Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(3): 1779–1798.
DOI:10.3934/math.2020120
Received: 02 November 2019
Accepted: 06 February 2020
Published: 14 February 2020

Research article

Elliptic problems with singular nonlinearities of indefinite sign

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Abstract: Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. We consider problems of the form $-\Delta u = \chi_{\{u>0\}}\left(au^{-\alpha} - g(.,u)\right)$ in Ω, $u = 0$ on $\partial\Omega$, $u \geq 0$ in Ω, where Ω is a bounded domain in $\mathbb{R}^n$, $0 \not\equiv a \in L^\infty(\Omega)$, $\alpha \in (0,1)$, and $g : \Omega \times [0,\infty) \to \mathbb{R}$ is a nonnegative Carathéodory function. We prove, under suitable assumptions on a and g, the existence of nontrivial and nonnegative weak solutions $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of the stated problem. Under additional assumptions, the positivity, *a.e.* in Ω, of the found solution u, is also proved.

Keywords: singular elliptic problems; nonnegative solutions; sub and supersolutions
Mathematics Subject Classification: Primary: 35J75; Secondary: 35D30, 35J20

1. Introduction and statement of the main results

Let Ω be a bounded and regular enough domain in $\mathbb{R}^n$, let $\alpha > 0$, and let $a : \Omega \to \mathbb{R}$ be a nonnegative and nonidentically zero function. Singular elliptic problems like to

$$\begin{cases} -\Delta u = au^{-\alpha} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.1}$$

arise in many applications to physical phenomena, for instance, in chemical catalysts process, non-Newtonian fluids, and in models for the temperature of electrical conductors (see e.g., [3,5,13,16] and the references therein). Starting with the pioneering works [6, 13, 16, 26], and [11], the existence of positive solutions of singular elliptic problems has been intensively studied in the literature.

Bifurcation problems whose model is $-\Delta u = au^{-\alpha} + f(.,\lambda u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, were studied by Coclite and Palmieri [4], under the assumptions $a \in C^1\left(\overline{\Omega}\right)$, $a > 0$ in $\overline{\Omega}$, $f \in C^1\left(\overline{\Omega} \times [0,\infty)\right)$ and $\lambda > 0$. Problems of the form $-\Delta u = Ku^{-\alpha} + \lambda s^p$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, were studied by Shi and Yao [35], when $p \in (0,1)$, K is a regular enough function that may change sign, and

$\lambda \in \mathbb{R}$. Ghergu and Rădulescu [19] addressed multi-parameter singular bifurcation problems of the form $-\Delta u = g(u) + \lambda |\nabla u|^p + \mu f(., u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where g is Hölder continuous, nonincreasingt and positive on $(0, \infty)$, and singular at the origin; $f : \overline{\Omega} \times [0, \infty) \to [0, \infty)$ is Hölder continuous, positive on $\overline{\Omega} \times (0, \infty)$, and such that $f(x, s)$ is nondecreasing with respect to s, $0 < p \leq 2$, and $\lambda > 0$. Dupaigne, Ghergu and Rădulescu [14] studied Lane–Emden–Fowler equations with convection and singular potential; and Rădulescu [32] addressed the existence, nonexistence, and uniqueness of blow-up boundary solutions of logistic equations and of singular Lane-Emden-Fowler equations with convection term. Cîrstea, Ghergu and Rădulescu [7] considered the problem of the existence of classical positive solutions for problems of the form $-\Delta u = a(x)h(u) + \lambda f(u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, in the case when Ω is a regular enough domain, f and h are positive Hölder continuous functions on $[0, \infty)$ and $(0, \infty)$ respectively satisfying some monotonicity assumptions, h singular at the origin, and $h(s) \leq cs^{-\alpha}$ for some positive constant c and some $\alpha \in (0, 1)$.

Multiplicity results for positive solutions of singular elliptic problems were obtained by Gasiński and Papageorgiou [17] and by Papageorgiou and G. Smyrlis [30]; in both articles the singular term of the considered nonlinearity has the form $a(x)s^{-\alpha}$, with $0 \leq a \in L^\infty(\Omega)$, $a \not\equiv 0$ in Ω, and α positive.

Recently, problem (1.1) has been addressed by Chu, Gao and Gao [8], under the assumption that $\alpha = \alpha(x)$ (i.e., with a singular nonlinearity with a variable exponent).

Concerning the existence of nonnegative solutions of singular elliptic problems, Dávila and Montenegro [9] studied the free boundary singular bifurcation problem

$$\begin{cases} -\Delta u = \chi_{\{u>0\}} \left(-u^{-\alpha} + \lambda f(., u)\right) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega, \ u \not\equiv 0 \text{ in } \Omega, \end{cases}$$

where $0 < \alpha < 1$, $\lambda > 0$, and $f : \Omega \times [0, \infty) \to [0, \infty)$ is a Carathéodory function f such that, for *a.e.* $x \in \Omega$, $f(x, s)$ is nondecreasing and concave in s, and satisfies $\lim_{s\to\infty} f(x, s)/s = 0$ uniformly on $x \in \Omega$. and where, for $h : \Omega \times (0, \infty) \to \mathbb{R}$, $\chi_{\{s>0\}}h(x, s)$ stands for the function defined on $\Omega \times [0, \infty)$ by $\chi_{\{s>0\}}h(x, s) := h(x, s)$ if $s > 0$, and $\chi_{\{s>0\}}h(x, s) := 0$ if $s = 0$. Let us mention also the work [10], where a related singular parabolic problem was treated.

For a systematic study of singular problems and additional references, we refer the reader to [18,32], see also [12].

Our aim in this work is to prove an existence result for nonnegative weak solutions of singular elliptic problems of the form

$$\begin{cases} -\Delta u = \chi_{\{u>0\}} \left(au^{-\alpha} - g(., u)\right) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega, \ u \not\equiv 0 \text{ in } \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, $\alpha \in (0, 1]$, $a : \Omega \to \mathbb{R}$, and $g : \Omega \times [0, \infty) \to \mathbb{R}$, with a and g satisfying the following conditions *h1)-h4)*:

h1) $0 \leq a \in L^\infty(\Omega)$ and $a \not\equiv 0$,
h2) $\{x \in \Omega : a(x) = 0\} = \Omega_0 \cup N$ for some (possibly empty) open set $\Omega_0 \subset \Omega$ and some measurable set $N \subset \Omega$ such that $|N| = 0$,
h3) g is a nonnegative Carathéodory function on $\Omega \times [0, \infty)$, i.e., $g(., s)$ is measurable for any

$s \in [0, \infty)$, and $g(x, .)$ is continuous on $[0, \infty)$ for *a.e.* $x \in \Omega$,
h4) $\sup_{0 \leq s \leq M} g(., s) \in L^{\infty}(\Omega)$ for any $M > 0$.

Here and below, $\chi_{\{u>0\}}(au^{-\alpha} - g(., u))$ stands for the function $h : \Omega \to \mathbb{R}$ defined by $h(x) := a(x) u^{-\alpha}(x) - g(x, u(x))$ if $u(x) \neq 0$, and $h(x) := 0$ otherwise; $u \not\equiv 0$ in Ω means $|\{x \in \Omega : u(x) \neq 0\}| > 0$ and, by a weak solution of (1.2), we mean a solution in the sense of the following:

Definition 1.1. Let $h : \Omega \to \mathbb{R}$ be a measurable function such that $h\varphi \in L^1(\Omega)$ for all φ in $H_0^1(\Omega) \cap L^{\infty}(\Omega)$. We say that $u : \Omega \to \mathbb{R}$ is a weak solution to the problem

$$\begin{cases} -\Delta u = h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

if $u \in H_0^1(\Omega)$, and $\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle = \int_{\Omega} h\varphi$ for all φ in $H_0^1(\Omega) \cap L^{\infty}(\Omega)$.
We will say that, in weak sense,

$$\begin{aligned} &-\Delta u \leq h \text{ in } \Omega \text{ (respectively } -\Delta u \geq h \text{ in } \Omega), \\ &\quad u = 0 \text{ on } \partial\Omega \end{aligned}$$

if $u \in H_0^1(\Omega)$, and $\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle \leq \int_{\Omega} h\varphi$ (respectively $\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle \geq \int_{\Omega} h\varphi$) for all nonnegative φ in $H_0^1(\Omega) \cap L^{\infty}(\Omega)$.

Our first result reads as follows:

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. Let $\alpha \in (0, 1]$, let $a : \Omega \to [0, \infty)$ and let $g : \Omega \times (0, \infty) \to \mathbb{R}$; and assume that a and g satisfy the conditions h1)-h4). Then there exists a nonnegative weak solution $u \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$, in the sense of Definition 1.1, to problem (1.2), and such that $u > 0$ a.e. in $\{a > 0\}$. In particular, $\chi_{\{u>0\}}(au^{-\alpha} - g(., u)) \not\equiv 0$ in Ω and $\chi_{\{u>0\}}(au^{-\alpha} - g(., u))\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$).*

Let us mention that in [21] it was proved the existence of weak solutions (in the sense of Definition 1.1) of problem (1.2), in the case when $0 \leq a \in L^{\infty}(\Omega)$, $a \not\equiv 0$, $0 < \alpha < 1$, and $g(., u) = -bu^p$, with $0 < p < \frac{n+2}{n-2}$, and $0 \leq b \in L^r(\Omega)$ for suitable values of r. In addition, existence results for weak solutions of problems of the form

$$\begin{cases} -\Delta u = \chi_{\{u>0\}} au^{-\alpha} - h(., u) \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega, \\ \quad u \geq 0 \text{ in } \Omega, \text{ and } u \not\equiv 0 \text{ in } \Omega, \end{cases} \tag{1.4}$$

were obtained, in [22] (see Remark 2.1 below), and in ([25], Theorem 1.2), for more general nonlinearities $h : \Omega \times [0, \infty) \to [0, \infty)(x, s)$, in the case when h is a Carathéodory function on $\Omega \times [0, \infty)$, which satisfies $h(., 0) \leq 0$ as well as some additional hypothesis. Then the result of Theorem 1.2 is not covered by those in [22] and [25] because, under the assumptions of Theorem 1.2, the condition $g(., 0) \leq 0$ is not required and $\chi_{\{s>0\}} g(., s)$ is not, in general, a Carathéodory function on $\Omega \times [0, \infty)$ (except when $g(., 0) \equiv 0$ in Ω).

Our next result says that if the condition *h4)* is replaced by the stronger condition *h4')* $a > 0$ *a.e.* in Ω and $\sup_{0<s\leq M} s^{-1}g(.,s) \in L^{\infty}(\Omega)$ for any $M > 0$,

then the solution u, given by Theorem 1.2, is positive *a.e.* in Ω and is a weak solution in the usual sense of $H_0^1(\Omega)$.

Theorem 1.3. *Let Ω, α, and a be as in Theorem 1.2, and let $g : \Omega\times(0,\infty) \to \mathbb{R}$. Assume the conditions h1)-h3) and h4'). Then the solution u of (1.2), given by Theorem 1.2, belongs to $C(\overline{\Omega}) \cap W_{loc}^{2,p}(\Omega)$ for any $p \in [1,\infty)$, there exist positive constants c, c' and τ such that $cd_\Omega \leq u \leq c'd_\Omega^\tau$ in Ω, and u is a weak solution, in the usual $H_0^1(\Omega)$ sense, of the problem*

$$\begin{cases} -\Delta u = au^{-\alpha} - g(.,u) \ \ in\ \Omega, \\ \qquad u = 0 \ on\ \partial\Omega, \\ \qquad u > 0 \ in\ \Omega \end{cases} \tag{1.5}$$

i.e., for any $\varphi \in H_0^1(\Omega)$, $(au^{-\alpha} - g(.,u))\varphi \in L^1(\Omega)$ and $\int_\Omega \langle\nabla u, \nabla\varphi\rangle = \int_\Omega (au^{-\alpha} - g(.,u))\varphi$.

Finally, our last result says that, if in addition to *h1)-h4)*, α is sufficiently small, the set where $a > 0$ is nice enough and, for any $s \geq 0$, $g(.,s) = 0$ *a.e.* in the set where $a > 0$, then the solution obtained in Theorem 1.2, is a weak solution in the usual sense of $H_0^1(\Omega)$, and that it is positive on some subset of Ω:

Theorem 1.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. Assume the hypothesis h1)-h4) of Theorem 1.2 and that $0 < \alpha < \frac{1}{2} + \frac{1}{n}$ when $n > 2$, and $\alpha \in (0,1]$ when $n \leq 2$. Let $A^+ := \{x \in \Omega : a(x) > 0\}$ and assume, in addition, the following two conditions:*

h5) $g(.,s) = 0$ a.e. in A^+ for any $s \geq 0$.
h6) $A^+ = \Omega^+ \cup N^+$ for some open set Ω^+ and a measurable set N^+ such that $|N^+| = 0$, and with Ω^+ such that Ω^+ has a finite number of connected components $\{\Omega_l^+\}_{1\leq l\leq N}$ and each Ω_l^+ is a $C^{1,1}$ domain.

Then the solution u of problem (1.2), given by Theorem 1.2, is a weak solution, in the usual $H_0^1(\Omega)$ sense, to the same problem, and there exist positive constants c, c' and τ such that $u \geq cd_{\Omega^+}$ a.e. in Ω^+, and $u \leq c'd_\Omega^\tau$ a.e. in Ω.

The article is organized as follows: In Section 2 we study, for $\varepsilon \in (0,1]$, the existence of weak solutions to the auxiliary problem

$$\begin{cases} -\Delta u = au^{-\alpha} - g_\varepsilon(.,u) \ \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega, \\ \qquad u > 0 \text{ in } \Omega. \end{cases} \tag{1.6}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, $\alpha \in (0,1]$, $a : \Omega \to [0,\infty)$ is a nonnegative function in $L^\infty(\Omega)$ such that $|\{x \in \Omega : a(x) > 0\}| > 0$, and $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ is a family of real valued functions defined on $\Omega \times [0,\infty)$ satisfying the following conditions *h7)-h9)*:

h7) g_ε is a nonnegative Carathéodory function on $\Omega \times [0,\infty)$ for any $\varepsilon \in (0,1]$.
h8) $\sup_{0<s\leq M} s^{-1}g_\varepsilon(.,s) \in L^\infty(\Omega)$ for any $\varepsilon \in (0,1]$ and $M > 0$.
h9) The map $\varepsilon \to g_\varepsilon(x,s)$ is nonincreasing on $(0,1]$ for any $(x,s) \in \Omega \times [0,\infty)$.

Lemma 2.2 observes that, as a consequence of a result of [22], the problem

$$\begin{cases} -\Delta u = \chi_{\{u>0\}} a u^{-\alpha} - g_{\varepsilon}(., u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega, \ u \not\equiv 0 \text{ in } \Omega \end{cases} \tag{1.7}$$

has (at least) a weak solution u (in the sense of Definition 1.1) which satisfies $u > 0$ *a.e.* in $\{a > 0\}$; and this assertion is improved in Lemmas 2.6 and 2.7, which state that any weak solution u (in the sense of Definition 1.1) of problem (1.7) is positive in Ω, belongs to $C\left(\overline{\Omega}\right)$, and is also a weak solution in the usual sense of $H_0^1(\Omega)$. By using a sub-supersolution theorem of [28] as well as an adaptation of arguments of [27], Lemma 2.15 shows that, for any $\varepsilon \in (0, 1]$, problem (1.6) has a solution $u_\varepsilon \in H_0^1(\Omega)$, which is a weak solution in the usual sense of $H_0^1(\Omega)$, and is maximal in the sense that, if v is a solution, in the sense of Definition 1.1, of problem (1.6) then $v \leq u_\varepsilon$. Lemma 2.16 states that $\varepsilon \to u_\varepsilon$ is nondecreasing, Lemma 2.17 says that $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $H_0^1(\Omega)$, and Lemma 2.18 says that the function $\boldsymbol{u} := \lim_{\varepsilon\to 0^+} u_\varepsilon$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and is positive in $\{a > 0\}$.

To prove Theorems 1.2–1.4 we consider, in Section 3, the family $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ defined by $g_\varepsilon(., s) := s(s+\varepsilon)^{-1} g(., s)$ and we show that, in each case, the corresponding function $\boldsymbol{u}$ defined above is a solution of problem (1.2) with the desired properties.

2. Preliminaries

We assume, from now on, that Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, $\alpha \in (0, 1]$ and $a : \Omega \to [0, \infty)$ is a nonnegative function in $L^\infty(\Omega)$ such that $|\{x \in \Omega : a(x) > 0\}| > 0$, and additional conditions will be explicitely impossed on a and α when necessary, at some steps of the paper. Our aim in this section is to study, for $\varepsilon \in (0, 1]$, the existence of weak solutions to problem (1.6), in the case when $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ is a family of functions satisfying the conditions *h7)-h9).*

In order to present, in the next remark, a need result of [22], we need to recall the notion of principal egenvalue with weight function: For $b \in L^\infty(\Omega)$ such that $b \not\equiv 0$, we say that $\lambda \in \mathbb{R}$ is a principal eigenvalue for $-\Delta$ on Ω, with weight function b and homogeneous Dirichlet boundary condition, if the problem $-\Delta u = \lambda b u$ in Ω, $u = 0$ on $\partial\Omega$ has a solution u wich is positive in Ω. If $b \in L^\infty(\Omega)$ and $b^+ \not\equiv 0$, it is well known that there exists a unique positive principal eigenvalue for the above problem, which we wiill denote by $\lambda_1(b)$. For a proof of this fact and for additional properties of principal eigenvalues and their associated principal eigenfunctions see, for instance [15].

Remark 2.1. (See [22], Theorem 1.2, or, in a more general setting, [25], Theorem 1.2) Let $\beta \in (0, 3)$, $\widetilde{a} : \Omega \to \mathbb{R}$ and $f : \Omega \times [0, \infty) \to \mathbb{R}$; and assume the following conditions H1)-H6):
H1) $0 \leq \widetilde{a} \in L^\infty(\Omega)$, and $\widetilde{a} \not\equiv 0$,
H2) f is a Carathéodory function on $\Omega \times [0, \infty)$,
H3) $\sup_{0\leq s\leq M} |f(., s)| \in L^1(\Omega)$ for any $M > 0$,
H4) One of the two following conditions holds:
H4') $\sup_{s>0} \frac{f(.,s)}{s} \leq b$ for some $b \in L^\infty(\Omega)$ such that $b^+ \not\equiv 0$, and $\lambda_1(b) > m$ for some integer $m \geq \max\{2, 1+\beta\}$,
H4") $f \in L^\infty(\Omega \times (0, \sigma))$ for all $\sigma > 0$, and $\overline{\lim}_{s\to\infty} \frac{f(.,s)}{s} \leq 0$ uniformly on Ω, i.e., for any $\varepsilon > 0$ there exists $s_0 > 0$ such that $\sup_{s\geq s_0} \frac{f(.,s)}{s} \leq \varepsilon$, *a.e.* in Ω,

H5) $f(.,0) \geq 0.$
Then the problem

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}\widetilde{a}u^{-\beta} + f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega, \ u \not\equiv 0 \text{ in } \Omega. \end{cases} \tag{2.1}$$

has a weak solution (in the sense of Definition 1.1) $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $u > 0$ *a.e.* in $\{\widetilde{a} > 0\}$.

Lemma 2.2. *Let $a \in L^\infty(\Omega)$ be such that $a \geq 0$ in Ω and $a \not\equiv 0$, let $\alpha \in (0,1]$, and let $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ be a family of functions defined on $\Omega \times [0,\infty)$ satisfying the conditions h7)-h9) stated at the introduction. Then, for any $\varepsilon \in (0,1]$, problem (1.7) has at least a weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, in the sense of Definition 1.1, such that $u > 0$ a.e. in $\{a > 0\}$.*

Proof. Notice that, since g_ε is a Carathéodory function, we have $g_\varepsilon(.,0) = \lim_{s\to 0^+} g_\varepsilon(.,s) = \lim_{s\to 0^+}\left(ss^{-1}g_\varepsilon(.,s)\right) = 0$, the last inequality by *h8).* Thus $g_\varepsilon(.,0) = 0$. Taking into account this fact and h7)-h9), the lemma follows immediately from Remark 2.1. □

Let us recall, in the next remark, the uniform Hopf maximum principle:

Remark 2.3. i) (see [2], Lemma 3.2) Suppose that $0 \leq h \in L^\infty(\Omega)$; and let $v \in \cap_{1\leq p<\infty}\left(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right)$ be the strong solution of $-\Delta v = h$ in Ω, $v = 0$ on $\partial\Omega$. Then $v \geq cd_\Omega \int_\Omega hd_\Omega$ *a.e.* in Ω, where $d_\Omega := dist(.,\partial\Omega)$, and c is a positive constant depending only on Ω.
ii) (see e.g., [25], Remark 8) Let Ψ be a nonnegative function in $L^1_{loc}(\Omega)$, and let v be a function in $H_0^1(\Omega)$ such that $-\Delta v \geq \Psi$ on Ω in the sense of distributions. Then

$$v(x) \geq cd_\Omega \int_\Omega \Psi d_\Omega \qquad a.e. \text{ in } \Omega, \tag{2.2}$$

where c is a positive constant depending only on Ω.

Remark 2.4. (See, e.g., [23], Lemmas 2.9, 2.10 and 2.12) Let $a \in L^\infty(\Omega)$ be such that $a \geq 0$ in Ω and $a \not\equiv 0$, and let let $\alpha \in (0,1]$. Then the problem

$$\begin{cases} -\Delta z = az^{-\alpha} \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z \geq 0 \text{ in } \Omega. \end{cases} \tag{2.3}$$

has a unique weak solution, in the sense of Definition 1.1, $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover:
i) $z \in C\left(\overline{\Omega}\right)$.
ii) There exists positive constants c_1, c_2 and $\tau > 0$ such that $c_1 d_\Omega \leq z \leq c_2 d_\Omega^\tau$ in Ω.
iii) z is a solution of problem (2.3) in the usual weak sense, i.e., for any $\varphi \in H_0^1(\Omega)$, $az^{-\alpha}\varphi \in L^1(\Omega)$ and $\int_\Omega \langle\nabla z, \nabla\varphi\rangle = \int_\Omega az^{-\alpha}\varphi$.

Lemma 2.5. *Let a, α, and $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ be as in Lemma 2.2, let z be as given in Remark 2.4; and let $\varepsilon \in (0,1]$. If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution, in the sense of Definition 1.1, of problem (1.7), then $u \leq z$ a.e. in Ω.*

Proof. By *h5)*, $g_\varepsilon(.,u) \geq 0$ and so, from Lemma 2.2 and Remark 2.4, we have, in the sense of Definition 1.1,

$$-\Delta(u-z) = au^{-\alpha} - g_\varepsilon(.,u) - az^{-\alpha} \leq a\left(u^{-\alpha} - z^{-\alpha}\right) \text{ in } \Omega,$$

Thus, taking $(u-z)^+$ as a test function, we get

$$\int_\Omega \left|\nabla(u-z)^+\right|^2 \leq \int_\Omega a\left(u^{-\alpha} - z^{-\alpha}\right)(u-z)^+ \leq 0$$

which implies $u \leq z$ *a.e. in* Ω. □

Lemma 2.6. *Let a, α, and $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ be as in Lemma 2.2. If $\varepsilon \in (0,1]$ and $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution, in the sense of Definition 1.1, of problem (1.7), then:*
i) There exists a positive constant c_1 (which may depend on ε) and constants c_2 and τ such that $c_1 d_\Omega \leq u \leq c_2 d_\Omega^\tau$ a.e. in Ω (and so, in particular, $u > 0$ in Ω).
ii) For any $\varphi \in H_0^1(\Omega)$ we have $(au^{-\alpha} - g_\varepsilon(.,u))\varphi \in L^1(\Omega)$ and

$$\int_\Omega \langle \nabla u, \nabla \varphi\rangle = \int_\Omega \left(au^{-\alpha} - g_\varepsilon(.,u)\right)\varphi,$$

i.e., u is a weak solution, in the usual sense of $H_0^1(\Omega)$, to the problem $-\Delta u = au^{-\alpha} - g_\varepsilon(.,u)$ in Ω, $u = 0$ on $\partial\Omega$.

Proof. We have, in the weak sense of Definition 1.1, $-\Delta u = \chi_{\{u>0\}} au^{-\alpha} - g_\varepsilon(.,u)$ in Ω, $u = 0$ on $\partial\Omega$. Also, $u \geq 0$ in Ω and $u \not\equiv 0$ in Ω. Let $a_0 : \Omega \to \mathbb{R}$ be defined by $a_0(x) = u^{-1}(x) g_\varepsilon(x, u(x))$ if $u(x) \neq 0$ and by $a_0(x) = 0$ otherwise. Since $u \in L^\infty(\Omega)$ and taking into account *h7)* and *h8)*, we have $0 \leq a_0 \in L^\infty(\Omega)$, and from the definition of a_0 we have $g_\varepsilon(.,u) = a_0 u$ *a.e.* in Ω. Therefore u satisfies, in the sense of Definition 1.1, $-\Delta u + a_0 u = \chi_{\{u>0\}} au^{-\alpha}$ in Ω, $u = 0$ on $\partial\Omega$. Thus, since u is nonidentically zero, it follows that $\chi_{\{u>0\}} au^{-\alpha}$ is nonidentically zero on Ω. Then there exist $\eta > 0$, and a measurable set $E \subset \Omega$, such that $|E| > 0$ and $\chi_{\{u>0\}} au^{-\alpha} \geq \eta \chi_E$ in Ω. Let $\psi \in \cap_{1\leq q<\infty} W^{2,,q}(\Omega) \cap W_0^{1,,q}(\Omega)$ be the solution of the problem $-\Delta\psi + a_0\psi = \eta\chi_E$ in Ω, $\psi = 0$ on $\partial\Omega$. By the Hopf maximum principle (as stated, e.g., in [34], Theorem 1.1) there exists a positive constant c_1 such that $\psi \geq c_1 d_\Omega$ in Ω; and, from (1.7) we have $-\Delta u + a_0 u \geq \eta\chi_E$ in $D'(\Omega)$. Then, by the weak maximum principle (as stated, e.g., in [20], Theorem 8.1), $u \geq \psi$ in Ω. Hence $u \geq c_1 d_\Omega$ in Ω. Also, by Lemma 2.5, $u \leq z$ *a.e.* in Ω, and so Remark 2.4 gives positive constants c_2 and τ (both independent of ε) such that $u \leq c_2 d_\Omega^\tau$ in Ω. Thus *i)* holds.
To see *ii)*, consider an arbitrary function $\varphi \in H_0^1(\Omega)$, and for $k \in \mathbb{N}$, let $\varphi_k^+ := \max\{k, \varphi^+\}$. Thus $\varphi_k^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\left\{\varphi_k^+\right\}_{k\in\mathbb{N}}$ converges to φ^+ in $H_0^1(\Omega)$ and, after pass to some subsequence if necessary, we can assume also that $\left\{\varphi_k^+\right\}_{k\in\mathbb{N}}$ converges to φ^+ *a.e.* in Ω. Since u is a weak solution, in the sense of Definition 1.1, of (1.7) and $u > 0$ *a.e.* in Ω, we have, for all $k \in \mathbb{N}$, $(au^{-\alpha} - g_\varepsilon(.,u))\varphi_k^+ \in L^1(\Omega)$, and, by *h6)*, $g_\varepsilon(.,u) \in L^\infty(\Omega)$. Thus $g_\varepsilon(.,u)\varphi_k^+ \in L^1(\Omega)$. Then $au^{-\alpha}\varphi_k^+ \in L^1(\Omega)$.
From (1.7),

$$\int_\Omega \langle \nabla u, \nabla \varphi_k^+\rangle + \int_\Omega g_\varepsilon(.,u)\varphi_k^+ = \int_\Omega au^{-\alpha}\varphi_k^+. \tag{2.4}$$

Now, $\lim_{k\to\infty} \int_\Omega \left\langle \nabla u, \nabla\varphi_k^+\right\rangle = \int_\Omega \langle \nabla u, \nabla\varphi^+\rangle$. Also, for any k,

$$0 \leq g_\varepsilon(.,u)\varphi_k^+ \leq \sup_{s\in\left[0,\|u\|_\infty\right]} g_\varepsilon(.,s)\varphi^+ \in L^1(\Omega),$$

then, by the Lebesgue dominated convergence theorem, $\lim_{k\to\infty}\int_\Omega g_\varepsilon(.,u)\varphi_k^+ = \int_\Omega g_\varepsilon(.,u)\varphi^+ < \infty$. Hence, by (2.4), $\lim_{k\to\infty}\int_\Omega au^{-\alpha}\varphi_k^+$ exists and is finite. Since $\left\{au^{-\alpha}\varphi_k^+\right\}_{k\in\mathbb{N}}$ is nondecreasing and converges to $au^{-\alpha}\varphi^+$ *a.e.* in Ω, the monotone convergence theorem gives $\lim_{k\to\infty}\int_\Omega au^{-\alpha}\varphi_k^+ = \int_\Omega au^{-\alpha}\varphi^+ < \infty$. Thus

$$(au^{-\alpha} - g_\varepsilon(.,u))\,\varphi^+ \in L^1(\Omega)$$

and

$$\int_\Omega \langle \nabla u, \nabla\varphi^+\rangle + \int_\Omega g_\varepsilon(.,u)\,\varphi^+ = \int_\Omega au^{-\alpha}\varphi^+. \tag{2.5}$$

Similarly, we have that $(au^{-\alpha} - g_\varepsilon(.,u))\,\varphi^- \in L^1(\Omega)$, and that (2.5) holds with φ^+ replaced by φ^- By writing $\varphi = \varphi^+ - \varphi^-$ the lemma follows. □

Lemma 2.7. *Let a, α, and $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ be as in Lemma 2.2. For any $\varepsilon \in (0,1]$, if $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is a weak solution, in the sense of Definition 1.1 (and so, by Lemma 2.6, also in the usual sense of $H_0^1((\Omega))$), of problem (1.7), then $u \in C\left(\overline{\Omega}\right)$.*

Proof. By Lemma 2.6 we have $u \geq cd_\Omega$ *a.e.* in Ω, with c a positive constant and, by h6), $0 \leq u^{-1}g_\varepsilon(.,u) \in L^\infty(\Omega)$. Thus $au^{-\alpha} - g_\varepsilon(.,u) \in L^\infty_{loc}(\Omega)$. Also, $u \in L^\infty(\Omega)$. Then, by the inner elliptic estimates (as stated, e.g., in [20], Theorem 8.24), $u \in W^{2,p}_{loc}(\Omega)$ for any $p \in [1,\infty)$. Thus $u \in C(\Omega)$, and, since $0 \leq u \leq z$, $z \in C\left(\overline{\Omega}\right)$ and $z = 0$ on $\partial\Omega$, it follows that u is also continuous at $\partial\Omega$. □

Definition 2.8. Let $C_0^\infty\left(\overline{\Omega}\right) := \left\{\varphi \in C^\infty\left(\overline{\Omega}\right) : \varphi = 0 \text{ on } \partial\Omega\right\}$. If $u \in L^1(\Omega)$ and $h \in L^1(\Omega)$, we will say that u is a solution, in the sense of $\left(C_0^\infty\left(\overline{\Omega}\right)\right)'$, of the problem $-\Delta u = h$ in Ω, $u = 0$ on $\partial\Omega$, if $-\int_\Omega u\Delta\varphi = \int_\Omega h\varphi$ for any $\varphi \in C_0^\infty\left(\overline{\Omega}\right)$.
We will say also that $-\Delta u \geq h$ in $\left(C_0^\infty\left(\overline{\Omega}\right)\right)'$ (respectively $-\Delta u \leq h$ in $\left(C_0^\infty\left(\overline{\Omega}\right)\right)'$) if $-\int_\Omega u\Delta\varphi \geq \int_\Omega h\varphi$ (resp. $-\int_\Omega u\Delta\varphi \leq \int_\Omega h\varphi$) for any nonnegative $\varphi \in C_0^\infty\left(\overline{\Omega}\right)$.

Remark 2.9. The following statements hold:
i) (Maximum principle, [31], Proposition 5.1) If $u \in L^1(\Omega)$, $0 \leq h \in L^1(\Omega)$, and $-\Delta u \geq h$ in the sense of $\left(C_0^\infty\left(\overline{\Omega}\right)\right)'$, then $u \geq 0$ *a.e.* in Ω.
ii) (Kato's inequality, [31], Proposition 5.7) If $h \in L^1(\Omega)$, $u \in L^1(\Omega)$ and if $-\Delta u \leq h$ in $D'(\Omega)$, then $-\Delta(u^+) \leq \chi_{\{u>0\}}h$ in $D'(\Omega)$.
iii) ([31], Proposition 3.5) For $\varepsilon > 0$, let $A_\varepsilon := \{x \in \Omega : dist(x,\partial\Omega) < \varepsilon\}$. If $h \in L^1(\Omega)$ and if $u \in L^1(\Omega)$ is a solution of $-\Delta u = h$, in the sense of Definition 2.8, then there exists a constant c such that, for all $\varepsilon > 0$, $\int_{A_\varepsilon}|u| \leq c\varepsilon^2\|h\|_1$. In particular, $\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{A_\varepsilon}|u| = 0$.
iv) ([31], Proposition 5.2) Let $u \in L^1(\Omega)$ and $h \in L^1(\Omega)$. If $-\Delta u \leq h$ (respectively $-\Delta u = h$) in $D'(\Omega)$ and $\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{A_\varepsilon}|u| = 0$ then $-\Delta u \leq h$ (resp. $-\Delta u = h$) in the sense of $\left(C_0^\infty\left(\overline{\Omega}\right)\right)'$.
v) ([31], Proposition 5.9) Let $f_1, f_2 \in L^1(\Omega)$. If $u_1, u_2 \in L^1(\Omega)$ are such that $\Delta u_1 \geq f_1$ and $\Delta u_2 \geq f_2$ in the sense of distributions in Ω, then $\Delta\max\{u_1,u_2\} \geq \chi_{\{u_1>u_2\}}f_1 + \chi_{\{u_2>u1\}}f_2 + \chi_{\{u_1=u_2\}}\frac{1}{2}(f_1+f_2)$ in the sense of distributions in Ω.

If $h : \Omega \to \mathbb{R}$ is a measurable function such that $h\varphi \in L^1(\Omega)$ for any $\varphi \in C_c^\infty(\Omega)$, we say that $u : \Omega \to \mathbb{R}$ is a subsolution (respectively a supersolution), in the sense of distributions, of the problem $-\Delta u = h$ in Ω, if $u \in L^1_{loc}(\Omega)$ and $-\int_\Omega u\Delta\varphi \leq \int_\Omega h\varphi$ (resp. $-\int_\Omega u\Delta\varphi \geq \int_\Omega h\varphi$) for any nonnegative $\varphi \in C_c^\infty(\Omega)$.

Remark 2.10. ([28], Theorem 2.4) Let $f : \Omega \times (0, \infty) \to \mathbb{R}$ be a Caratheodory function, and let $\underline{w}$ and $\overline{w}$ be two functions, both in $L^\infty_{loc}(\Omega) \cap W^{1,2}_{loc}(\Omega)$, and such that $f\left(., \underline{w}\right)$ and $f(., \overline{w})$ belong to $L^1_{loc}(\Omega)$. Suppose that $\underline{w}$ is a subsolution and $\overline{w}$ is a supersolution, both in the sense of distributions, of the problem

$$-\Delta w = f(., w) \text{ in } \Omega. \tag{2.6}$$

Suppose in addition that $0 < \underline{w}(x) \le \overline{w}(x)$ *a.e.* $x \in \Omega$, and that there exists $h \in L^\infty_{loc}(\Omega)$ such that $\sup_{s \in [\underline{w}(x), \overline{w}(x)]} |f(x, s)| \le h(x)$ *a.e.* $x \in \Omega$. Then (2.6) has a solution w, in the sense of distributions, which satisfies $\underline{w} \le w \le \overline{w}$ *a.e.* in Ω. Moreover, as observed in [28], if in addition $f(., w) \in L^\infty_{loc}(\Omega)$, then, by a density argument, the equality $\int_\Omega \langle \nabla w, \nabla \varphi \rangle = \int_\Omega f(., w)\varphi$ holds also for any $\varphi \in W^{1,2}_{loc}(\Omega)$ with compact support.

Remark 2.11. Let us recall the Hardy inequality (as stated, e.g., in [29], Theorem 1.10.15, see also [1], p. 313): There exists a positive constant c such that $\left\| \frac{\varphi}{d_\Omega} \right\|_{L^2(\Omega)} \le c \|\nabla \varphi\|_{L^2(\Omega)}$ for all $\varphi \in H^1_0(\Omega)$.

Remark 2.12. Let a and $\{g_\varepsilon\}_{\varepsilon \in (0,1]}$ be as in Lemma 2.2 and assume that $\alpha \in (0, 1]$. Let $\varepsilon \in (0, 1]$. If $u \in L^\infty(\Omega)$ and if, for some positive constant c, $u \ge c d_\Omega$ *a.e.* in Ω, then $au^{-\alpha} - g_\varepsilon(., u) \in \left(H^1_0(\Omega)\right)'$. Indeed, for $\varphi \in H^1_0(\Omega)$ we have $|au^{-\alpha}\varphi| \le c^{-\alpha} d_\Omega^{1-\alpha} \left|\frac{\varphi}{d_\Omega}\right|$. Since $d_\Omega^{1-\alpha} \in L^\infty(\Omega)$ (because $\alpha \le 1$), the Hardy inequality gives a positive constant c' independent of φ such that $\|au^{-\alpha}\varphi\|_1 \le c' \|\nabla\varphi\|_2$. Also, since $u \in L^\infty(\Omega)$, from *h6)* and the Hardy inequality, $\|g_\varepsilon(., u)\varphi\|_1 \le c'' \|\nabla\varphi\|_2$, with c'' a positive constant independent of φ.

Lemma 2.13. *Let a and $\{g_\varepsilon\}_{\varepsilon \in (0,1]}$ be as in Lemma 2.2 and assume that $\alpha \in (0, 1]$. Let $\varepsilon \in (0, 1]$. Suppose that $u \in W^{1,2}_{loc}(\Omega) \cap L^\infty(\Omega)$ is a solution, in the sense of distributions, of the problem*

$$-\Delta u = au^{-\alpha} - g_\varepsilon(., u) \text{ in } \Omega, \tag{2.7}$$

and that there exist positive constants c, c' and γ such that $c' d_\Omega \le u \le c d_\Omega^\gamma$ a.e. in Ω. Then $u \in H^1_0(\Omega) \cap C\left(\overline{\Omega}\right)$, and u is a weak solution, in the usual sense of $H^1_0(\Omega)$, of problem (1.6).

Proof. Since $u \in L^\infty(\Omega)$ and $u \ge c' d_\Omega$, we have $au^{-\alpha} - g_\varepsilon(., u) \in L^\infty_{loc}(\Omega)$. Thus, from the inner elliptic estimates in ([20], Theorem 8.24), $u \in C(\Omega)$ and, from the inequalities $c' d_\Omega \le u \le c d_\Omega^\gamma$ *a.e.* in Ω, u is also continuous on $\partial\Omega$. Then $u \in C\left(\overline{\Omega}\right)$

The proof of that $u \in H^1_0(\Omega)$ and that u is a weak solution, in the usual sense of $H^1_0(\Omega)$, of problem (1.6), is a slight variation of the proof of ([24], Lemma 2.4). For the convenience of the reader, we give the details: For $j \in \mathbb{N}$, let $h_j : \mathbb{R} \to \mathbb{R}$ be the function defined by $h_j(s) := 0$ if $s \le \frac{1}{j}$, $h_j(s) := -3j^2 s^3 + 14 j s^2 - 19 s + \frac{8}{j}$ if $\frac{1}{j} < s < \frac{2}{j}$ and $h(s) = s$ for $\frac{2}{j} \le s$. Then $h_j \in C^1(\mathbb{R})$, $h'_j(s) = 0$ for $s < \frac{1}{j}$, $h'_j(s) \ge 0$ for $\frac{1}{j} < s < \frac{2}{j}$ and $h'_j(s) = 1$ for $\frac{2}{j} \le s$. Moreover, for $s \in \left(\frac{1}{j}, \frac{2}{j}\right)$ we have $s^{-1} h_j(s) = -3j^2 s^2 + 14 j s - 19 + \frac{8}{js} < -3j^2 s^2 + 14 j s - 11 < 5$ (the last inequality because $-3t^2 + 14t - 16 < 0$ whenever $t \notin \left[\frac{8}{3}, 2\right]$). Thus $0 \le h_j(s) \le 5s$ for any $j \in \mathbb{N}$ and $s \ge 0$.

Let $h_j(u) := h_j \circ u$. Then, for all j, $\nabla\left(h_j(u)\right) = h'_j(u)\nabla u$. Since $u \in W^{1,2}_{loc}(\Omega)$, we have $h_j(u) \in W^{1,2}_{loc}(\Omega)$, and since $h_j(u)$ has compact support, Remark 2.10 gives, for all $j \in \mathbb{N}$, $\int_\Omega \left\langle \nabla u, \nabla\left(h_j(u)\right)\right\rangle = \int_\Omega (au^{-\alpha} - g_\varepsilon(., u)) h_j(u)$, i.e.,

$$\int_{\{u>0\}} h'_j(u) |\nabla u|^2 = \int_\Omega (au^{-\alpha} - g_\varepsilon(., u)) h_j(u). \tag{2.8}$$

Now, $h_j'(u)|\nabla u|^2$ is a nonnegative function and $\lim_{j\to\infty} h_j'(u)|\nabla u|^2 = |\nabla u|^2$ *a.e.* in Ω, and so, by (2.8) and the Fatou's lemma,

$$\int_\Omega |\nabla u|^2 \le \underline{\lim}_{j\to\infty} \int_\Omega (au^{-\alpha} - g_\varepsilon(.,u))\, h_j(u).$$

Also,

$$\lim_{j\to\infty} (au^{-\alpha} - g_\varepsilon(.,u))\, h_j(u) = au^{1-\alpha} - ug_\varepsilon(.,u) \;\; a.e. \text{ in } \Omega.$$

Now, $0 \le au^{-\alpha}h_j(u) \le 5au^{1-\alpha}$. Since a and u belong to $L^\infty(\Omega)$ and $\alpha \le 1$, we have $au^{1-\alpha} \in L^1(\Omega)$. Also,

$$0 \le g_\varepsilon(.,u)\, h_j(u) \le 5ug_\varepsilon(.,u) \le 5\|u\|_\infty^2 \sup_{0<s\le\|u\|_\infty} s^{-1} g_\varepsilon(.,s) \;\; a.e. \text{ in } \Omega,$$

and, by *h6)*, $\sup_{0<s\le\|u\|_\infty} s^{-1}g_\varepsilon(.,s) \in L^\infty(\Omega)$. Then, by the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty} \int_\Omega (au^{-\alpha} - g_\varepsilon(.,u))\, h_j(u) = \int_\Omega \left(au^{1-\alpha} - ug_\varepsilon(.,u)\right) < \infty.$$

Thus $\int_\Omega |\nabla u|^2 < \infty$, and so $u \in H^1(\Omega)$. Since $u \in C\left(\overline{\Omega}\right)$ and $u = 0$ on $\partial\Omega$, we conclude that $u \in H_0^1(\Omega)$. Also, by Remark 2.12, $au^{-\alpha} - g_\varepsilon(.,u) \in \left(H_0^1(\Omega)\right)'$. Then, by a density argument, the equality

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle = \int_\Omega (au^{-\alpha} - g_\varepsilon(.,u))\, \varphi$$

which holds for $\varphi \in C_c^\infty(\Omega)$, holds also for any $\varphi \in H_0^1(\Omega)$. □

Lemma 2.14. *Let a, α, and $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ be as in Lemma 2.2. Let $\varepsilon \in (0,1]$ and let $f_\varepsilon : \Omega \times [0,\infty) \to \mathbb{R}$ be defined by $f_\varepsilon(.,s) := \chi_{(0,\infty)}(s)\, as^{-\alpha} - g_\varepsilon(.,s)$. Let v_1 and v_2 be two nonnegative functions in $L^\infty(\Omega) \cap H_0^1(\Omega)$ such that $f_\varepsilon(.,v_i) \in L^1_{loc}(\Omega)$ for $i = 1,2$; and let $v := \max\{v_1, v_2\}$. Then:*
i) $f_\varepsilon(.,v) \in L^1_{loc}(\Omega)$.
ii) If v_1 and v_2 are subsolutions, in the sense of distributions, to problem (1.7), then v is also a subsolution, in the sense of distributions, to the problem

$$-\Delta u = \chi_{\{u>0\}} au^{-\alpha} - g_\varepsilon(.,u) \;\; in \; \Omega.$$

Proof. Since $0 \le v \in L^\infty(\Omega)$, from *h7)* and *h8)* it follows that $g_\varepsilon(.,v) \in L^1(\Omega)$. Similarly, $g_\varepsilon(.,v_1)$ and $g_\varepsilon(.,v_2)$ belong to $L^1(\Omega)$ and so, since $f_\varepsilon(.,v_i) \in L^1_{loc}(\Omega)$ for $i = 1,2$; we get that $\chi_{\{v_1>0\}} av_1^{-\alpha}$ and $\chi_{\{v_2>0\}} av_2^{-\alpha}$ belong to $L^1_{loc}(\Omega)$. Therefore, to prove *i)* it suffices to see that $\chi_{\{v>0\}} av^{-\alpha} \in L^1_{loc}(\Omega)$. Note that if $x \in \Omega$ and $v(x) > 0$ then either $v_1(x) > 0$ or $v_2(x) > 0$. Now, $\chi_{\{v>0\}} av^{-\alpha} = av^{-\alpha} \le av_1^{-\alpha} = \chi_{\{v_1>0\}} av_1^{-\alpha}$ in $\{v_1 > 0\}$, and similarly, $\chi_{\{v>0\}} av^{-\alpha} \le \chi_{\{v_2>0\}} av_2^{-\alpha}$ in $\{v_2 > 0\}$. Also, $\chi_{\{v>0\}} av^{-\alpha} = 0$ in $\{v = 0\}$. Then $\chi_{\{v>0\}} av^{-\alpha} \le \chi_{\{v_1>0\}} av_1^{-\alpha} + \chi_{\{v_2>0\}} av_2^{-\alpha}$ in Ω and so $\chi_{\{v>0\}} av^{-\alpha} \in L^1_{loc}(\Omega)$. Thus *i)* holds.
To see *ii)*, suppose that $-\Delta v_i \le f_\varepsilon(.,v_i)$ in $D'(\Omega)$ for $i = 1,2$; and let φ be a nonnegative function in $C_c^\infty(\Omega)$. Let Ω' be a $C^{1,1}$ subdomain of Ω, such that $supp(\varphi) \subset \Omega'$ and $\overline{\Omega'} \subset \Omega$. Consider the restrictions (still denoted by v_1 and v_2) of v_1 and v_2 to Ω'. For each $i = 1,2$, we have $v_i \in L^1(\Omega')$, $f_\varepsilon(.,v_i) \in L^1(\Omega')$ and $-\Delta v_i \le f_\varepsilon(.,v_i)$ in $D'(\Omega')$. Thus, from Remark 2.9 *v)*,

$$\begin{aligned} -\Delta v &\le \chi_{\{v_1>v_2\}} f_\varepsilon(.,v_1) + \chi_{\{v_2>v_1\}} f_\varepsilon(.,v_2) + \chi_{\{v_1=v_2\}} \frac{1}{2}\left(f_\varepsilon(.,v_1) + f_\varepsilon(.,v_2)\right) \\ &= f_\varepsilon(.,v) \text{ in } D'(\Omega') \end{aligned}$$

and then $-\int_\Omega v\Delta\varphi \le \int_\Omega f_\varepsilon(.,v)\,\varphi$. □

Lemma 2.15. *Let a, α, and $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ be as in Lemma 2.2. Then for any $\varepsilon \in (0,1]$ there exists a weak solution u_ε, in the sense of Definition 1.1, of problem (1.7), which is maximal in the following sense: If v is a weak solution, in the sense of Definition 1.1, of problem (1.7), then $v \leq u_\varepsilon$ a.e. in Ω. Moreover, u_ε is a solution, in the usual sense of $H_0^1(\Omega)$, of problem (1.7).*

Proof. Let z be as given in Remark 2.4, and let $\mathcal{S}$ be the set of the nonidentically zero weak solutions, in the sense of Definition 1.1, of problem (1.7). By Lemma 2.2, $\mathcal{S} \neq \emptyset$ and, for any $u \in \mathcal{S}$, by Lemma 2.5 we have $u \leq z$ in Ω and, by Lemma 2.6, there exists a positive constant c such that $u \geq cd_\Omega$ in Ω. Then $0 < \int_\Omega u \leq \int_\Omega z < \infty$ for any $u \in \mathcal{S}$. Let $\beta := \sup\left\{\int_\Omega u : u \in \mathcal{S}\right\}$. Thus $0 < \beta < \infty$. Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{S}$ be a sequence such that $\lim_{k\to\infty}\int_\Omega u_k = \beta$. For $k \in \mathbb{N}$, let $w_k := \max\left\{u_j : 1 \leq j \leq k\right\}$. Thus $\{w_k\}_{k\in\mathbb{N}}$ is a nondecreasing sequence in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and a repeated use of Lemma 2.14 gives that each w_k is a subsolution, in the sense of $D'(\Omega)$, of the problem

$$-\Delta u = au^{-\alpha} - g_\varepsilon(.,u) \text{ in } \Omega. \tag{2.9}$$

Since $w_k \in L^\infty(\Omega)$ and $w_k \geq u_1 \geq c_1 d_\Omega$ *a.e.* in Ω, Remark 2.12 gives that $aw_k^{-\alpha} - g_\varepsilon(.,w_k) \in \left(H_0^1(\Omega)\right)'$. Then, by a density argument, the inequality

$$\int_\Omega \langle\nabla w_k, \nabla\varphi\rangle \leq \int_\Omega \left(aw_k^{-\alpha} - g_\varepsilon(.,w_k)\right)\varphi, \tag{2.10}$$

which holds for $\varphi \in C_c^\infty(\Omega)$, holds also for any $\varphi \in H_0^1(\Omega)$, i.e., w_k is a subsolution, in the usual sense of $H_0^1(\Omega)$, of problem (2.9)

Note that $\left\{\int_{\{a>0\}} aw_k^{1-\alpha}\right\}_{k\in\mathbb{N}}$ is bounded. Indeed, since $u_k \leq z$ *a.e.* in Ω for any $k \in \mathbb{N}$, we have $w_k \leq z$ *a.e.* in Ω for all k, and so $\int_{\{a>0\}} aw_k^{1-\alpha} \leq \int_\Omega az^{1-\alpha} < \infty$. Moreover, $\{w_k\}_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. In fact, taking w_k as a test function in (2.10) we get, for any $k \in \mathbb{N}$,

$$\int_\Omega |\nabla w_k|^2 + \int_\Omega g_\varepsilon(.,w_k)\,w_k \leq \int_{\{a>0\}} aw_k^{1-\alpha} \tag{2.11}$$

Then, after pass to a subsequence if necessary, we can assume that there exists $w \in H_0^1(\Omega)$ such that $\{w_k\}_{k\in\mathbb{N}}$ converges in $L^2(\Omega)$ and *a.e.* in Ω to w; and $\{\nabla w_k\}_{k\in\mathbb{N}}$ converges weakly in $L^2(\Omega,\mathbb{R}^n)$ to ∇w. Let us show that w is a subsolution, in the sense of distributions of problem (2.9). Let φ be a nonnegative function in $C_c^\infty(\Omega)$ and let $k \in \mathbb{N}$. Since w_k is a subsolution, in the sense of distributions, of (2.9), we have

$$\int_\Omega \langle\nabla w_k, \nabla\varphi\rangle + \int_\Omega g_\varepsilon(.,w_k)\,\varphi \leq \int_\Omega aw_k^{-\alpha}\varphi. \tag{2.12}$$

Since $\{\nabla w_k\}_{k\in\mathbb{N}}$ converges weakly in $L^2(\Omega,\mathbb{R}^n)$ to ∇w, we have

$$\lim_{k\to\infty}\int_\Omega \langle\nabla w_k, \nabla\varphi\rangle = \int_\Omega \langle\nabla w, \nabla\varphi\rangle.$$

Also, since $\{g_\varepsilon(.,w_k)\,\varphi\}_{k\in\mathbb{N}}$ converges to $g_\varepsilon(.,w)\,\varphi$ *a.e.* in Ω, and

$$|g_\varepsilon(.,w_k)\,\varphi| \leq \sup_{s\in\left[0,\|z\|_\infty\right]}\left(s^{-1}g_\varepsilon(.,s)\right)w_k\,|\varphi| \in L^1(\Omega),$$

the Lebesgue dominated convergence theorem gives

$$\lim_{k\to\infty}\int_{\Omega} g_{\varepsilon}(.,w_k)\varphi = \int_{\Omega} g_{\varepsilon}(.,w)\varphi.$$

On the other hand, $\left\{aw_k^{-\alpha}\varphi\right\}_{k\in\mathbb{N}}$ converges to $aw^{-\alpha}\varphi$ *a.e.* in Ω; and $w_k \geq u_1 \geq cd_{\Omega}$ *a.e.* in Ω, and so $\left|aw_k^{-\alpha}\varphi\right| \leq c^{-\alpha}ad_{\Omega}^{1-\alpha}\left|d_{\Omega}^{-1}\varphi\right|$ *a.e.* in Ω; and, since $d_{\Omega}^{1-\alpha} \in L^{\infty}(\Omega)$, the Hardy inequality gives that $ad_{\Omega}^{1-\alpha}\left|d_{\Omega}^{-1}\varphi\right| \in L^1(\Omega)$. Then, by the Lebesgue dominated convergence theorem, $\lim_{k\to\infty}\int_{\Omega} aw_k^{-\alpha}\varphi = \int_{\Omega} aw^{-\alpha}\varphi < \infty$. Hence, from (2.12),

$$\int_{\Omega}\langle\nabla w,\nabla\varphi\rangle + \int_{\Omega} g_{\varepsilon}(.,w)\varphi \leq \int_{\Omega} aw^{-\alpha}\varphi,$$

and so w is a subsolution, in the sense of distributions to problem (2.9). Note that z is a supersolution, in the sense of distributions, of problem (2.9) and that $w \leq z$ *a.e.* in Ω (because $u_k \leq z$ for all $k \in \mathbb{N}$). Also, for some positive constant c and for any k, $w \geq w_k \geq u_1 \geq cd_{\Omega}$ *a.e.* in Ω. Then there exists a positive constant c' such that

$$\sup_{s\in[w(x),z(x)]}\left(\chi_{\{s>0\}}a(x)s^{-\alpha} - g_{\varepsilon}(x,s)\right) \leq c'd_{\Omega}^{-\alpha} \text{ for } a.e\ x\in\Omega$$

and so, by Remark 2.10, there exists a solution $u_{\varepsilon} \in W_{loc}^{1,2}(\Omega)$, in the sense of distributions, of (2.9) such that $w \leq u_{\varepsilon} \leq z$ *a.e. a.e.* in Ω. Therefore, by Remark 2.4, $cd_{\Omega} \leq u_{\varepsilon} \leq c'd_{\Omega}^{\tau}$ *a.e.* in Ω, with c, c' and τ positive constants. Then, by Lemma 2.13, $u_{\varepsilon} \in H_0^1(\Omega)\cap C\left(\overline{\Omega}\right)$ and u_{ε} is a weak solution, in the sense of Definition 1.1, of problem (1.7). Also, $u_{\varepsilon} \geq w \geq w_k \geq u_k$ *a.e.* in Ω for any $k \in \mathbb{N}$, and so $\int_{\Omega} u_{\varepsilon} \geq \beta$ which, by the definition of β, implies $\int_{\Omega} u_{\varepsilon} = \beta$.

Let us show that u_{ε} is the maximal solution of problem (1.7), in the sense required by the lemma. Suppose that w^* is a nonidentically zero weak solution, in the sense of Definition 1.1, of (1.7). By Lemmas 2.5, 2.7 and 2.6, $w^* \leq z$ in Ω, $w^* \in C\left(\overline{\Omega}\right)$ and $w^* \geq cd_{\Omega}$ *a.e.* in Ω with c a positive constant c. Let $w^{**} := \max\{u_{\varepsilon}, w^*\}$. Thus w^{**} is a subsolution, in the sense of distributions, of problem (2.9), Remark 2.10 applies to obtain a solution $\widetilde{w}$, in the sense of distributions, of problem (1.7), such that $w^{**} \leq \widetilde{w} \leq z$, and Lemma 2.13 applies to obtain that $\widetilde{w} \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$ and that $\widetilde{w}$ is a weak solution, in the sense of Definition 1.1, to problem (1.7). Then $\int_{\Omega}\widetilde{w} \leq \beta$. Since $u_{\varepsilon} \leq w^{**} \leq \widetilde{w}$ we get $\beta = \int_{\Omega} u_{\varepsilon} \leq \int_{\Omega} w^{**} \leq \int_{\Omega}\widetilde{w} \leq \beta$, and so $u_{\varepsilon} = w^{**}$. Thus $u_{\varepsilon} \geq w^*$. □

For $\varepsilon \in (0,1]$, let u_{ε} be the maximal weak solution to problem (1.7) given by Lemma 2.15.

Lemma 2.16. *Let a, α, and $\{g_{\varepsilon}\}_{\varepsilon\in(0,1]}$ be as in Lemma 2.2. Then the map $\varepsilon \to u_{\varepsilon}$ is nondecreasing on* $(0,1]$.

Proof. For $0 < \varepsilon < \eta$ we have, in the sense of definition 1.1,

$$-\Delta u_{\varepsilon} = au_{\varepsilon}^{-\alpha} - g_{\varepsilon}(.,u_{\varepsilon}) \leq au_{\varepsilon}^{-\alpha} - g_{\eta}(.,u_{\varepsilon}) \text{ in } \Omega,$$

and so $u_{\varepsilon} \in H_0^1(\Omega)\cap C\left(\overline{\Omega}\right)$ is a subsolution, in the sense of distributions, to the problem

$$-\Delta u = au^{-\alpha} - g_{\eta}(.,u) \text{ in } \Omega. \tag{2.13}$$

Let z be as in Remark 2.4. Thus z is a supersolution, in the sense of distributions, of problem (2.9), and $z \leq cd_{\Omega}^{\tau}$ *a.e.* in Ω, with c and τ positive constants c. Taking into account that, for some positive constant c, $u_{\varepsilon} \geq cd_{\Omega}$ *a.e.* in Ω, Remark 2.10 applies, as before, to obtain a weak solution, in the sense of distributions, $\widetilde{u}_{\eta} \in W_{loc}^{1,2}(\Omega)$ of (2.13) such that $u_{\varepsilon} \leq \widetilde{u}_{\eta} \leq z$. Now, Lemma 2.13 gives that $\widetilde{u}_{\eta} \in H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$ and that $\widetilde{u}_{\eta}$ is a weak solution, in the sense of Definition 1.1, of problem (2.13), which implies $\widetilde{u}_{\eta} \leq u_{\eta}$. Thus $u_{\varepsilon} \leq u_{\eta}$. □

Lemma 2.17. *Let a, α, and $\{g_{\varepsilon}\}_{\varepsilon\in(0,1]}$ be as in Lemma 2.2. Then $\{u_{\varepsilon}\}_{\varepsilon\in(0,1]}$ is bounded in $H_0^1(\Omega)$.*

Proof. Let z be as in Remark 2.4. by Lemma 2.5 $u_{\varepsilon} \leq z$ in Ω and so, since $0 < \alpha \leq 1$, we have $\int_{\{a>0\}} au_{\varepsilon}^{1-\alpha} \leq \int_{\Omega} az^{1-\alpha} < \infty$. By taking u_{ε} as a test function in (1.7) we get, for any $\varepsilon \in (0,1]$,

$$\int_{\Omega} |\nabla u_{\varepsilon}|^2 + \int_{\Omega} u_{\varepsilon} g_{\varepsilon}(., u_{\varepsilon}) = \int_{\{a>0\}} au_{\varepsilon}^{1-\alpha}.$$

Then $\int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq \int_{\Omega} az^{1-\alpha} < \infty$. □

Lemma 2.18. *Let a, α, and $\{g_{\varepsilon}\}_{\varepsilon\in(0,1]}$ be as in Lemma 2.2. Let $\boldsymbol{u} := \lim_{\varepsilon\to0^+} u_{\varepsilon}$. Then:*
i) $\boldsymbol{u} \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$.
ii) $\boldsymbol{u} > 0$ a.e. in $\{a > 0\}$.
iii) $\chi_{\{\boldsymbol{u}>0\}} a\boldsymbol{u}^{-\alpha}\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$.
iv) If $\left\{\varepsilon_j\right\}_{j\in\mathbb{N}}$ is a decreasing sequence in $(0,1]$ such that $\lim_{j\to\infty} \varepsilon_j = 0$ then $\lim_{j\to\infty} \int_{\{a>0\}} au_{\varepsilon_j}^{-\alpha}\varphi = \int_{\{a>0\}} a\boldsymbol{u}^{-\alpha}\varphi$ for any $\varphi \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$.

Proof. To see *i)*, consider a nonincreasing sequence $\left\{\varepsilon_j\right\}_{j\in\mathbb{N}} \subset (0,1]$ such that $\lim_{j\to\infty} \varepsilon_j = 0$. By Lemma 2.17, $\left\{u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and so, after pass to a subsequence if necessary, $\left\{u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ converges, strongly in $L^2(\Omega)$, and *a.e.* in Ω, to some $\widetilde{u} \in H_0^1(\Omega)$, and $\left\{\nabla u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ converges weakly in $L^2(\Omega, \mathbb{R}^n)$ to $\nabla\widetilde{u}$. Since u_{ε_j} converges to $\boldsymbol{u}$ *a.e.* in Ω we have $\boldsymbol{u} = \widetilde{u}$ *a.e.* in Ω, and so $\boldsymbol{u} \in H_0^1(\Omega)$. Also, $0 \leq \boldsymbol{u} \leq u_{\varepsilon_1} \in L^{\infty}(\Omega)$ and then $\boldsymbol{u} \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$. Thus *i)* holds.
To see *ii)* and *iii)*, consider an arbitrary nonnegative function $\varphi \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$. From (1.7) we have, for each j,

$$\int_{\Omega} \left\langle \nabla u_{\varepsilon_j}, \nabla\varphi \right\rangle + \int_{\Omega} g_{\varepsilon_j}\left(., u_{\varepsilon_j}\right)\varphi = \int_{\Omega} au_{\varepsilon_j}^{-\alpha}\varphi. \tag{2.14}$$

$\left\{\nabla u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ converges weakly in $L^2(\Omega, \mathbb{R}^n)$ to $\nabla\boldsymbol{u}$, and thus

$$\lim_{j\to\infty} \int_{\Omega} \left\langle \nabla u_{\varepsilon_j}, \nabla\varphi \right\rangle = \int_{\Omega} \langle \nabla\boldsymbol{u}, \nabla\varphi \rangle.$$

By Lemma 2.16, $\left\{au_{\varepsilon_j}^{-\alpha}\varphi\right\}_{j\in\mathbb{N}}$ is nondecreasing, then, by the monotone convergence theorem, $\lim_{j\to\infty} \int_{\Omega} au_{\varepsilon_j}^{-\alpha}\varphi = \lim_{j\to\infty} \int_{\{a>0\}} au_{\varepsilon_j}^{-\alpha}\varphi = \int_{\{a>0\}} a\boldsymbol{u}^{-\alpha}\varphi$.
Let z be as in Lemma 2.5. Then $u_{\varepsilon_j} \leq z$ in Ω and so, taking into account *h4)*, $\int_{\Omega} g_{\varepsilon_j}\left(., u_{\varepsilon_j}\right)\varphi \leq \int_{\Omega} \sup_{0\leq s\leq\|z\|_{\infty}} g(., s)\varphi < \infty$. Thus

$$\int_{\{a>0\}} a\boldsymbol{u}^{-\alpha}\varphi = \lim_{j\to\infty} \int_{\Omega} au_{\varepsilon_j}^{-\alpha}\varphi = \lim_{j\to\infty} \left(\int_{\Omega} \left\langle \nabla u_{\varepsilon_j}, \nabla\varphi \right\rangle + \int_{\Omega} g_{\varepsilon_j}\left(., u_{\varepsilon_j}\right)\varphi \right)$$

$$\leq \overline{\lim}_{j\to\infty} \int_\Omega \left\langle \nabla u_{\varepsilon_j}, \nabla\varphi \right\rangle + \overline{\lim}_{j\to\infty} \int_\Omega g_{\varepsilon_j}\left(., u_{\varepsilon_j}\right)\varphi$$

$$\leq \int_\Omega \langle \nabla \boldsymbol{u}, \nabla\varphi\rangle + \int_\Omega \sup_{0\leq s\leq \|z\|_\infty} g(., s)\,\varphi < \infty.$$

Therefore $\int_{\{a>0\}} a\boldsymbol{u}^{-\alpha}\varphi < \infty$. Since this holds for any nonnegative $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we conclude that $\boldsymbol{u} > 0$ *a.e.* in $\{a > 0\}$. Thus *ii)* holds. Now,

$$\int_\Omega \chi_{\{\boldsymbol{u}>0\}} a\boldsymbol{u}^{-\alpha}\varphi = \int_{\{a>0\}} \chi_{\{\boldsymbol{u}>0\}} a\boldsymbol{u}^{-\alpha}\varphi = \int_{\{a>0\}} a\boldsymbol{u}^{-\alpha}\varphi < \infty,$$

and then *iii)* holds for any nonnegative $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Hence, by writing $\varphi = \varphi^+ - \varphi^-$, *iii)* holds also for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Finally, observe that, in the case when $\varphi \geq 0$, the monotone convergence theorem gives *iv)*. Then, by writing $\varphi = \varphi^+ - \varphi^-$, *iv)*, holds also for an arbitrary $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. □

Remark 2.19. Assume that a satisfies the conditions *h1)*, *h2)* and also the condition *h6)* of Theorem 1.4; and let Ω^+ be as in *h6)*. Taking into account *h6)*, Remark 2.4 (applied in each connected component of Ω^+) gives that the problem

$$\begin{cases} -\Delta\zeta = a\zeta^{-\alpha} \text{ in } \Omega^+, \\ \zeta = 0 \text{ on } \partial\Omega^+, \\ \zeta > 0 \text{ in } \Omega^+, \end{cases} \tag{2.15}$$

has a unique weak solution, in the sense of Definition 1.1, $\zeta \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and that it satisfies:
i) $\zeta \in C\left(\overline{\Omega^+}\right)$.
ii) There exists a positive constant c such that $\zeta \geq c d_{\Omega^+}$ in Ω^+.
iii) ζ is also a solution of problem (2.15) in the usual sense of $H_0^1(\Omega^+)$, i.e., $a\zeta^{-\alpha}\varphi \in L^1(\Omega)$ and $\int_\Omega \langle\nabla\zeta, \nabla\varphi\rangle = \int_\Omega a\zeta^{-\alpha}\varphi$ for any $\varphi \in H_0^1(\Omega^+)$.

Lemma 2.20. *Assume that a and g satisfy the conditions h1)-h4) and also the condition h6) of Theorem 1.4. Let Ω^+ and A^+ be as in the statement of Theorem 1.4 and assume, in addition, that $g(., s) = 0$ a.e. in A^+ for any $s \geq 0$. Let ζ be as in Remark 2.19, let $\varepsilon \in (0, 1]$, and let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution, in the sense of Definition 1.1, of problem (1.5). Then $u \geq \zeta$ in Ω^+.*

Proof. By Remark 2.19 *i)*, $\zeta \in C\left(\overline{\Omega^+}\right)$ and, by Lemma 2.7, $u \in C\left(\overline{\Omega}\right)$. Also, since $g(., s) = 0$ *a.e.* in Ω^+ for $s \geq 0$, we have $-\Delta(u - \zeta) = a(u^{-\alpha} - \zeta^{-\alpha}) \geq 0$ in $D'(\Omega^+)$. We claim that $u \geq \zeta$ in Ω^+. To prove this fact we proceed by the way of contradiction: Let $U := \{x \in \Omega^+ : u(x) < \zeta(x)\}$ and suppose that $U \neq \emptyset$. Then U is an open subset of Ω^+ and $-\Delta(u - \zeta) = a(u^{-\alpha} - \zeta^{-\alpha}) \geq 0$ in $D'(U)$. Notice that $u - \zeta \geq 0$ on ∂U. In fact, if $u(x) < \zeta(x)$ for some $x \in \partial U$ we would have, either $x \in \Omega^+$ or $x \in \partial\Omega^+$; if $x \in \Omega^+$ then, since u and ζ are continuous on Ω^+, we would have $u < \zeta$ on some ball around x, contradicting the fact that $x \in \partial U$, and if $x \in \partial\Omega^+$, then $u(x) \geq 0 = \zeta(x)$ contradicting our assumption that $u(x) < \zeta(x)$. Then $U = \emptyset$ and so $u \geq \zeta$ in Ω^+; and then, by continuity, also $u \geq \zeta$ on $\partial\Omega^+$. Therefore, from the weak maximum principle, $u \geq \zeta$ in Ω^+. □

3. Proof of the main results

Observe that if $g : \Omega \times [0,\infty) \to \mathbb{R}$ satisfies the conditions *h3)* and *h4)* stated at the introduction, and if, for $\varepsilon \in (0,1]$, $g_\varepsilon : \Omega \times [0,\infty) \to \mathbb{R}$ is defined by

$$g_\varepsilon(.,s) := s(s+\varepsilon)^{-1} g(.,s), \tag{3.1}$$

then, for any $s > 0$, $g(.,s) = \lim_{\varepsilon\to 0^+} g_\varepsilon(.,s)$ *a.e.* in Ω; and the family $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ satisfies the conditions *h7)-h9)*. Therefore all the results of the Section 2 hold for such a family $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$.

Lemma 3.1. *Let $a : \Omega \to \mathbb{R}$ and $g : \Omega \times [0,\infty) \to \mathbb{R}$ satisfying the conditions h1)-h4) and, for $\varepsilon \in (0,1]$, let $g_\varepsilon : \Omega \times [0,\infty) \to \mathbb{R}$ be defined by (3.1), let u_ε be as given by Lemma 2.15, and let $\boldsymbol{u} := \lim_{\varepsilon\to 0^+} u_\varepsilon$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1]$ be a nonincreasing sequence such that $\lim_{j\to\infty} \varepsilon_j = 0$ and, for $j \in \mathbb{N}$, let u_{ε_j} be as given by Lemma 2.15. Let $\theta_j := u_{\varepsilon_j}\left(u_{\varepsilon_j} + \varepsilon_j\right)^{-1}$. Then there exist a nonnegative function $\theta^* \in L^\infty(\Omega)$ and a sequence $\{w_m\}_{m\in\mathbb{N}} \subset L^2(\Omega,\mathbb{R}^n) \times L^2(\Omega)$ with the following properties:*
i) for each $m \in \mathbb{N}$, $w_m = \sum_{l\in\mathcal{F}_m} \gamma_{l,m}\left(\nabla u_{\varepsilon_l}, \theta_l g(.,u_{\varepsilon_l})\right)$, where each $\mathcal{F}_m$ is a finite subset of $\mathbb{N}$ satisfying $\lim_{m\to\infty}\min\mathcal{F}_m = \infty$; $\gamma_{l,m} \in [0,1]$ for any $m \in \mathbb{N}$ and $l \in \mathcal{F}_m$; and $\sum_{l\in\mathcal{F}_m}\gamma_{l,m} = 1$ for any $m \in \mathbb{N}$.
ii) $\{w_m\}_{m\in\mathbb{N}}$ converges strongly in $L^2(\Omega,\mathbb{R}^n)\times L^2(\Omega)$ to $(\nabla\mathbf{u},\theta^*)$.
iii) $\lim_{m\to\infty}\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\theta_l g(.,u_{\varepsilon_l}) = \theta^*$ a.e. in Ω.
iv) $\theta^* = \chi_{\{\mathbf{u}>0\}} g(.,\mathbf{u})$ a.e. in $\{\mathbf{u}>0\}$.

Proof. By Lemma 2.17 $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Then, after pass to a subsequence if necessary, we can assume that $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ converges to $\mathbf{u}$ in $L^2(\Omega)$ and that $\{\nabla u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ converges weakly to $\nabla\mathbf{u}$ in $L^2(\Omega,\mathbb{R}^n)$. Moreover, by Lemma 2.5, $u_{\varepsilon_j} \le z$ *a.e.* in Ω for all j, and so $\mathbf{u} \le z$ *a.e.* in Ω. Since, for any j, $0 < \theta_j < 1$ *a.e.* in Ω, and, by *h3)* and *h4)*, $0 \le g\left(.,u_{\varepsilon_j}\right) \le \sup_{s\in[0,\|z\|_\infty]} g(.,s) \in L^\infty(\Omega)$, we have that $\left\{\theta_j g\left(.,u_{\varepsilon_j}\right)\right\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. Thus, after pass to a further subsequence, we can assume that $\left\{\theta_j g\left(.,u_{\varepsilon_j}\right)\right\}_{j\in\mathbb{N}}$ is weakly convergent in $L^2(\Omega)$ to a function $\theta^* \in L^2(\Omega)$, and that $\{\nabla u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ is weakly convergent in $L^2(\Omega,\mathbb{R}^n)$ to $\nabla\mathbf{u}$. Then $\left\{\left(\nabla u_{\varepsilon_j}, \theta_j g\left(.,u_{\varepsilon_j}\right)\right)\right\}_{j\in\mathbb{N}}$ is weakly convergent to $(\nabla\mathbf{u},\theta^*)$ in $L^2(\Omega,\mathbb{R}^n)\times L^2(\Omega)$. Thus (see e.g., [33] Theorem 3.13) there exists a sequence $\{w_m\}_{m\in\mathbb{N}}$ of the form $w_m = \sum_{l\in\mathcal{F}_m}\gamma_{l,m}\left(\nabla u_{\varepsilon_l}, \theta_l g(.,u_{\varepsilon_l})\right)$, where each $\mathcal{F}_m$ is a finite subset of $\mathbb{N}$ such that $\lim_{m\to\infty}\min\mathcal{F}_m = \infty$, $\gamma_{l,m} \in [0,1]$ for any $m \in \mathbb{N}$ and $l \in \mathcal{F}_m$, for each m, $\sum_{l\in\mathcal{F}_m}\gamma_{l,m} = 1$ and such that $\{w_m\}_{m\in\mathbb{N}}$ converges strongly in $L^2(\Omega,\mathbb{R}^n)\times L^2(\Omega)$ to $(\nabla\mathbf{u},\theta^*)$. Then *i)* and *ii)* hold, and $\left\{\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\theta_l g(.,u_{\varepsilon_l})\right\}_{m\in\mathbb{N}}$ converges in $L^2(\Omega)$ to θ^*. Therefore, after pass to a further subsequence, we can assume that $\lim_{m\to\infty}\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\theta_l g(.,\boldsymbol{u}_{\varepsilon_l}) = \theta^*$ *a.e.* in Ω and, since $\left\{\theta_j g\left(.,u_{\varepsilon_j}\right)\right\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, we have that $\theta^* \in L^\infty(\Omega)$. Thus *iii)* holds. Also $\{\theta_j\}_{j\in\mathbb{N}}$ and $\left\{g\left(.,u_{\varepsilon_j}\right)\right\}_{j\in\mathbb{N}}$ converge, *a.e.* in$\{\mathbf{u}>0\}$, to $\chi_{\{\mathbf{u}>0\}}$ and to $g(.,\mathbf{u})$ respectively, and then *iv)* follows from *iii)*. □

Proof of Theorem 1.2. Let $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1)$ be a nonincreasing sequence such that $\lim_{j\to\infty}\varepsilon_j = 0$, let θ^* and $\{w_m\}_{m\in\mathbb{N}} \subset L^2(\Omega,\mathbb{R}^n)\times L^2(\Omega)$ be as given by Lemma 3.1, and let $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$. Assume temporarily that $\varphi \ge 0$ in Ω. Then $\left\{\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\theta_l g(.,u_{\varepsilon_l})\varphi\right\}_{m\in\mathbb{N}}$ and $\left\{\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\langle\nabla u_{\varepsilon_l},\nabla\varphi\rangle\right\}_{m\in\mathbb{N}}$ converge in $L^1(\Omega)$ to $\theta^*\varphi$ and $\langle\nabla\mathbf{u},\nabla\varphi\rangle$ respectively. Thus

$$\lim_{m\to\infty}\int_\Omega\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\theta_l g\left(.,u_{\varepsilon_l}\right)\varphi=\int_\Omega\theta^*\varphi, \tag{3.2}$$

$$\lim_{m\to\infty}\int_\Omega\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\left\langle\nabla u_{\varepsilon_l},\nabla\varphi\right\rangle=\int_\Omega\langle\nabla\mathbf{u},\nabla\varphi\rangle \tag{3.3}$$

and both limits are finite. Since $\left\{u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ is nonincreasing we have, for $m\in\mathbb{N}$ and $l\in\mathcal{F}_m$,

$$au_{\varepsilon_{L_m}}^{-\alpha}\varphi\le a\sum_{l\in\mathcal{F}_m}\gamma_{l,m}u_{\varepsilon_l}^{-\alpha}\varphi\le au_{\varepsilon_{L_m^*}}^{-\alpha}\varphi, \tag{3.4}$$

where $L_m:=\max\mathcal{F}_m$ and $L_m^*:=\min\mathcal{F}_m$. Also, by the monotone convergence theorem,

$$\lim_{j\to\infty}\int_\Omega au_{\varepsilon_j}^{-\alpha}\varphi=\lim_{j\to\infty}\int_{\{a>0\}}au_{\varepsilon_j}^{-\alpha}\varphi=\int_{\{a>0\}}a\mathbf{u}^{-\alpha}\varphi=\int_\Omega\chi_{\{\mathbf{u}>0\}}a\mathbf{u}^{-\alpha}\varphi, \tag{3.5}$$

the last equality because, by Lemma 2.18, $\mathbf{u}>0$ *a.e.* in $\{a>0\}$. Then, since $\lim_{m\to\infty}L_m^*=\infty$, (3.4) and (3.5) give

$$\lim_{m\to\infty}\int_{\{a>0\}}a\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\boldsymbol{u}_{\varepsilon_l}^{-\alpha}\varphi=\int_\Omega\chi_{\{\mathbf{u}>0\}}a\mathbf{u}^{-\alpha}\varphi. \tag{3.6}$$

(notice that, by Lemma 2.18, $\int_\Omega\chi_{\{\mathbf{u}>0\}}a\mathbf{u}^{-\alpha}\varphi<\infty$). Since $\theta_l g\left(.,u_{\varepsilon_l}\right)=g_{\varepsilon_l}\left(.,u_{\varepsilon_l}\right)$ we have, for any $m\in\mathbb{N}$, and in the sense of definition 1.1,

$$\begin{cases} -\Delta\left(\sum_{l\in\mathcal{F}_m}\gamma_{l,m}u_{\varepsilon_l}\right) \\ =a\sum_{l\in\mathcal{F}_m}\gamma_{l,m}u_{\varepsilon_l}^{-\alpha}-\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\theta_l g\left(.,u_{\varepsilon_l}\right)\text{ in }\Omega, \\ \sum_{l\in\mathcal{F}_m}\gamma_{l,m}u_{\varepsilon_l}=0\text{ on }\partial\Omega \end{cases} \tag{3.7}$$

and so

$$\begin{aligned} &\int_\Omega\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\left\langle\nabla u_{\varepsilon_l},\nabla\varphi\right\rangle \\ &=\int_\Omega a\sum_{l\in\mathcal{F}_m}\gamma_{l,m}u_{\varepsilon_l}^{-\alpha}\varphi-\int_\Omega\sum_{l\in\mathcal{F}_m}\gamma_{l,m}\theta_l g\left(.,u_{\varepsilon_l}\right)\varphi. \end{aligned} \tag{3.8}$$

Taking the limit as $m\to\infty$ in (3.8), and using (3.2), (3.3), (3.6) and recalling that, by Lemma 3.1 *iv)*, $\theta^*=\chi_{\{\mathbf{u}>0\}}g(.,\mathbf{u})$ *a.e.* in $\{\mathbf{u}>0\}$, we get that

$$\begin{aligned} \int_\Omega\langle\nabla\mathbf{u},\nabla\varphi\rangle&=\int_\Omega\chi_{\{\mathbf{u}>0\}}a\mathbf{u}^{-\alpha}\varphi-\int_\Omega\theta^*\varphi \\ &=\int_\Omega\chi_{\{\mathbf{u}>0\}}a\mathbf{u}^{-\alpha}\varphi-\int_\Omega\chi_{\{\mathbf{u}>0\}}g(.,\mathbf{u})\varphi-\int_{\{\mathbf{u}=0\}}\theta^*\varphi. \end{aligned} \tag{3.9}$$

for any nonnegative $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$, and by writing $\varphi=\varphi^+-\varphi^-$ it follows that (3.9) holds also for any $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$.

Let Ω_0 be as in *h3)*. If $\Omega_0 = \emptyset$ then $\mathbf{u} > 0$ *a.e.* in Ω (because $\mathbf{u} > 0$ *a.e.* in $\{a > 0\}$) and thus, by (3.9), $\mathbf{u}$ is a solution, in the sense of Definition 1.1, of problem (1.2). Consider now the case when $\Omega_0 \neq \emptyset$. We claim that, in this case, $\mathbf{u} \in W^{2,p}_{loc}(\Omega_0)$ for any $p \in [1,\infty)$. Indeed, let Ω_0' be a an arbitrary $C^{1,1}$ subdomain of Ω_0 such that $\overline{\Omega_0'} \subset \Omega_0$. We have $\chi_{\{\mathbf{u}>0\}} a\mathbf{u}^{-\alpha} = 0$ on Ω_0, and so, from (3.9), $-\Delta\mathbf{u} = -\chi_{\{\mathbf{u}>0\}} g(.,\mathbf{u}) - \theta^*$ in $D'(\Omega_0)$. Also, the restrictions to Ω_0 of $\mathbf{u}$ and θ^* belong to $L^\infty(\Omega_0)$ and so, from the inner elliptic estimates (as stated e.g., in [20], Theorem 8.24), $\mathbf{u} \in W^{2,p}\left(\Omega_0'\right)$. Then $\mathbf{u} \in W^{2,p}_{loc}(\Omega_0)$ for any $p \in [1,\infty)$. Thus, for any $p \in [1,\infty)$, $\mathbf{u}$ is a strong solution in $W^{2,p}_{loc}(\Omega_0)$ of $-\Delta\mathbf{u} = -\chi_{\{\mathbf{u}>0\}} g(.,\mathbf{u}) - \theta^*$ in Ω_0.
Taking into account (3.9), in order to complete the proof of the theorem it is enough to see that the set $E := \{\mathbf{u} = 0\} \cap \{\theta^* > 0\}$ has zero measure. Suppose that $|E| > 0$. Since $\mathbf{u} > 0$ *a.e.* in $\{a > 0\}$, from *h5)* it follows that $E \subset \Omega_0 \cup V$, for some measurable $V \subset \Omega$ such that $|V| = 0$. Since $|E| > 0$, there exists a subdomain Ω', with $\overline{\Omega'} \subset \Omega_0$, and such that $E' := E \cap \Omega'$ has positive measure. Since $\mathbf{u} = 0$ *a.e.* in E' and $\mathbf{u} \in W^{1,p}(\Omega')$ we have $\nabla\mathbf{u} = 0$ *a.e.* in E' (see [20], Lemma 7.7). Thus $\frac{\partial\mathbf{u}}{\partial x_i} = 0$ *a.e.* in E' for each $i = 1,2,...,n$; and since $\frac{\partial\mathbf{u}}{\partial x_i} \in W^{1,p}\left(\Omega_0'\right)$ the same argument gives that also the second order derivatives $\frac{\partial^2\mathbf{u}}{\partial x_i\partial x_j}$ vanish *a.e.* in E'. Then $\Delta\mathbf{u} = 0$ *a.e.* in E', which, taking into account that $g(.,\mathbf{u})$ is nonnegative and $\theta^* > 0$ in E', contradicts the fact that $-\Delta\mathbf{u} = -\chi_{\{\mathbf{u}>0\}} g(.,\mathbf{u}) - \theta^*$ *a.e.* in Ω_0. □

Proof of Theorem 1.3. Notice that the condition *h4')* is stronger than *h4)* and so Theorem 1.2 gives a weak solution $\boldsymbol{u}$, in the sense of definition 1.1, of problem (1.2) which satisfies $\boldsymbol{u} > 0$ *a.e.* in $\{a > 0\}$, and so, since $a > 0$ *a.e.* in Ω, by Lemma 2.18, we have $\boldsymbol{u} > 0$ *a.e.* in Ω. Thus $\boldsymbol{u}$ is a weak solution, in the sense of Definition 1.1, of the problem

$$\begin{cases} -\Delta\boldsymbol{u} = a\boldsymbol{u}^{-\alpha} - g(.,\boldsymbol{u}) \text{ in } \Omega, \\ \boldsymbol{u} = 0 \text{ on } \partial\Omega. \end{cases}$$

Let $a_0 := \boldsymbol{u}^{-1} g(.,\boldsymbol{u})$. Since $g \geq 0$ and $\boldsymbol{u} \in L^\infty(\Omega)$, *h4')* gives $0 \leq a_0 \in L^\infty(\Omega)$. Now, in the sense of Definition 1.1, $-\Delta\boldsymbol{u} + a_0\boldsymbol{u} = a\boldsymbol{u}^{-\alpha}$ in Ω, $\boldsymbol{u} = 0$ on $\partial\Omega$, and $\boldsymbol{u} > 0$ *a.e.* in Ω; Then, for some $\eta > 0$ and some measurable set $E \subset \Omega$ with $|E| > 0$, we have $\chi_{\{u>0\}} a\boldsymbol{u}^{-\alpha} \geq \eta\chi_E$ *a.e.* in Ω. Let $\psi \in \cap_{1\leq q<\infty} W^{2,,q}(\Omega) \cap W_0^{1,,q}(\Omega)$ be the solution of the problem $-\Delta\psi + a_0\psi = \eta\chi_E$ in Ω, $\psi = 0$ on $\partial\Omega$. By the Hopf maximum principle (as stated, e.g., in [34], Theorem 1.1) there exists a positive constant c_1 such that $\psi \geq c_1 d_\Omega$ in Ω; and, from (1.7) we have $-\Delta\boldsymbol{u} + a_0\boldsymbol{u} \geq \eta\chi_E$ in $D'(\Omega)$. Then, by the weak maximum principle (as stated, e.g., in [20], Theorem 8.1), $\boldsymbol{u} \geq \psi$ *a.e.* in Ω. Therefore, $\boldsymbol{u} \geq c_1 d_\Omega$ *a.e.* in Ω. Thus, for some positive constant c', $a\boldsymbol{u}^{-\alpha} \leq c' d_\Omega^{-\alpha}$ *a.e.* in Ω . Also, $g(.,\boldsymbol{u}) \in L^\infty(\Omega)$ and so, for a larger c' if necessary, we have $|a\boldsymbol{u}^{-\alpha} - g(.,\boldsymbol{u})| \leq c' d_\Omega^{-\alpha}$ *a.e.* in Ω. Then, taking into account that $\alpha \leq 1$, the Hardy inequality gives, for any $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega \left|(a\boldsymbol{u}^{-\alpha} - g(.,\boldsymbol{u}))\varphi\right| \leq \int_\Omega c' d_\Omega^{1-\alpha} \left|d_\Omega^{-1}\varphi\right| \leq c'' \|\varphi\|_{H_0^1(\Omega)}.$$

with c'' a positive constant independent of φ. Thus $a\boldsymbol{u}^{-\alpha} - g(.,\boldsymbol{u}) \in \left(H_0^1(\Omega)\right)'$. Let z be as in Lemma 2.5. Since $\boldsymbol{u} \leq u_{\varepsilon_j} \leq z$, Lemma 2.5 gives that $\boldsymbol{u} \leq c''' d_\Omega^\tau$ for some positive constants c''' and τ. Therefore, by Lemma 2.13, $\boldsymbol{u}$ is a weak solution, in the usual sense of $H_0^1(\Omega)$, of problem (1.2). Moreover, since

$$c d_\Omega \leq \boldsymbol{u} \leq c''' d_\Omega^\tau \text{ a.e. in } \Omega, \tag{3.10}$$

then $a\boldsymbol{u}^{-\alpha} - g(., \boldsymbol{u}) \in L^{\infty}_{loc}(\Omega)$, also $\boldsymbol{u} \in L^{\infty}(\Omega)$ and then, by the inner elliptic estimates, $\boldsymbol{u} \in W^{2,p}_{loc}(\Omega)$ for any $p \in [1, \infty)$. Thus $\boldsymbol{u} \in C(\Omega)$ and from (3.10), u is also continuous at $\partial\Omega$. Thus $u \in C\left(\overline{\Omega}\right)$. □

Proof of Theorem 1.4. Suppose that $0 < \alpha < \frac{1}{2} + \frac{1}{n}$ when $n > 2$, that and $0 < \alpha \leq 1$ when $n \leq 2$. Assume also that $g(., s) = 0$ on Ω^+ and that *h1)-h4)* and *h5)* hold. Let z be as in Remark 2.4, let $\left\{\varepsilon_j\right\}_{j\in\mathbb{N}} \subset (0, 1)$ be a nonincreasing sequence such that $\lim_{j\to\infty} \varepsilon_j = 0$, and let $\left\{u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ be as in Theorem 1.2. Let $\boldsymbol{u} := \lim_{j\to\infty} u_{\varepsilon_j}$. By Lemma 2.5 we have, $u_{\varepsilon_j} \leq z$ in Ω for all $j \in \mathbb{N}$, and so $\boldsymbol{u} \leq z$ *a.e.* in Ω. Thus, by Remark 2.4, there exist positive constants c and τ such that $\boldsymbol{u} \leq c d^{\tau}_{\Omega}$ *a.e.* in Ω. Let Ω^+ as given by *h6)*, and let $\zeta : \Omega^+ \to \mathbb{R}$ be as given by Remark 2.19. Thus, by Remark 2.19 *ii)*, there exists a positive constant c' such that $\zeta \geq c' d_{\Omega^+}$ in Ω^+, and by Remark 2.20, $u_{\varepsilon_j} \geq \zeta$ in Ω^+ for all $j \in \mathbb{N}$. Then $u_{\varepsilon_j} \geq c' d_{\Omega^+}$ in Ω^+ for all j, and so $\boldsymbol{u} \geq c d_{\Omega^+}$ *a.e.* in Ω^+.
Let $\varphi \in H^1_0(\Omega)$ and, for $k \in \mathbb{N}$, let $\varphi_k : \Omega \to \mathbb{R}$ be defined by $\varphi_k(x) = \varphi(x)$ if $|\varphi(x)| \leq k$, $\varphi_k(x) = k$ if $\varphi(x) > k$ and $\varphi_k(x) = -k$ if $\varphi(x) < -k$. Thus $\varphi_k \in H^1_0(\Omega) \cap L^{\infty}(\Omega)$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ converges to φ in $H^1_0(\Omega)$. By Theorem 1.2, u is a weak solution, in the sense of definition 1.1, of problem (1.2). Then, for all $k \in \mathbb{N}$,

$$\begin{aligned} \int_{\Omega} \langle \nabla \boldsymbol{u}, \nabla \varphi_k \rangle &= \int_{\Omega} \chi_{\{\boldsymbol{u}>0\}} \left(a\boldsymbol{u}^{-\alpha} - g(., \boldsymbol{u})\right) \varphi_k \\ &= \int_{\Omega} \left(a\boldsymbol{u}^{-\alpha} - \chi_{\{\boldsymbol{u}>0\}} g(., \boldsymbol{u})\right) \varphi_k \\ &= \int_{\Omega} \left(\chi_{\{a>0\}} a\boldsymbol{u}^{-\alpha} - \chi_{\{\boldsymbol{u}>0\}} g(., \boldsymbol{u})\right) \varphi_k. \end{aligned} \tag{3.11}$$

Note that $\chi_{\{a>0\}} a\boldsymbol{u}^{-\alpha} - \chi_{\{\boldsymbol{u}>0\}} g(., \boldsymbol{u}) \in \left(H^1_0(\Omega)\right)'$. Indeed, by *h4)*, $\chi_{\{\boldsymbol{u}>0\}} g(., \boldsymbol{u}) \in L^{\infty}(\Omega) \subset \left(H^1_0(\Omega)\right)'$, and, since $\boldsymbol{u} \geq c d_{\Omega^+}$ *a.e.* in Ω^+ and $a = 0$ *a.e.* in $\Omega \setminus \Omega^+$, we have $\chi_{\{a>0\}} a\boldsymbol{u}^{-\alpha} \in L^{(2^*)'}(\Omega) \subset \left(H^1_0(\Omega)\right)'$ when $n > 2$ (because $0 < \alpha < \frac{1}{2} + \frac{1}{n}$ if $n > 2$), and, in the case $n \leq 2$, $\chi_{\{a>0\}} a\boldsymbol{u}^{-\alpha} \in L^{\frac{1}{\alpha} - \eta}(\Omega) \subset \left(H^1_0(\Omega)\right)'$ for η positive and small enough, (because $0 < \alpha \leq 1$ if $n \leq 2$). Now, we take $\lim_{k\to\infty}$ in (3.11), to obtain

$$\begin{aligned} \int_{\Omega} \langle \nabla \boldsymbol{u}, \nabla \varphi \rangle &= \int_{\Omega} \left(\chi_{\{a>0\}} a\boldsymbol{u}^{-\alpha} - \chi_{\{\boldsymbol{u}>0\}} g(., \boldsymbol{u})\right) \varphi \\ &= \int_{\Omega} \chi_{\{\boldsymbol{u}>0\}} \left(a\boldsymbol{u}^{-\alpha} - g(., \boldsymbol{u})\right) \varphi, \end{aligned}$$

the last equality because $u > 0$ *a.e.* in $\{a > 0\}$. □

Acknowledgment

The author wish to thank an anonymous referee for his/her helpful suggestions and critical comments, which led to a substantial improvement of the paper.

Conflict of interest

The author declare no conflicts of interest in this paper

References

1. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.
2. H. Brezis and X Cabre, *Some simple nonlinear PDE's without solutions,* Bollettino Della Unione Matematica Italiana, **2** (1998), 223–262.
3. A. Callegari and A. Nashman, *A nonlinear singular boundary-value problem in the theory of pseudoplastic fluids,* SIAM J. Appl. Math., **38** (1980), 275–281.
4. M. M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem,* Comm. Part. Differ. Equat., **14** (1989), 1315–1327.
5. D. S. Cohen and H. B. Keller, *Some positive problems suggested by nonlinear heat generators,* J. Math. Mech., **16** (1967), 1361–1376.
6. M. G. Crandall, P. H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Part. Differ. Equations, **2** (1977), 193–222.
7. F. Cîrstea, M. Ghergu and V. Rădulescu, *Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane–Emden–Fowler type,* J. Math. Pures Appl., **84** (2005), 493–508.
8. Y. Chu, Y. Gao and W. Gao, *Existence of solutions to a class of semilinear elliptic problem with nonlinear singular terms and variable exponent*, J. Funct. Spaces, **2016** (2016), 1–11.
9. J. Dávila and M. Montenegro, *Positive versus free boundary solutions to a singular elliptic equation,* J. Anal. Math., **90** (2003), 303–335.
10. J. Dávila and M. Montenegro, *Existence and asymptotic behavior for a singular parabolic equation,* Trans. Amer. Math. Soc., **357** (2005), 1801–1828
11. M. A. del Pino, *A global estimate for the gradient in a singular elliptic boundary value problem,* Proc. R. Soc. Edinburgh Sect. A, **122** (1992), 341–352.
12. J. I. Díaz and J. Hernández, *Positive and free boundary solutions to singular nonlinear elliptic problems with absorption; An overview and open problems,* Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems, Electron. J. Differ. Eq., Conf., **21** (2014), 31–44.
13. J. I. Díaz, J. M. Morel and L. Oswald, *An elliptic equation with singular nonlinearity,* Comm. Part. Diff. Eq., **12** (1987), 1333–1344.
14. L. Dupaigne, M. Ghergu and V. Rădulescu, *Lane-Emden-Fowler equations with convection and singular potential,* J. Math. Pures Appl., **87** (2007), 563–581.
15. D. G. De Figueiredo, *Positive solutions of semilinear elliptic equations,* in: Guedes de Figueiredo D., Hönig C.S. (eds) Differential Equations. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, **957** (1982), 34–87
16. W. Fulks and J. S. Maybee, *A singular nonlinear equation*, Osaka Math. J., **12** (1960), 1–19.
17. L. Gasiński and N. S. Papageorgiou, *Nonlinear Elliptic Equations with Singular Terms and Combined Nonlinearities,* Ann. Henri Poincaré, **13** (2012), 481–512.
18. M. Ghergu and V. D. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, 2008.

19. M. Ghergu and V. D. Rădulescu, *Multi-parameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term,* Proc. Royal Soc. Edinburgh, Sect. A, **135** (2005), 61–84.
20. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
21. T. Godoy and A. Guerin, *Existence of nonnegative solutions for singular elliptic problems*, Electron. J. Differential Equations, **191** (2016), 1–16.
22. T. Godoy and A. Guerin, *Nonnegative solutions to some singular semilinear elliptic problems*, Journal of Nonlinear Functional Analysis, **2017** (2017), 1–23.
23. T. Godoy and A. Guerin, *Positive weak solutions of elliptic Dirichlet problems with singularities in both the dependent and the independent variables*, Electron. J. Qual. Theory Differ. Equ., **54** (2019), 1–17.
24. T. Godoy and A. Guerin, *Regularity of the lower positive branch for singular elliptic bifurcation problems*, Electron. J. Differential Equations, **49** (2019), 1–32.
25. T. Godoy and A. Guerin, *Existence of nonnegative solutions to singular elliptic problems, a variational approach,* Discrete Contin. Dyn. Syst., Series A, **38** (2018), 1505–1525.
26. A. C. Lazer and P. J. McKenna, *On a singular nonlinear elliptic boundary value problem*, Proc. Amer. Math. Soc., **111** (1991), 721–730.
27. M. Montenegro and A. Ponce, *The sub-supersolution method for weak solutions,* Proc. Amer. Math. Soc., **136** (2008), 2429–2438.
28. N. H. Loc and K. Schmitt, *Boundary value problems for singular elliptic equations*, Rocky Mt. J. Math., **41** (2011), 555–572.
29. N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, *Nonlinear Analysis–Theory and Methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.
30. N. S. Papageorgiou and G. Smyrlis, *Nonlinear elliptic equations with singular reaction,* Osaka J. Math., **53** (2016), 489–514.
31. A. C. Ponce, *Selected problems on elliptic equations involving measures,* arXiv:1204.0668 [v3], 2017.
32. V. D. Rădulescu, *Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities,* in: Handbook of Differential Equations: Stationary Partial Differential Equations, (M. Chipot, Editor), North-Holland Elsevier Science, Amsterdam, **4** (2007), 483–591.
33. W. Rudin, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.
34. J. C. Sabina de Lis, *Hopf maximum principle revisited*, Electron. J. Differential Equations, **115** (2015), 1–9.
35. J. Shi and M. Yao, *On a singular nonlinear semilinear elliptic problem,* Proc. R. Soc. Edinburgh, Sect A, **128** (1998), 1389–1401.

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