 0, & \textit{otherwise,} \end{cases}$$

that is, $G_{u,v,j}^{(p,r)} =$

$$\begin{bmatrix} ru(x) & -i & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{r+1}{2}u(x) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{r+p-1}{p}u(x) & -i & \dots & 0 \\ rv(x)(i)^p & 0 & \dots & 0 & \frac{r+p}{p+1}u(x) & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots \frac{(p+1)r-p+j-1}{j}v(x)(i)^p \dots & 0 & 0 & \dots & \frac{r+j-1}{j}u(x) \end{bmatrix},$$

where $i = \sqrt{-1}$. *Then*

$$per(G_{u,v,j}^{(p,r)}) = L_{u,v,j+1}^{(p,r)}(x). \tag{4.1}$$

Proof. By the induction on j the result is true for $j = 1$. Let us consider the result is true for all positive integers less than or equal to $j-1$, i.e. $per(G_{u,v,j}^{(p,r)}) = L_{u,v,j+1}^{(p,r)}(x)$. Then by using Lemma 4.1, we have

$$\begin{aligned} per(G_{u,v,j+1}^{(p,r)}) &= g_{j+1,j+1}per(G_{u,v,j}^{(p,r)}) + \sum_{t=1}^{j}(a_{j+1,t}\prod_{s=t}^{j}a_{s,s+1}per(G_{t-1})) \\ &= \frac{r+j}{j+1}u(x)per(G_{u,v,j}^{(p,r)}) + \sum_{t=1}^{j-p}(a_{j+1,t}\prod_{s=t}^{j}a_{s,s+1}per(G_{t-1})) \\ &\quad +a_{j+1,j-p+1}\prod_{s=j-p+1}^{j}(-i)per(G_{j-p}) + \sum_{t=j-p+2}^{j}(a_{j+1,t}\prod_{s=t}^{j}a_{s,s+1}per(G_{t-1})) \\ &= \frac{r+j}{j+1}u(x)per(G_{u,v,j}^{(p,r)}) + \frac{(p+1)r-p+j}{j+1}v(x)(i)^p(-i)^p per(G_{j-p}) \\ &= \frac{r+j}{j+1}u(x)L_{u,v,j+1}^{(p,r)}(x) + \frac{(p+1)r-p+j}{j+1}v(x)L_{u,v,j-p+1}^{(p,r)}(x), \end{aligned}$$

which is true by (2.4). This hence the proof. □

Theorem 4.3. *Let* $H_{u,v,j}^{(p,r)} = (h_{st})$ *be* $j \times j$ *Hessenberg matrix, given by*

$$h_{st} = \begin{cases} \frac{r+s-1}{s}u(x), & \text{if } t = s; \\ \frac{(p+1)r+s-p-1}{s}v(x), & \text{if } s-t = p; \\ 1, & \text{if } t-s = 1; \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$H_{u,v,j}^{(p,r)} = \begin{bmatrix} ru(x) & 1 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{r+1}{2}u(x) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{r+p-1}{p}u(x) & 1 & \dots & 0 \\ rv(x) & 0 & \dots & 0 & \frac{r+p}{p+1}u(x) & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots \frac{(p+1)r-p+j-1}{j}v(x) \dots & 0 & 0 & \dots & \frac{r+j-1}{j}u(x) \end{bmatrix}.$$

Then

$$per(H_{u,v,j}^{(p,r)}) = L_{u,v,j+1}^{(p,r)}(x). \tag{4.2}$$

Proof. The proof is analogous to the proof of Theorem 4.2.

□

To better understand the above theorems, let us consider the following examples.

Example 4.4. *We calculate the polynomial $L_{u,v,j+1}^{(p,r)}(x)$ for $(p, r) = (4, 3)$ and $j = 5$ by using Theorem 4.2.*

$$L_{u,v,6}^{(4,3)}(x) = per\begin{bmatrix} 3u(x) & -i & 0 & 0 & 0 \\ 0 & 2u(x) & -i & 0 & 0 \\ 0 & 0 & \frac{5}{3}u(x) & -i & 0 \\ 0 & 0 & 0 & \frac{3}{2}u(x) & -i \\ 3v(x) & 0 & 0 & 0 & \frac{7}{5}u(x) \end{bmatrix}_{5\times 5}$$

$$\begin{aligned} &= \sum_{\sigma\in s_5}\prod_{i=1}^{5} a_{i,\sigma(i)} = a_{11}a_{22}a_{33}a_{44}a_{55} + a_{12}a_{23}a_{34}a_{45}a_{51} \\ &= 21u^5(x) + 3v(x). \end{aligned}$$

Example 4.5. *We calculate the polynomial $L_{u,v,j+1}^{(p,r)}(x)$ with $(p, r) = (3, 3)$ and $j = 4$ by using Theorem 4.3.*

$$L_{u,v,5}^{(3,3)}(x) = per\begin{bmatrix} 3u(x) & 1 & 0 & 0 \\ 0 & 2u(x) & 1 & 0 \\ 0 & 0 & \frac{5}{3}u(x) & 1 \\ 3v(x) & 0 & 0 & \frac{3}{2}u(x) \end{bmatrix}_{4\times 4}$$

$$\begin{aligned} &= \sum_{\sigma\in s_4}\prod_{i=1}^{4} a_{i,\sigma(i)} = a_{11}a_{22}a_{33}a_{44} + a_{12}a_{23}a_{34}a_{41} \\ &= 15u^4(x) + 3v(x). \end{aligned}$$

At the end of this section, we present two important results concerning convolved (u, v)-Lucas first kind p-polynomials. We omit the proofs of these results because they are similar to the methods which are adopted in Theorem 9 of [6].

Theorem 4.6. *Let* $\tilde{F}^{(p,r)}_{u,v,j+1}$ *be the* $(j+1)-by-(j+1)$ *non singular matrix given by*

$$\tilde{F}^{(p,r)}_{u,v,j+1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ & & & & & \vdots \\ & F^{(p,r)}_{u,v,j} & & & & 0 \\ & & & & & 0 \\ & & & & & 1 \end{bmatrix},$$

where $F^{(p,r)}_{u,v,j}$ *is the Hessenberg matrix of order j defined in Theorem 3.2. Then the first column of* $(\tilde{F}^{(p,r)}_{u,v,j+1})^{-1}$ *is*

$$\begin{bmatrix} L^{(p,r)}_{u,v,1}(x) \\ iL^{(p,r)}_{u,v,2}(x) \\ \vdots \\ i^{j-1}L^{(p,r)}_{u,v,j}(x) \\ i^{j+1}L^{(p,r)}_{u,v,j+1}(x) \end{bmatrix},$$

where $i = \sqrt{-1}$ *and* $L^{(p,r)}_{u,v,j}(x)$ *is the convolved* (u, v)*-Lucas first kind p-polynomials.*

Theorem 4.7. *Let* $\tilde{D}^{(p,r)}_{u,v,j+1}$ *be the* $(j+1)-by-(j+1)$ *non singular matrix given by*

$$\tilde{D}^{(p,r)}_{u,v,j+1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ & & & & & \vdots \\ & D^{(p,r)}_{u,v,j} & & & & 0 \\ & & & & & 0 \\ & & & & & 1 \end{bmatrix},$$

where $D^{(p,r)}_{u,v,j}$ *is the Hessenberg matrix of order j defined in Theorem 3.3. Then the first column of* $(\tilde{D}^{(p,r)}_{u,v,j+1})^{-1}$ *is*

$$\begin{bmatrix} L^{(p,r)}_{u,v,1}(x) \\ L^{(p,r)}_{u,v,2}(x) \\ \vdots \\ L^{(p,r)}_{u,v,j}(x) \\ -L^{(p,r)}_{u,v,j+1}(x) \end{bmatrix},$$

where $L^{(p,r)}_{u,v,j}(x)$ *is the convolved* (u, v)*-Lucas first kind p-polynomials.*

In order to verify these theorems, we need the following results of [2]. The first result is

$$F\alpha + fe_j = 0, \tag{4.3}$$

which is obtained from $\tilde{F} \cdot \tilde{F}^{-1} = I_{j+1}$ and the second result is

$$det(F) = (-1)^j f \cdot det(\tilde{F}), \tag{4.4}$$

where

$$F = \begin{bmatrix} f_{11} & f_{12} & 0 & \dots & 0 \\ f_{21} & f_{22} & f_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ f_{(j-1)1} & f_{(j-1)2} & f_{(j-1)3} & \dots & f_{(j-1)j} \\ f_{j1} & f_{j2} & f_{j3} & \dots & f_{jj} \end{bmatrix},$$

$$\tilde{F} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ f_{11} & f_{12} & 0 & \dots & 0 & 0 \\ f_{21} & f_{22} & f_{23} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ f_{(j-1)1} & f_{(j-1)2} & f_{(j-1)3} & \dots & f_{(j-1)j} & 0 \\ f_{j1} & f_{j2} & f_{j3} & \dots & f_{jj} & 1 \end{bmatrix} = \begin{bmatrix} e_1^T & 0 \\ F & e_j \end{bmatrix} \text{and}$$

$\tilde{F}^{-1} = \begin{bmatrix} \alpha & L \\ f & \beta^T \end{bmatrix}$ with $\alpha,\ L,\ f$ and β^T are of order $j \times j,\ j \times j,\ 1 \times 1$ and $1 \times j$ respectively.

Example 4.8. *We verify the Theorem 4.6 by taking $(p, r) = (2, 3)$ and $j = 5$*

$$\tilde{F}^{(2,3)}_{u,v,6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3u(x) & i & 0 & 0 & 0 & 0 \\ 0 & 2u(x) & i & 0 & 0 & 0 \\ -3v(x) & 0 & \frac{5}{3}u(x) & i & 0 & 0 \\ 0 & \frac{-5}{2}v(x) & 0 & \frac{3}{2}u(x) & i & 0 \\ 0 & 0 & \frac{-11}{5}v(x) & 0 & \frac{7}{5}u(x) & 1 \end{bmatrix}.$$

Let us consider

$$(\tilde{F}^{(2,3)}_{u,v,6})^{-1} = \begin{bmatrix} [\alpha]_{j\times 1} & [L]_{j\times j} \\ [f]_{1\times 1} & [\beta^T]_{1\times j} \end{bmatrix}.$$

Using (4.4), we have

$$det(F^{(2,3)}_{u,v,5}) = (-1)^5 f \cdot det(\tilde{F}^{(2,3)}_{u,v,6}),$$

and further applying (3.1), we get

$$f = -L^{(2,3)}_{u,v,6}(x).$$

Using (4.3), we get

$$\alpha = (F^{(2,3)}_{u,v,5})^{-1} L^{(2,3)}_{u,v,6}(x) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = adj(F^{(2,3)}_{u,v,5}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i3u(x) \\ i^2 6u^2(x) \\ i^3[10u^3(x) + 3v(x)] \\ i^4[15u^4(x) + 12u(x)v(x)] \end{bmatrix}.$$

Hence by Table 1, it is verified that the first column of $(\tilde{F}^{(2,3)}_{u,v,6})^{-1}$ *is*

$$\begin{bmatrix}\alpha\\ f\end{bmatrix} = \begin{bmatrix} L^{(2,3)}_{u,v,1}(x)\\ iL^{(2,3)}_{u,v,2}(x)\\ i^2L^{(2,3)}_{u,v,3}(x)\\ i^3L^{(2,3)}_{u,v,4}(x)\\ i^4L^{(2,3)}_{u,v,5}(x)\\ i^6L^{(2,3)}_{u,v,6}(x)\end{bmatrix}.$$

Example 4.9. *We verify the Theorem 4.7 by taking* $(p, r) = (4, 3)$ *and* $j = 5$

$$\tilde{D}^{(4,3)}_{u,v,6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 3u(x) & -1 & 0 & 0 & 0 & 0\\ 0 & 2u(x) & -1 & 0 & 0 & 0\\ 0 & 0 & \frac{5}{3}u(x) & -1 & 0 & 0\\ 0 & 0 & 0 & \frac{3}{2}u(x) & -1 & 0\\ 3v(x) & 0 & 0 & 0 & \frac{7}{5}u(x) & 1\end{bmatrix}.$$

Let us consider

$$(\tilde{D}^{(4,3)}_{u,v,6})^{-1} = \begin{bmatrix} [\alpha]_{j\times 1} & [L]_{j\times j}\\ [d]_{1\times 1} & [\beta^T]_{1\times j}\end{bmatrix}.$$

Using (4.4), we have

$$det(D^{(4,3)}_{u,v,5}) = (-1)^5 d \cdot det(\tilde{D}^{(4,3)}_{u,v,6}),$$

and further applying (3.2), we get

$$d = -L^{(4,3)}_{u,v,6}(x).$$

Using (4.3), we get

$$\alpha = (D^{(4,3)}_{u,v,5})^{-1} L^{(4,3)}_{u,v,6}(x)\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = adj(D^{(4,3)}_{u,v,5})\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\ 3u(x)\\ 6u^2(x)\\ 10u^3(x)\\ 15u^4(x)\end{bmatrix}.$$

Hence by Table 1, it is verified that the first column of $(\tilde{D}^{(4,3)}_{u,v,6})^{-1}$ *is*

$$\begin{bmatrix}\alpha\\ d\end{bmatrix} = \begin{bmatrix} L^{(4,3)}_{u,v,1}(x)\\ L^{(4,3)}_{u,v,2}(x)\\ L^{(4,3)}_{u,v,3}(x)\\ L^{(4,3)}_{u,v,4}(x)\\ L^{(4,3)}_{u,v,5}(x)\\ -L^{(4,3)}_{u,v,6}(x)\end{bmatrix}.$$

Conflict of interest

The authors declare there are no conflicts of interest in this paper.

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