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Research article

The least common multiple of consecutive terms in a cubic progression

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Abstract: Let k be a positive integer and $f(x)$ a polynomial with integer coefficients. Associated to the least common multiple $\text{lcm}_{0\le i\le k}\{f(n+i)\}$, we define the function $\mathcal{G}_{k,f}$ for all positive integers $n \in \mathbb{N}^* \setminus Z_{k,f}$ by $\mathcal{G}_{k,f}(n) := \frac{\prod_{i=0}^{k}|f(n+i)|}{\text{lcm}_{0\le i\le k}\{f(n+i)\}}$, where $Z_{k,f} := \bigcup_{i=0}^{k}\{n \in \mathbb{N}^* : f(n+i) = 0\}$. If $f(x) = x$, then Farhi showed in 2007 that $\mathcal{G}_{k,f}$ is periodic with $k!$ as its period. Consequently, Hong and Yang improved Farhi's period $k!$ to $\text{lcm}(1, ..., k)$. Later on, Farhi and Kane confirmed a conjecture of Hong and Yang and determined the smallest period of $\mathcal{G}_{k,f}$. For the general linear polynomial $f(x)$, Hong and Qian showed in 2011 that $\mathcal{G}_{k,f}$ is periodic and got a formula for its smallest period. In 2015, Hong and Qian characterized the quadratic polynomial $f(x)$ such that $\mathcal{G}_{k,f}$ is almost periodic and also arrived at an explicit formula for the smallest period of $\mathcal{G}_{k,f}$. If $\deg f(x) \ge 3$, then one naturally asks the following interesting question: Is the arithmetic function $\mathcal{G}_{k,f}$ almost periodic and, if so, what is the smallest period? In this paper, we asnwer this question for the case $f(x) = x^3 + 2$. First of all, with the help of Hua's identity, we prove that $\mathcal{G}_{k,x^3+2}$ is periodic. Consequently, we use Hensel's lemma, develop a detailed p-adic analysis to $\mathcal{G}_{k,x^3+2}$ and particularly investigate arithmetic properties of the congruences $x^3 + 2 \equiv 0 \pmod{p^e}$ and $x^6 + 108 \equiv 0 \pmod{p^e}$, and with more efforts, its smallest period is finally determined. Furthermore, an asymptotic formula for $\log \text{lcm}_{0\le i\le k}\{(n+i)^3 + 2\}$ is given.

Keywords: cubic progression; least common multiple; p-adic valuation; almost periodic function; the smallest period; Hensel's lemma
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1. Introduction and statements of the main results

The least common multiple of consecutive positive integers was investigated by Chebyshev for the first significant attempt to prove the prime number theorem in [2]. Since then, the least common multiple of any given sequence of positive integers has been an interesting and important topic. Hanson

[9] and Nair [20] got the upper and lower bound of $\mathrm{lcm}_{1\le i\le n}\{i\}$, respectively. Then Nair's lower bound was extended in [3–5, 10, 11, 14, 17, 22, 25]. Goutziers [8] investigated the asymptotic behavior on the least common multiple of a set of integers not exceeding N. Bateman, Kalb and Stenger [1] obtained an asymptotic estimate for the least common multiple of arithmetic progressions. Hong, Qian and Tan [15] got an asymptotic estimate for the least common multiple of the sequence of product of linear polynomials. Qian and Hong [23] studied the asymptotic behavior of the least common multiple of consecutive terms in arithmetic progressions. In [6], Farhi found an interesting relation between the least common multiple and the derivative of integr-valued polynomials. Actually, Farhi [6] showed that $\mathrm{lcm}_{1\le i\le n}\{i\}$ is the smallest positive integer c_n satisfying the property: For any $P(x)\in E_n$, one has $c_nP'(x)\in E_n$, where E_n is the set of all the integer-valued polynomials of degree no more than n.

Let k be a positive integer and $f(x)$ be a polynomial with integer coefficients. Associated to the least common multiple $\mathrm{lcm}_{0\le i\le k}\{f(n+i)\}$ of any $k+1$ consecutive terms in the sequence $\{f(n)\}_{n=1}^{\infty}$, we define the arithmetic function $\mathcal{G}_{k,f}$ for all positive integers $n\in\mathbb{N}^*\setminus Z_{k,f}$ by

$$\mathcal{G}_{k,f}(n) := \frac{\prod_{i=0}^{k}|f(n+i)|}{\mathrm{lcm}_{0\le i\le k}\{f(n+i)\}},$$

where

$$Z_{k,f} := \bigcup_{i=0}^{k}\{n\in\mathbb{N}^* : f(n+i)=0\}.$$

If $f(x)=x$, then Farhi [4] showed that $\mathcal{G}_{k,f}$ is periodic with $k!$ as its period. Consequently, Hong and Yang [16] improved Farhi's period $k!$ to $\mathrm{lcm}(1,...,k)$. Later on, Farhi and Kane [7] confirmed the Hong-Yang conjecture in [16] and determined the smallest period of $\mathcal{G}_{k,f}$. For the general linear polynomial $f(x)$, Hong and Qian [12] showed that $\mathcal{G}_{k,f}$ is periodic and got a formula for its smallest period. In 2012, Qian, Tan and Hong [24] proved that if $f(x)=x^2+1$, then $\mathcal{G}_{k,f}$ is periodic with determining its exact period. In 2015, Hong and Qian [13] characterized the quadratic polynomial $f(x)$ such that $\mathcal{G}_{k,f}$ is almost periodic and also arrived at an explicit formula for the smallest period of $\mathcal{G}_{k,f}$. One naturally asks the following problem: Let $\deg f(x)\ge 3$. Is the arithmetic function $\mathcal{G}_{k,f}$ almost periodic and, if so, what is the smallest period? This is a nontrivial and interesting question. In general, it is hard to answer it because few are known about the roots of higher degree congruences modulo prime powers that play key roles in this topic. Among the higher degree sequences, $\{n^3+2\}_{n=1}^{\infty}$ is the first example one would like to understand. For instance, a renowned conjecture in number theory states that there are infinitely many primes in the cubic sequence $\{n^3+2\}_{n=1}^{\infty}$.

In this paper, our main goal is to study the above problem for the cubic sequence $\{n^3+2\}_{n=1}^{\infty}$. In what follows, for brevity, we write $\mathcal{G}_k := \mathcal{G}_{k,f}$ when $f(x)=x^3+2$. That is, we have

$$\mathcal{G}_k(n) := \frac{\prod_{i=0}^{k}((n+i)^3+2)}{\mathrm{lcm}_{0\le i\le k}\{(n+i)^3+2\}}. \tag{1.1}$$

Assume that $\mathcal{G}_k$ is periodic and let P_k denote its smallest period. Now we can use P_k to give a formula for $\mathrm{lcm}_{0\le i\le k}\{(n+i)^3+2\}$ as follows: For any positive integer n, one has

$$\mathrm{lcm}_{0\le i\le k}\{(n+i)^3+2\} = \frac{\prod_{i=0}^{k}((n+i)^3+2)}{\mathcal{G}_k(\langle n\rangle_{P_k})},$$

where $\langle n\rangle_{P_k}$ means the least positive integer congruent to n modulo P_k. So it is significant to determine the exact value of the smallest period P_k.

Before stating the main results of this paper, let us briefly explain the basic new ideas and key techniques in this paper. To deal with the cubic sequence $\{n^3+2\}_{n=1}^{\infty}$, we have to develop furthermore the techniques presented in previous works [7,12,16,24] and [13] that are far not enough for the current case. In fact, we use Hua's identity [18] to show the periodicity and establish the factorization result to reduce the computation of the smallest period P_k to that of the local periods $P_{p,k}$. For the small primes $p=2$ and 3, we calculate directly $P_{p,k}$. For the prime $p\geq 5$, we consider two cases $p\equiv 1 \pmod 6$ and $p\equiv 5 \pmod 6$. The former case is much more difficult to treat than the latter one. More complicated analysis will be needed in the former case. But for both cases, one needs to study in detail the arithmetic properties of roots of the congruences $x^3+2\equiv 0 \pmod{p^e}$ and $x^6+108\equiv 0 \pmod{p^e}$, where $e\geq 1$ is an integer and the famous Hensel's lemma is used to lift the roots of the two congruences from modulo p to modulo p^e.

To state our main results, as usual, for any prime number p, we let v_p be the normalized p-adic valuation of $\mathbb{Q}$, that is, $v_p(a)=b$ if $p^b \parallel a$. We also let $\gcd(a,b)$ denote the greatest common divisor of any integers a and b. For any real number x, by $\lfloor x\rfloor$ we denote the largest integer no more than x, we denote by $\lceil x\rceil$ the smallest integer no less than x. Further, by $\{x\}$ we denote the fractional part of x, i.e. $\{x\}:=x-\lfloor x\rfloor$. Obviously, $0\leq\{x\}<1$. For any positive integer k, we define the positive integers R_k and Q_k by

$$R_k := \mathrm{lcm}_{1\leq i\leq k}\{i(i^6+108)\} \tag{1.2}$$

and

$$Q_k := 2^{\frac{(-1)^k+1}{2}} 3^{\lceil\{\frac{k+1}{3}\}\rceil} \cdot \frac{R_k}{2^{v_2(R_k)}3^{v_3(R_k)} \prod\limits_{\substack{p\mid R_k \\ p\equiv 1 \pmod 6 \\ 2^{\frac{p-1}{3}}\not\equiv 1 \pmod p}} p^{v_p(R_k)}}. \tag{1.3}$$

Evidently, $v_p(Q_k)=v_p(R_k)$ for any prime $p\equiv 5 \pmod 6$. The main result of this paper can be stated as follows.

Theorem 1.1. *Let k be a positive integer. The arithmetic function $\mathcal{G}_k$ is periodic, and its smallest period is equal to $R_1=109$ if $k=1$, and if $k\geq 2$, then its smallest period equals Q_k except that $v_p(k+1)\geq v_p(Q_k)\geq 1$ for at most one prime $p\geq 5$, in which case its smallest period is equal to $\frac{Q_k}{p^{v_p(Q_k)}}$.*

Obviously, Theorem 1.1 answers affirmatively the above question for the case where $f(x)=x^3+2$. Moreover, from Theorem 1.1 one can easily deduce the following interesting asymptotic result.

Theorem 1.2. *Let k be any given positive integer. Then the following asymptotic formula holds:*

$$\log \mathrm{lcm}_{0\leq i\leq k}\{(n+i)^3+2\} \sim 3(k+1)\log n \ \ as\ n\to\infty.$$

This paper is organized as follows. In Section 2, we first use an identity of Hua [18] to show that the arithmetic function $\mathcal{G}_k$ is periodic with R_k as a period of it. Then we factor the smallest period P_k of $\mathcal{G}_k$ into the product of local period $P_{p,k}$ with p running over all prime divisors of R_k, and determine the local periods $P_{2,k}$ and $P_{3,k}$. Later on, with a little more effort, we show that Q_k is a period of $\mathcal{G}_k$ (see Theorem 2.7 below). Subsequently, we develop in Section 3 a p-adic analysis of the periodic function $\mathcal{G}_k$, and determine the local period $P_{p,k}$ of $\mathcal{G}_k$ for the case $p\equiv 5 \pmod 6$. In Section 4, we evaluate the local period $P_{p,k}$ when $p \equiv 1 \pmod 6$ and $2^{\frac{p-1}{3}} \equiv 1 \pmod p$. In the process, we need to explore the arithmetic properties of the congruences $x^3+2 \equiv 0 \pmod{p^e}$ and $x^6+108 \equiv 0 \pmod{p^e}$. Particularly, we express the smallest positive root of $x^6 + 108 \equiv 0 \pmod{p^e}$ in the terms of the roots of $x^3 + 2 \equiv 0 \pmod{p^e}$. In the final section, we provide the proofs of Theorems 1.1 and 1.2. Two examples are also given to illustrate the validity of our main result.

2. Periodicity of the arithmetic function $\mathcal{G}_k$

In this section, by using a well-known theorem of Hua in [18], we first prove that $\mathcal{G}_k$ is periodic. In the meantime, we also arrive at a nontrivial period of $\mathcal{G}_k$.

Lemma 2.1. *The arithmetic function $\mathcal{G}_k$ is periodic, and R_k is a period of $\mathcal{G}_k$.*

Proof. For any positive integer n, it follows from Theorem 7.3 of [18] (see page 11 of [18]) that

$$\mathcal{G}_k(n) = \prod_{t=1}^{k} \prod_{0\le i_0<...<i_t\le k} (\gcd((n+i_0)^3+2,...,(n+i_t)^3+2))^{(-1)^{t-1}}$$

and

$$\mathcal{G}_k(n+R_k) = \prod_{t=1}^{k} \prod_{0\le i_0<...<i_t\le k} (\gcd((n+i_0+R_k)^3+2,...,(n+i_t+R_k)^3+2))^{(-1)^{t-1}}.$$

Claim that $\mathcal{G}_k(n+R_k) = \mathcal{G}_k(n)$ from which one can read that $\mathcal{G}_k$ is periodic with R_k as its period. It remains to show the claim that will be done in what follows. It is clear that the claim follows if one can prove the following identity holds for all integers i and j with $0 \le i < j \le k$:

$$\gcd((n+i+R_k)^3+2,(n+j+R_k)^3+2) = \gcd((n+i)^3+2,(n+j)^3+2). \tag{2.1}$$

One can easily check that

$$u(n,i,j)((n+i)^3+2) - u(n,j,i)((n+j)^3+2) = (j-i)((j-i)^6+108),$$

where $u(n,i,j) := -4(i-j)^4 - 3(n+2j-i)((2n+i+j)(i-j)^2-6)$. So one has

$$\gcd((n+i)^3+2,(n+j)^3+2)|(j-i)((j-i)^6+108).$$

It then follows from $(j-i)((j-i)^6+108)|R_k$ that

$$\gcd((n+i)^3+2,(n+j)^3+2)|R_k. \tag{2.2}$$

This implies that

$$\gcd((n+i)^3+2,(n+j)^3+2)|(n+i\pm R_k)^3+2$$

and

$$\gcd((n+i)^3+2,(n+j)^3+2)|(n+j\pm R_k)^3+2.$$

One then derives that

$$\gcd((n+i)^3+2,(n+j)^3+2)|\gcd((n+i+R_k)^3+2,(n+j+R_k)^3+2) \tag{2.3}$$

and

$$\gcd\left((n+i)^3+2,(n+j)^3+2\right) \mid \gcd\left((n+i-R_k)^3+2,(n+j-R_k)^3+2\right). \tag{2.4}$$

On the other hand, replacing n by $n+R_k$ in (2.4) gives us that

$$\gcd\left((n+i+R_k)^3+2,(n+j+R_k)^3+2\right) \mid \gcd\left((n+i)^3+2,(n+j)^3+2\right). \tag{2.5}$$

Therefore (2.1) follows immediately from (2.3) and (2.5). Hence the claim is proved.

This completes the proof of Lemma 2.1. □

For any given prime p, we define the arithmetic function $\mathcal{G}_{p,k}$ for any positive integer n by $\mathcal{G}_{p,k}(n) := v_p(\mathcal{G}_k(n))$. Since $\mathcal{G}_k$ is a periodic function, $\mathcal{G}_{p,k}$ is periodic for each prime p and P_k is a period of $\mathcal{G}_{p,k}$. Let $P_{p,k}$ be the smallest period of $\mathcal{G}_{p,k}$. Then $P_{p,k}$ is called *p-adic period* of $\mathcal{G}_k$. All the p-adic periods $P_{p,k}$ are called *local period* of $\mathcal{G}_k$. The following result says that $P_{p,k}$ is a power of p, and P_k equals the product of all the p-adic periods $P_{p,k}$ with p being a prime divisor of R_k.

Lemma 2.2. *For any prime p, $P_{p,k}$ divides $p^{v_p(R_k)}$. Furthermore, one has*

$$P_k = \prod_{p \mid R_k} P_{p,k}.$$

Proof. At first, we show that for any prime p, $p^{v_p(R_k)}$ is a period of $\mathcal{G}_{p,k}$. To do so, it is enough to prove that for any given positive integer n and integers i and j with $0 \le i < j \le k$, one has

$$v_p(\gcd((n+i+p^{v_p(R_k)})^3+2,(n+j+p^{v_p(R_k)})^3+2)) = v_p(\gcd((n+i)^3+2,(n+j)^3+2)). \tag{2.6}$$

In the following we show that (2.6) is true. Clearly, (2.2) infers that $v_p(\gcd((n+i)^3+2,(n+j)^3+2)) \le v_p(R_k)$. In other words, we have $v_p((n+i)^3+2) \le v_p(R_k)$ or $v_p((n+j)^3+2) \le v_p(R_k)$. From this one can deduce that $v_p((n+i)^3+2) \le v_p((n+i \pm p^{v_p(R_k)})^3+2)$ or $v_p((n+j)^3+2) \le v_p((n+j \pm p^{v_p(R_k)})^3+2)$. It follows that

$$\begin{aligned}
& v_p(\gcd((n+i)^3+2,(n+j)^3+2)) \\
&= \min(v_p((n+i)^3+2), v_p((n+j)^3+2))) \\
&\le \min\{v_p((n+i+p^{v_p(R_k)})^3+2), v_p((n+j+p^{v_p(R_k)})^3+2)\} \\
&= v_p(\gcd((n+i+p^{v_p(R_k)})^3+2,(n+j+p^{v_p(R_k)})^3+2))
\end{aligned} \tag{2.7}$$

and

$$v_p(\gcd((n+i)^3+2,(n+j)^3+2)) \le v_p(\gcd((n+i-p^{v_p(R_k)})^3+2,(n+j-p^{v_p(R_k)})^3+2)). \tag{2.8}$$

Replacing n by $n+p^{v_p(R_k)}$ in (2.8) gives us that

$$v_p(\gcd((n+i+p^{v_p(R_k)})^3+2,(n+j+p^{v_p(R_k)})^3+2)) \le v_p(\gcd((n+i)^3+2,(n+j)^3+2))). \tag{2.9}$$

Therefore (2.6) follows immediately from (2.7) and (2.9).

Now using (2.6) and Hua's identity (Theorem 7.3 of [18]), we can deduce that for any given prime p, $\mathcal{G}_{p,k}(n) = \mathcal{G}_{p,k}(n+p^{v_p(R_k)})$ holds for any positive integer n. Hence $p^{v_p(R_k)}$ is a period of $\mathcal{G}_{p,k}$ which implies that $P_{p,k}|p^{v_p(R_k)}$. Therefore $P_{p,k}$ are pairwise relatively prime for different prime numbers p and $P_{p,k} = 1$ for those primes $p \nmid R_k$. So $\prod_{\text{prime } q|R_k} P_{q,k}\Big|P_k$ since $P_{p,k}|P_k$ for each prime p.

On the other hand, by the definition of $P_{p,k}$, we know that for any positive integer n, $v_p(\mathcal{G}_k(n + \prod_{\text{prime } q|R_k} P_{q,k})) = v_p(\mathcal{G}_k(n))$ holds for all primes p. Hence $\mathcal{G}_k(n+\prod_{\text{prime } q|R_k} P_{q,k}) = \mathcal{G}_k(n)$ holds for any positive integer n. That is, $\prod_{p|R_k} P_{p,k}$ is a period of $\mathcal{G}_k$, which implies that $P_k\Big|\prod_{\text{prime } p|R_k} P_{p,k}$. Hence $P_k = \prod_{p|R_k} P_{p,k}$ as desired. Lemma 2.2 is proved. □

In order to determine the smallest period P_k of $\mathcal{G}_k$, by Lemma 2.2 it is enough to determine the exact value of $P_{p,k}$ for all prime divisors p of R_k. In what follows, we show that Q_k is a period of $\mathcal{G}_k$. For arbitrary positive integers n and k, let

$$S_k(n) := \{n^3 + 2, (n + 1)^3 + 2, ..., (n + k)^3 + 2\} \tag{2.10}$$

be the set of any $k + 1$ consecutive terms in the cubic progression $\{m^3 + 2\}_{m\in\mathbb{N}}$, and

$$S_k^{(e)}(n) := \{m \in S_k(n) : p^e|m\}. \tag{2.11}$$

Then

$$\mathcal{G}_{p,k}(n) = \sum_{m\in S_k(n)} v_p(m) - \max_{m\in S_k(n)}\{v_p(m)\} \tag{2.12}$$

$$= \sum_{e=1}^{\infty} |S_k^{(e)}(n)| - \sum_{e=1}^{\infty}(1 \text{ if } p^e|m \text{ for some } m \in S_k(n))$$

$$= \sum_{e=1}^{\infty} \max(0, |S_k^{(e)}(n)| - 1). \tag{2.13}$$

We denote $l_p := v_p(R_k)$ for any prime p. If there is at most one element divisible by p^{l_p+1} in $S_k(n)$ for any positive integer n, then all the terms on the right-hand side of (2.13) are 0 if $e \geq l_p + 1$. It then follows that for any positive integer n, if there is at most one element divisible by p^{l_p+1} in $S_k(n)$, then

$$\mathcal{G}_{p,k}(n) = \sum_{e=1}^{l_p} f_e(n) = \sum_{e=1}^{l_p-1} f_e(n) + f_{l_p}(n), \tag{2.14}$$

where

$$f_e(n) := \max(0, |S_k^{(e)}(n)| - 1). \tag{2.15}$$

Clearly, one has that $f_e(n) = 0$ if $|S_k^{(e)}(n)| \leq 1$, and $f_e(n) = |S_k^{(e)}(n)| - 1$ if $|S_k^{(e)}(n)| > 1$.

Lemma 2.3. *We have* $P_{2,k} = 2^{\frac{(-1)^k+1}{2}}$.

Proof. Clearly, for any odd integer n, we have $v_2(n^3 + 2) = 0$. For any even integer n, letting $n = 2m$ gives us that $v_2(n^3 + 2) = v_2((2m)^3 + 2) = v_2(8m^3 + 2) = 1$. So for any positive integer k, one has that $\max_{m\in S_k(n)}\{v_2(m)\} = 1$ and

$$\sum_{m\in S_k(n)} v_2(m) = \sum_{i=0}^{k} v_2((n + i)^3 + 2)$$

$$=v_2((n+k)^3+2)\delta_k+\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}(v_2((n+2i)^3+2)+v_2((n+2i+1)^3+2)). \tag{2.16}$$

where $\delta_k := 1$ if $2|k$, and 0 otherwise. One can easily see that for any positive integers k and n, exactly one of $v_2((n+2i)^3+2)$ and $v_2((n+2i+1)^3+2)$ equals 1 and another one is 0 for all integers i with $0 \le i \le \lfloor\frac{k-1}{2}\rfloor$. This implies that $v_2((n+2i)^3+2)+v_2((n+2i+1)^3+2)=1$. Also $v_2((n+k)^3+2)\delta_k=1$ happens only when both of k and n are even, and 0 otherwise. Then by (2.12) and (2.16), we derive that

$$\begin{aligned}
g_{2,k}(n)=&\sum_{m\in S_k(n)}v_2(m)-\max_{m\in S_k(n)}\{v_p(m)\}\\
=&\Big\lfloor\frac{k-1}{2}\Big\rfloor+v_2((n+k)^3+2)\delta_k\\
=&\begin{cases}\lfloor\frac{k-1}{2}\rfloor+1, & \text{if } k \text{ and } n \text{ are even},\\ \lfloor\frac{k-1}{2}\rfloor, & \text{otherwise}\end{cases}\\
=&\begin{cases}\frac{k-1}{2}, & \text{if } k \text{ is odd},\\ \frac{k}{2}, & \text{if } k \text{ and } n \text{ are even},\\ \frac{k}{2}-1, & \text{otherwise.}\end{cases}
\end{aligned}$$

Therefore $P_{2,k}=1$ if $2\nmid k$, and if k is even, then for any positive integer n, one has $g_{2,k}(n+2)=g_{2,k}(n)$ and $g_{2,k}(n+1)\neq g_{2,k}(n)$. It follows that $P_{2,k}=2$ if $2|k$ as required. So Lemma 2.3 is proved. □

Lemma 2.4. *We have that $P_{3,1}=1$ and $P_{3,k}=3^{\lceil\{\frac{k+1}{3}\}\rceil}$ if $k\ge 2$.*

Proof. First of all, we have

$$v_3(n^3+2)=\begin{cases}1, & \text{if } n\equiv 1 \pmod 3,\\ 0, & \text{otherwise.}\end{cases}$$

Since 3 does not divide the difference of $(n+1)^3+2$ and n^3+2, at most one of them is divisible by 3. This implies that $g_{3,1}(n)=0$ for all positive integers n. So $P_{3,1}=1$ as desired.

Now let $k\ge 2$. One has that $\max_{m\in S_k(n)}\{v_3(m)\}=1$ and

$$\begin{aligned}
\sum_{m\in S_k(n)}v_3(m)=&\sum_{i=0}^{k}v_3((n+i)^3+2)\\
=&\sum_{i=0}^{\lfloor\frac{k-2}{3}\rfloor}(v_3((n+3i)^3+2)+v_3((n+3i+1)^3+2)+v_3((n+3i+2)^3+2))\\
&+v_3((n+k)^3+2)\delta_k^{(1)}+v_3((n+k-1)^3+2)\delta_k^{(2)},
\end{aligned} \tag{2.17}$$

where

$$\delta_k^{(1)}=\begin{cases}1, & \text{if } k\equiv 0 \text{ or } 1 \pmod 3,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\delta_k^{(2)} = \begin{cases} 1, & \text{if } k \equiv 1 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

One can easily see that for any positive integers k and n, exactly one of $v_3((n+3i)^3+2)$, $v_3((n+3i+1)^3+2)$ and $v_3((n+3i+2)^3+2)$ equals 1 and the remaining two are 0 for all integers i with $0 \le i \le \lfloor\frac{k-2}{3}\rfloor$. This implies that $v_3((n+3i)^3+2) + v_3((n+3i+1)^3+2) + v_3((n+3i+2)^3+2 = 1$. Then by (2.12) applied to $p = 3$ and (2.17), we derive that

$$g_{3,k}(n) = \Big\lfloor\frac{k-2}{3}\Big\rfloor + v_3((n+k)^3+2)\delta_k^{(1)} + v_3((n+k-1)^3+2)\delta_k^{(2)}. \tag{2.18}$$

If $k \equiv 2 \pmod 3$, then $\delta_k^{(1)} = \delta_k^{(2)} = 0$. So by (2.18), one has $g_{3,k}(n) = \lfloor\frac{k-2}{3}\rfloor = \frac{k-2}{3}$, which infers that $P_{3,k} = 1$ if $k \equiv 2 \pmod 3$.

If $k \equiv 0 \pmod 3$, then $\delta_k^{(1)} = 1$ and $\delta_k^{(2)} = 0$. If $n \equiv 1 \pmod 3$, then $v_3((n+k)^3+2) = 1$, and 0 otherwise. Again by (2.18), we have

$$\begin{aligned} g_{3,k}(n) &= \Big\lfloor\frac{k-2}{3}\Big\rfloor + v_3((n+k)^3+2) \\ &= \begin{cases} \lfloor\frac{k-2}{3}\rfloor + 1, & \text{if } n \equiv 1 \pmod 3, \\ \lfloor\frac{k-2}{3}\rfloor, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{k}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{k}{3} - 1, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that for any integer t, one has that $g_{3,k}(t+3) = g_{3,k}(t)$ and $g_{3,k}(3t) \neq g_{3,k}(3t+1)$. That is $P_{3,k} = 3$ if $k \equiv 0 \pmod 3$.

If $k \equiv 1 \pmod 3$, then $\delta_k^{(1)} = \delta_k^{(2)} = 1$. If $n \equiv 0$ or $1 \pmod 3$, then $v_3((n+k-1)^3+2)+v_3((n+k)^3+2) = 1$, and 0 otherwise. Then by (2.18), one has

$$\begin{aligned} g_{3,k}(n) &= \Big\lfloor\frac{k-2}{3}\Big\rfloor + v_3((n+k-1)^3+2) + v_3((n+k)^3+2) \\ &= \begin{cases} \lfloor\frac{k-2}{3}\rfloor + 1, & \text{if } n \equiv 0 \text{ or } 1 \pmod 3, \\ \lfloor\frac{k-2}{3}\rfloor, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{k-1}{3}, & \text{if } n \equiv 0 \text{ or } 1 \pmod 3, \\ \frac{k-4}{3}, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence for any integer t, one has that $g_{3,k}(t+3) = g_{3,k}(t)$ and $g_{3,k}(3t+1) \neq g_{3,k}(3t+2)$. Therefore $P_{3,k} = 3$ if $k \equiv 1 \pmod 3$.

Finally, we can conclude that if $k \ge 2$, then

$$P_{3,k} = \begin{cases} 1, & \text{if } k \equiv 2 \pmod 3, \\ 3, & \text{otherwise.} \end{cases}$$

Namely, $P_{3,k} = 3^{\lceil\{\frac{k+1}{3}\}\rceil}$ if $k \ge 2$. This finishes the proof of Lemma 2.4. □

Lemma 2.5. *Let p be a prime such that $p \equiv 1 \pmod 6$. Then each of the following is true:*

(i). *The congruence $x^3 + 2 \equiv 0 \pmod p$ is solvable if and only if $2^{\frac{p-1}{3}} \equiv 1 \pmod p$.*

(ii). *The congruence $x^6 + 108 \equiv 0 \pmod p$ is solvable if and only if $2^{\frac{p-1}{3}} \equiv 1 \pmod p$.*

Proof. (i). Let m and a be positive integers with $\gcd(a, p) = 1$. It is well known that $x^m \equiv a \pmod p$ has a solution if and only if $a^{\frac{p-1}{\gcd(p-1,m)}} \equiv 1 \pmod p$. Since $p \equiv 1 \pmod 6$, letting $m = 3$ and $a = -2$ gives that $x^3 \equiv -2 \pmod p$ has a solution if and only if $2^{\frac{p-1}{3}} \equiv 1 \pmod p$. So part (i) is proved.

(ii). Likewise, $x^6 \equiv -108 \pmod p$ has a solution if and only if $(-108)^{\frac{p-1}{\gcd(6,p-1)}} = (-108)^{\frac{p-1}{6}} \equiv 1 \pmod p$. It is sufficient to show that $(-108)^{\frac{p-1}{6}} \equiv 2^{\frac{p-1}{3}} \pmod p$.

Let $(\frac{a}{p})$ be the Legendre symbol. By the Gauss quadratic reciprocity law, one has

$$\left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{3-1}{2}}\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{3}{p}\right).$$

Then by Euler's criterion, one has

$$(-3)^{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} 3^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}}\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) \equiv \left(\frac{1}{3}\right) \equiv 1 \pmod p$$

since $p \equiv 1 \pmod 6$. Therefore $(-108)^{\frac{p-1}{6}} = 2^{\frac{p-1}{3}} \cdot (-3)^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{3}} \pmod p$ as desired. This completes the proof of Lemma 2.5. □

Lemma 2.6. *If p is a prime number such that $p \equiv 1 \pmod 6$ and $2^{\frac{p-1}{3}} \not\equiv 1 \pmod p$, then $P_{p,k} = 1$.*

Proof. Since $2^{\frac{p-1}{3}} \not\equiv 1 \pmod p$, by Lemma 2.5, $x^3 + 2 \equiv 0 \pmod p$ has no solution. Thus for any positive integer n, one has $v_p(n^3 + 2) = 0$. It then follows that $\mathcal{G}_{p,k}(n) = v_p(\mathcal{G}_k(n)) = 0$. So $P_{p,k} = 1$ as desired. Lemma 2.6 is proved. □

Theorem 2.7. *Let k be a positive integer with $k \geq 2$. Then Q_k is a period of $\mathcal{G}_k$.*

Proof. By Lemmas 2.2-2.6, one has

$$\begin{aligned} P_k =& P_{2,k}P_{3,k}\Bigg(\prod_{\substack{p|R_k \\ p\equiv 1 \pmod 6 \\ 2^{\frac{p-1}{3}} \not\equiv 1 \pmod p}} P_{p,k}\Bigg)\Bigg(\prod_{\substack{p|R_k \\ p\equiv 5 \pmod 6}} P_{p,k}\Bigg)\Bigg(\prod_{\substack{p|P_k \\ p\equiv 1 \pmod 6 \\ 2^{\frac{p-1}{3}} \equiv 1 \pmod p}} P_{p,k}\Bigg) \\ =& 2^{\frac{(-1)^k+1}{2}} 3^{\lceil\{\frac{k+1}{3}\}\rceil}\Bigg(\prod_{\substack{p|R_k \\ p\equiv 5 \pmod 6}} P_{p,k}\Bigg)\Bigg(\prod_{\substack{p|R_k \\ p\equiv 1 \pmod 6 \\ 2^{\frac{p-1}{3}} \equiv 1 \pmod p}} P_{p,k}\Bigg). \end{aligned} \tag{2.19}$$

Since $P_{p,k}$ is a power of p for each prime p, the product

$$\Bigg(\prod_{\substack{p|R_k \\ p\equiv 5 \pmod 6}} P_{p,k}\Bigg)\Bigg(\prod_{\substack{p|R_k \\ p\equiv 1 \pmod 6 \\ 2^{\frac{p-1}{3}} \equiv 1 \pmod p}} P_{p,k}\Bigg)$$

divides the integer

$$\frac{R_k}{2^{v_2(R_k)}3^{v_3(R_k)}\prod\limits_{\substack{p|R_k\\ p\equiv 1(\mathrm{mod} 6)\\ 2^{\frac{p-1}{3}}\not\equiv 1(\mathrm{mod} p)}} p^{v_p(R_k)}}.$$

Thus $P_k|Q_k$ and Q_k is a period of $\mathcal{G}_k$. This concludes the proof of Theorem 2.7. □

Consequently, we prove a result which may be of independently interest.

Lemma 2.8. *There is at most one prime p such that $v_p(k+1) \geq v_p(R_k) \geq 1$.*

Proof. Suppose that there are two distinct primes p and q such that $v_p(k+1) \geq v_p(R_k) \geq 1$ and $v_q(k+1) \geq v_q(R_k) \geq 1$. Then $k+1 \geq p$ and $k+1 \geq q$. Furthermore, we have

$$k+1 \geq p^{v_p(R_k)}q^{v_q(R_k)} \geq \max(pq, p^{v_p(L_k)}q^{v_q(L_k)}),$$

where $L_k := \mathrm{lcm}_{1\leq i\leq k}\{i\}$.

If $v_p(L_k) = 0$ or $v_q(L_k) = 0$, then $p \geq k+1$ or $q \geq k+1$. Thus $k+1 = p$ or q since $k+1 \geq p$ and $k+1 \geq q$. We arrive at a contradiction since $k+1 \geq pq$.

If $v_p(L_k) \geq 1$ and $v_q(L_k) \geq 1$, then

$$k+1 \geq p^{v_p(L_k)}q^{v_q(L_k)} > \min(p^{v_p(L_k)+1}, q^{v_q(L_k)+1}). \tag{2.20}$$

But the inequality $p^{v_p(L_k)+1} = p^{\lfloor \log_p k\rfloor+1} > p^{\log_p k} = k$ together with $q^{v_q(L_k)+1} = q^{\lfloor \log_q k\rfloor+1} > q^{\log_q k} = k$ implies that

$$\min(p^{v_p(L_k)+1}, q^{v_q(L_k)+1}) > k. \tag{2.21}$$

Clearly, (2.20) is contradict to (2.21). So the assumption is not true. Thus Lemma 2.8 is proved. □

In the conclusion of this section, we state two well-known results that are also needed in this paper.

Lemma 2.9. *(Hensel's lemma) (see, for example, [19] or [21]) Let p be a prime and $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n with leading coefficient not divisible by p. If there exists an integer x_1 such that $f(x_1) \equiv 0 \pmod p$ and $f'(x_1) \not\equiv 0 \pmod p$, then for every integer $k \geq 2$, there exists an integer x_k suck that $f(x_k) \equiv 0 \pmod{p^k}$ and $x_k \equiv x_{k-1} \pmod{p^{k-1}}$.*

Lemma 2.10. *(Theorem 7.2 of [18]) Let p be a prime and let a and m be positive integers such that $\gcd(p,a) = 1$. Then either the congruence $x^m \equiv a \pmod p$ has no solution or it has $\gcd(m, p-1)$ solutions.*

3. Local periods for the case $p \equiv 5 \pmod 6$

Throughout this section, we always assume that p is a prime number such $p|R_k$ and $p \equiv 5 \pmod 6$.

Lemma 3.1. *Then the congruence $x^3+2 \equiv 0 \pmod p$ has exactly one solution in any complete residue system modulo p which is given by $x \equiv \left(\frac{-2}{p}\right)(-2)^{\frac{p+1}{6}} \pmod p$.*

Proof. By the Euler's criterion, one has $(-2)^{\frac{p-1}{2}} \equiv (\frac{-2}{p}) \pmod p$. Since -2 is coprime to p, one has $(\frac{-2}{p}) = \pm 1$. Then by the Fermat's little theorem, we have

$$\Big(\Big(\frac{-2}{p}\Big)(-2)^{\frac{p+1}{6}}\Big)^3 = \Big(\frac{-2}{p}\Big)(-2)^{\frac{p+1}{2}} \equiv (-2)^{\frac{p-1}{2}+\frac{p+1}{2}} \equiv (-2)^p \equiv (-2) \pmod p.$$

So $\left(\frac{-2}{p}\right)(-2)^{\frac{p+1}{6}}$ is a solution of the congruence $x^3 \equiv -2 \pmod p$.

Since $p \equiv 5 \pmod 6$, one has $\gcd(3, p-1) = 1$. Then by Lemma 2.10, one has that the congruence $x^3 \equiv -2 \pmod p$ has only one solution. Hence Lemma 3.1 is proved. □

Lemma 3.2. *The congruence $x^6 + 108 \equiv 0 \pmod p$ has no solution.*

Proof. At first, we have $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$. Then by the Gauss quadratic reciprocity law, one has

$$\left(\frac{-108}{p}\right) = \left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = \left(\frac{-1}{p}\right)(-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right) = \left(\frac{p}{3}\right) = \left(\frac{2}{3}\right) = -1$$

since $p \equiv 5 \pmod 6$. In other words, $x^2 + 108 \equiv 0 \pmod p$ has no solution. This implies that the congruence $x^6 + 108 \equiv 0 \pmod p$ has no solution. So Lemma 3.2 is proved. □

Lemma 3.3. *Let e and n be any given positive integers. Each of the following holds:*

(i). *There exists exactly one term divisible by p^e in any p^e consecutive terms of the cubic progression $\{(n+i)^3+2\}_{i\in\mathbb{N}}$.*

(ii). *There is at most one element divisible by $p^{v_p(R_k)+1}$ in $S_k(n)$.*

Proof. (i). By Lemma 3.1, for any prime $p \equiv 5 \pmod 6$, $x^3+2 \equiv 0 \pmod p$ has exactly one solution in any given complete residue system modulo p. It then follows immediately from Lemma 2.9 (Hensel's lemma) that for any positive integer e, the congruence $x^3 + 2 \equiv 0 \pmod{p^e}$ has exactly one solution in a complete residue system modulo p^e. So there exist exactly one term divisible by p^e in any p^e consecutive terms of the cubic progression $\{(n+i)^3+2\}_{i\in\mathbb{N}}$. Part (i) is proved.

(ii). Let $l_p := v_p(R_k)$. Then by Lemmas 3.1 and 3.2, one has

$$l_p = v_p(\mathrm{lcm}_{1\le i\le k}\{i(i^6+108)\}) = v_p(\mathrm{lcm}_{1\le i\le k}\{i\}) = \max_{1\le i\le k}\{v_p(i)\} \tag{3.1}$$

for all primes $p \equiv 5 \pmod 6$. By (3.1), one derives that

$$p^{l_p} \le k < p^{l_p+1} \tag{3.2}$$

Suppose that there exist integers i_1 and i_2 such that $0 \le i_1 < i_2 \le k$ and $(n+i_1)^3+2 \equiv (n+i_2)^3+2 \equiv 0 \pmod{p^{l_p+1}}$. By part (i), one has $i_2 - i_1 \ge p^{l_p+1}$. This is a contradiction since $0 \le i_1 < i_2 \le k < p^{l_p+1}$. Therefore there is at most one term divisible by p^{l_p+1} in $S_k(n)$. This finishes the proof of part (ii). □

Lemma 3.4. *Let p be a prime number such that $p \equiv 5 \pmod 6$ and $p|R_k$. Then $P_{p,k} = p^{v_p(R_k)}$ except that $v_p(k+1) \ge v_p(R_k) \ge 1$, in which case one has $P_{p,k} = 1$.*

Proof. First of all, part (ii) of Lemma 3.3 tells us that there is at most one element divisible by p^{l_p+1} in $S_k(n)$ for any positive integer n.

Now let $v_p(k+1) \geq v_p(R_k) \geq 1$. The inequality (3.2) implies that $p^{l_p} < k+1 \leq p^{l_p+1}$. Thus we can write $k+1 = vp^{l_p}$ for some integer v with $2 \leq v \leq p$. For any positive integers n and e with $1 \leq e \leq l_p$, it follows from Lemma 3.3 (i) that $|S_k^{(e)}(n)| = vp^{l_p-e}$. Then $|S_k^{(e)}(n)| = |S_k^{(e)}(n+1)|$. By (2.15), one has $f_e(n) = f_e(n+1)$. Thus $P_{p,k} = 1$ as desired. Lemma 3.4 is true if $v_p(k+1) \geq v_p(R_k) = l_p$.

In what follows, we let $v_p(k+1) < v_p(R_k) = l_p$. Then we can write $k+1 \equiv r \pmod{p^{l_p}}$ for some integer r with $p^{v_p(k+1)} \leq r \leq p^{l_p} - 1$. So one can write $k+1 = vp^{l_p} + r$ for some integer v. By (3.2), one has $1 \leq v < p$. Since for all positive integers n and t, $(n+tp^{l_p-1})^3 + 2 \equiv n^3 + 2 \pmod{p^e}$ holds for any integer e with $1 \leq e \leq l_p - 1$, we deduce that $|S_k^{(e)}(n)| = |S_k^{(e)}(n+tp^{l_p-1})|$. It follows that

$$\sum_{e=1}^{l_p-1} f_e(n+tp^{l_p-1}) = \sum_{e=1}^{l_p-1} f_e(n). \tag{3.3}$$

By Lemma 2.2, $P_{p,k} | p^{l_p}$. Then $P_{p,k} = p^{l_p}$ if and only if p^{l_p-1} is not a period of $\mathcal{G}_{p,k}$. So it is sufficient to prove that there exist a positive integers n_0 such that $\mathcal{G}_{p,k}(n_0) \neq \mathcal{G}_{p,k}(n_0 + p^{l_p-1})$. By Lemma 3.3, (2.14) and (3.3), this is equivalent to showing that

$$f_{l_p}(n_0) \neq f_{l_p}(n_0 + p^{l_p-1}). \tag{3.4}$$

Clearly, to prove (3.4), it is enough to show that there is a positive integer n' such that

$$f_{l_p}(n') > f_{l_p}(n' + p^{l_p-1}). \tag{3.5}$$

By Lemma 3.3 (i), the congruence $x^3 + 2 \equiv 0 \pmod{p^{l_p}}$ has exactly one solution in a complete residue system modulo p^{l_p}. Let n_1 be a positive integer such that $n_1^3 + 2 \equiv 0 \pmod{p^{l_p}}$. Then any term divisible by p^{l_p} in the cubic progression $\{n^3+2\}_{n\in\mathbb{N}}$ must be of the form $(n_1 + tp^{l_p})^3 + 2$ for some integer t. In the following we show that (3.5) is true. For this purpose, we consider the following two cases:

Case 1. $p^{v_p(k+1)} \leq r \leq p^{l_p} - p^{l_p-1}$. Then $n_1 + k = n_1 + vp^{l_p} + r - 1 \geq n_1 + vp^{l_p}$. It infers that the set of all the terms divisible by p^{l_p} in the set $S_k(n_1)$ contains the set $\{n_1^3 + 2, (n_1 + p^{l_p})^3 + 2, \cdots, (n_1 + vp^{l_p})^3 + 2\}$. Therefore $f_{l_p}(n_1) = |S_k^{(l_p)}(n_1)| - 1 \geq (v+1) - 1 \geq v$.

On the other hand, since $r \leq p^{l_p} - p^{l_p-1}$, we have $n_1 + p^{l_p-1} + k = n_1 + vp^{l_p} + p^{l_p-1} + r - 1 < n_1 + (v+1)p^{l_p}$. But $n_1 + p^{l_p-1} > n_1$. Hence the set of the terms divisible by p^{l_p} in the set $S_k(n_1 + p^{l_p-1})$ is contained in the set $\{(n_1 + p^{l_p})^3 + 2, (n_1 + 2p^{l_p})^3 + 2, \cdots, (n_1 + vp^{l_p})^3 + 2\}$. So $|S_k^{(l_p)}(n_1 + p^{l_p-1})| \leq v$. It follows that $f_{l_p}(n_1 + p^{l_p-1}) = \max\{0, |S_k^{(l_p)}(n_1 + p^{l_p-1})| - 1\} \leq v - 1$. Then picking $n' = n_1$ gives us the desired result (3.5). The claim is proved in this case.

Case 2. $p^{l_p} - p^{l_p-1} + 1 \leq r \leq p^{l_p} - 1$. Let $n_2 := n_1 + p^{l_p} - p^{l_p-1} + 1$. Then $n_2 \leq n_1 + p^{l_p}$. Since p is a prime number with $p \equiv 5 \pmod 6$, one has $p^{l_p} > 2p^{l_p-1}$. Then we have

$$n_2 + k = n_1 + (v+1)p^{l_p} + r - p^{l_p-1} + 1 > n_1 + (v+2)p^{l_p} - 2p^{l_p-1} > n_1 + (v+1)p^{l_p}.$$

So the set of all the terms divisible by p^{l_p} in the set $S_k(n_2)$ contains the set $\{(n_1 + p^{l_p})^3 + 2, \cdots, (n_1 + (v+1)p^{l_p})^3 + 2\}$. Therefore $f_{l_p}(n_2) = |S_k^{(l_p)}(n_2)| - 1 \geq (v+1) - 1 \geq v$.

However, one has $n_2 + p^{l_p-1} = n_1 + p^{l_p} + 1 > n_1 + p^{l_p}$ and

$$n_2 + p^{l_p-1} + k = n_1 + (v+1)p^{l_p} + r \leq n_1 + (v+1)p^{l_p} - 1 < n_1 + (v+2)p^{l_p}.$$

Then the set of all the terms divisible by p^{l_p} in the set $S_k(n_2 + p^{l_p-1})$ is contained in the set $\{(n_1 + 2p^{l_p})^3 + 2, (n_1 + 3p^{l_p})^3 + 2, \cdots, (n_1 + (v + 1)p^{l_p})^3 + 2\}$. It implies that $|S_k^{(l_p)}(n_2 + p^{l_p-1})| \le v$. Thus $f_{l_p}(n_2 + p^{l_p-1}) = |S_k^{(l_p)}(n_2 + p^{l_p-1})| - 1 \le v - 1$. Hence $f_{l_p}(n_2) > f_{l_p}(n_2 + p^{l_p-1})$. So letting $n' = n_2$ gives us the desired result (3.5). The claim is still true in this case.

Finally, by (3.5), we have $P_{p,k} \nmid p^{l_p-1}$. That is, p^{l_p-1} is not a period of $\mathcal{G}_{p,k}$. Thus $P_{p,k} = p^{v_p(R_k)}$ if $v_p(k + 1) < v_p(R_k)$.

The proof of Lemma 3.4 is complete. □

4. Local periods for the case $p \equiv 1 \pmod 6$

Throughout this section, we always assume that p is a prime number such that $p \equiv 1 \pmod 6$, $p|R_k$ and $2^{\frac{p-1}{3}} \equiv 1 \pmod p$. Then $\gcd(p, 108) = 1$. We begin with the following result.

Lemma 4.1. *Let e and n be positive integers. Each of the following is true:*

(i). *There exist exactly three terms divisible by p^e in any p^e consecutive terms of the cubic progression $\{(n + i)^3 + 2\}_{i\in\mathbb{N}}$.*

(ii). *There exist exactly six terms divisible by p^e in any p^e consecutive terms of the cubic progression $\{(n + i)^6 + 108\}_{i\in\mathbb{N}}$.*

Proof. (i). Since $p \equiv 1 \pmod 6$ and $2^{\frac{p-1}{3}} \equiv 1 \pmod p$, by Lemmas 2.5 and 2.10, the congruence $x^3 + 2 \equiv 0 \pmod p$ has exactly $\gcd(3, p - 1) = 3$ solutions. It then follows immediately from Lemma 2.9 (Hensel's lemma) that for any positive integer e, the congruence $x^3 + 2 \equiv 0 \pmod{p^e}$ has exactly 3 solutions. Hence there are exactly 3 terms divisible by p^e in any p^e consecutive terms of the cubic progression $\{(n + i)^3 + 2\}_{i\in\mathbb{N}}$. Part (i) is proved.

(ii). Likewise, by Lemmas 2.5 to 2.10 we know that the congruence $x^6 + 108 \equiv 0 \pmod{p^e}$ has exactly $\gcd(6, p-1) = 6$ solutions. Hence there are exactly 6 terms divisible by p^e in any p^e consecutive terms of the cubic progression $\{(n + i)^3 + 2\}_{i\in\mathbb{N}}$. Part (ii) is proved. □

For any positive integer e, we define d_{p^e} to be the smallest positive root of $x^6 + 108 \equiv 0 \pmod{p^e}$. By Lemma 4.1, one has $d_{p^e} < p^e$ for any positive integer e. For the simplicity, we write $r_0 := \max_{1\le i\le k}\{v_p(i)\} = v_p(L_k)$. Then $d_{p^{r_0}} < p^{r_0} \le k$ and $v_p((d_{p^{r_0}})^6 + 108) \ge r_0 = \max_{1\le i\le k}\{v_p(i)\}$. Since $v_p(\gcd(i, i^6 + 108)) = v_p(\gcd(i, 108)) = 0$ for any positive integer i and $v_p((d_{p^{r_0}})^6 + 108) \le \max_{1\le i\le k}\{v_p(i^6 + 108)\}$, we have

$$
\begin{aligned}
l_p =& v_p(R_k) = \max_{1\le i\le k}\{v_p(i(i^6 + 108))\} \\
&= \max_{1\le i\le k}\{v_p(\gcd(i, i^6 + 108)) + v_p(\mathrm{lcm}(i, i^6 + 108))\} \\
&= \max_{1\le i\le k}\{v_p(\mathrm{lcm}(i, i^6 + 108))\} \\
&= \max_{1\le i\le k}\{\max(v_p(i^6 + 108), v_p(i))\} \\
&= \max(\max_{1\le i\le k}\{v_p(i^6 + 108)\}, \max_{1\le i\le k}\{v_p(i)\}) \\
&= \max_{1\le i\le k}\{v_p(i^6 + 108)\}. \qquad (4.1)
\end{aligned}
$$

So $j^6 + 108 \equiv 0 \pmod{p^{l_p}}$ for some integer j with $1 \le j \le k$. By the definition of $d_{p^{l_p}}$, one has $k \ge j \ge d_{p^{l_p}}$ and $v_p(d^6_{p^{l_p}} + 108) \ge l_p$. Then by the fact that $k \ge d_{p^{l_p}}$ and (4.1), we have $v_p(d^6_{p^{l_p}} + 108) \le l_p$. Therefore $l_p = v_p(d^6_{p^{l_p}} + 108)$.

Since $v_p(d^6_{p^{l_p+1}} + 108) \ge l_p + 1$ and $l_p = \max_{1\le i\le k}\{v_p(i^6 + 108)\}$, one deduces that $k < d_{p^{l_p+1}}$. Thus we arrive at

$$d_{p^{l_p}} \le k < d_{p^{l_p+1}}. \tag{4.2}$$

Lemma 4.2. *Let p be an odd prime number with $2^{\frac{p-1}{3}} \equiv 1 \pmod p$ and e be a positive integer. If x_1 and x_2 are distinct roots of the congruence $x^3 + 2 \equiv 0 \pmod{p^e}$ in the interval $[1, p^e]$, then $x_1^2x_2 + x_1x_2^2 \equiv 2 \pmod{p^e}$ and $x_2 - x_1$ is a root of the congruence $x^6 + 108 \equiv 0 \pmod{p^e}$.*

Proof. Obviously, one has $\gcd(x_2, p) = \gcd(x_1, p) = 1$.

First, we show that $\gcd(x_2 - x_1, p) = 1$. Clearly, $\gcd(x_2 - x_1, p) = 1$ if $e = 1$. If $e > 1$, then by Lemma 2.9 (Hensel's Lemma), there exist two distinct integers x_1' and x_2' in the interval $[1, p]$ such that $x_1 - x_1' \equiv x_2 - x_2' \equiv 0 \pmod p$ and $x_1'^3 + 2 \equiv x_2'^3 + 2 \equiv 0 \pmod p$. Therefore one has that $\gcd(x_2 - x_1, p) \equiv \gcd(x_2' - x_1', p) \equiv 0 \pmod p$. Thus we arrive at $\gcd(x_2 - x_1, p) = 1$.

Subsequently, we prove that

$$x_1^2x_2 + x_1x_2^2 \equiv 2 \pmod{p^e}. \tag{4.3}$$

In fact, by $x_1^3 + 2 \equiv x_2^3 + 2 \equiv 0 \pmod{p^e}$, we deduce that $x_1(x_2^3 - x_1^3) = x_1(x_2 - x_1)(x_2^2 + x_1x_2 + x_1^2) \equiv (x_2 - x_1)(x_1x_2^2 + x_1^2x_2 - 2) \equiv 0 \pmod{p^e}$. Since $\gcd(x_2 - x_1, p) = 1$, we have $x_1x_2^2 + x_1^2x_2 - 2 \equiv 0 \pmod{p^e}$. This implies that (4.3) is true.

Finally, we show that $(x_2 - x_1)^6 + 108 \equiv 0 \pmod{p^e}$. Actually, by (4.3) and noticing that $x_1^3 \equiv x_2^3 \equiv -2 \pmod{p^e}$, one derives that

$$\begin{aligned}(x_2 - x_1)^6 + 108 &= x_2^6 - 6x_2^5x_1 + 15x_2^4x_1^2 - 20x_2^3x_1^3 + 15x_2^2x_1^4 - 6x_2x_1^5 + x_1^6 + 108\\ &\equiv 4 + 12x_1x_2^2 - 30x_2x_1^2 - 80 - 30x_1x_2^2 + 12x_2x_1^2 + 4 + 108\\ &\equiv 18(2 - x_1x_2^2 - x_1^2x_2) \equiv 0 \pmod{p^e}.\end{aligned}$$

In other words, we have $x_2 - x_1$ is a root of the congruence $x^6 + 108 \equiv 0 \pmod{p^e}$ as desired. This concludes the proof of Lemma 4.2. □

Lemma 4.3. *There is at most one element divisible by p^{l_p+1} in $S_k(n)$ for any positive integer n.*

Proof. For any positive integer e, by Lemma 4.1, we know that the congruence $x^3 + 2 \equiv 0 \pmod{p^e}$ holds three distinct roots in the interval $[1, p^e]$. We write x_{1e}, x_{2e} and x_{3e} as the three roots of $x^3 + 2 \equiv 0 \pmod{p^e}$ in the interval $[1, p^e]$. Furthermore, by Lemma 2.9, we have $x_{i1} \equiv x_{i2} \equiv \cdots \equiv x_{il_p} \pmod p$ for $1 \le i \le 3$. We may let $x_{11} < x_{21} < x_{31}$. By Lemma 4.2, one has $\gcd(x_{i_1j} - x_{i_2j}, p) = \gcd(x_{i_1j}, p) = 1$ for $1 \le i_1 \ne i_2 \le 3$ and $1 \le j \le l_p$.

Suppose that there exist positive integers n_1 and i_0 with $1 \le i_0 \le k$ such that

$$n_1^3 + 2 \equiv (n_1 + i_0)^3 + 2 \equiv 0 \pmod{p^{l_p+1}}. \tag{4.4}$$

By Lemma 2.9 (Hensel's lemma), we know that the terms divisible by p^{l_p+1} in the cubic progression $\{n^3+2\}_{n\in\mathbb{N}}$ must be of the form $(x_{1p^{l_p}} + tp^{l_p})^3 + 2$, $(x_{2p^{l_p}} + tp^{l_p})^3 + 2$ or $(x_{3p^{l_p}} + tp^{l_p})^3 + 2$ for some integer t. So there exist $j_1, j_2 \in \{1, 2, 3\}$ and integers $t_1, t_2 \ge 0$ such that $n_1 = x_{j_1l_p} + t_1p^{l_p}$ and $n_1 + i_0 = x_{j_2l_p} + t_2p^{l_p}$.

First, we show that $\gcd(i_0, p) = 1$. If $j_1 = j_2$, then $i_0 = (t_2 - t_1)p^{l_p} = tp^{l_p}$ where $t := t_2 - t_1$. By $i_0 \le k < d_{p^{l_p+1}} < p^{l_p+1}$, one has $tp^{l_p} < p^{l_p+1}$ which implies that $t \le p - 1$. Then $(n_1 + i_0)^3 + 2 = n_1^3 + 3n_1^2 tp^{l_p} + 3n_1 t^2 p^{2l_p} + t^3 p^{3l_p} + 2 \equiv 3n_1^2 tp^{l_p} \not\equiv 0 \pmod{p^{l_p+1}}$. This is a contradiction with the fact that $(n_1 + i_0)^3 + 2 \equiv 0 \pmod{p^{l_p+1}}$. Therefore we must have $j_1 \ne j_2$. Then by Lemma 4.2, one has $\gcd(i_0, p) = \gcd(x_{j_2 l_p} - x_{j_1 l_p}, p) = 1$. Thus $n_1 + i_0 \not\equiv n_1 \pmod{p^{l_p+1}}$ and so $n_1 + i_0$ and n_1 are distinct roots of the congruence $x^3 + 2 \equiv 0 \pmod{p^{l_p+1}}$.

By Lemma 4.2, one has i_0 is a root of the congruence $x^6 + 108 \equiv 0 \pmod{p^{l_p+1}}$, which implies $i_0 \ge d_{p^{l_p+1}}$ It contradicts with the assumption that $1 \le i_0 \le k < d_{p^{l_p+1}}$. So the assumption that there exist positive integers n_1 and i_0 with $1 \le i_0 \le k$ such that (4.4) holds is false. This proves Lemma 4.3. □

Lemma 4.4. *Let e be a positive integer and let x_{1e}, x_{2e} and x_{3e} be the three roots of $x^3+2 \equiv 0 \pmod{p^e}$ such that $x_{1e} < x_{2e} < x_{3e}$ and $x_{3e} - x_{1e} < p^e$. Then $d_{p^e} = \min(x_{2e} - x_{1e}, x_{3e} - x_{2e}, x_{1e} - x_{3e} + p^e)$ and $d_{p^e} \le \frac{p^e-4}{3}$.*

Proof. Let $y_1 := x_{2e} - x_{1e}$, $y_2 := x_{3e} - x_{2e}$, $y_3 := x_{3e} - x_{1e}$, $y_4 := p^e - y_1$, $y_5 := p^e - y_2$ and $y_6 := p^e - y_3$. Then $1 \le y_i < p^e$ for any integer i with $1 \le i \le 6$.

By Lemma 4.2, one knows that y_1, y_2 and y_3 are the roots of the congruence $x^6 + 108 \equiv 0 \pmod{p^e}$. We then deduce that y_4, y_5 and y_6 are the roots of $x^6 + 108 \equiv 0 \pmod{p^e}$ in the interval $[1, p^e]$. So all the y_i $(1 \le i \le 6)$ are the roots of the congruence $x^6 + 108 \equiv 0 \pmod{p^e}$ in the interval $[1, p^e]$.

Now we show that $y_i \not\equiv y_j \pmod{p^e}$ for all integers i and j with $1 \le i \ne j \le 6$. It is obvious that $y_1 \not\equiv y_3 \pmod{p^e}$, $y_2 \not\equiv y_3 \pmod{p^e}$, $y_4 \not\equiv y_6 \pmod{p^e}$ and $y_5 \not\equiv y_6 \pmod{p^e}$. Now we show that $y_1 \not\equiv y_2 \pmod{p^e}$. If $y_1 \equiv y_2 \pmod{p^e}$, then $x_{1e} + x_{3e} \equiv 2x_{2e} \pmod{p^e}$. By Lemma 4.2, one has $x_{1e}^2 x_{3e} + x_{1e} x_{3e}^2 \equiv 2 \pmod{p^e}$. It follows that

$$\begin{aligned}(x_{1e} + x_{3e})^3 - (2x_{2e})^3 &= x_{1e}^3 + x_{3e}^3 - 8x_{2e}^3 + 3x_{1e}^2 x_{3e} + 3x_{1e} x_{3e}^2 \\ &\equiv 12 + 3(x_{1e}^2 x_{3e} + x_{1e} x_{3e}^2) \equiv 18 \pmod{p^e}.\end{aligned}$$

In other words, $18 | p^e$, which contradicts with the assumption $p \equiv 1 \pmod 6$. So $x_{1e} + x_{3e} \not\equiv 2x_{2e} \pmod{p^e}$, i.e. $y_1 \not\equiv y_2 \pmod{p^e}$. This implies that $y_4 \not\equiv y_5 \pmod{p^e}$.

Likewise, one has $x_{2e} + x_{3e} \not\equiv 2x_{1e} \pmod{p^e}$. Then $y_6 - y_1 = p^e - (x_{3e} + x_{2e} - 2x_{1e}) \not\equiv 0 \pmod{p^e}$. Also we have $y_4 - y_1 = p^e - 2(x_{2e} - x_{1e}) \not\equiv 0 \pmod{p^e}$ and $y_5 - y_1 = p^e - (x_{3e} - x_{1e}) \not\equiv 0 \pmod{p^e}$. That is, $y_1 \not\equiv y_i \pmod{p^e}$ for any integer i with $4 \le i \le 6$. Similarly, one has $y_2 \not\equiv y_i \pmod{p^e}$ and $y_3 \not\equiv y_i \pmod{p^e}$ for any integer i with $4 \le i \le 6$. Hence $y_i \not\equiv y_j \pmod{p^e}$ for any integers i and j with $1 \le i \ne j \le 6$ as one desires. So all the y_i $(1 \le i \le 6)$ are pairwise distinct solutions of the congruence $x^6 + 108 \equiv 0 \pmod{p^e}$ in the interval $[1, p^e]$.

Consequently, we show that $d_{p^e} = \min(y_1, y_2, y_6)$. In fact, one has $y_3 = y_1 + y_2$ and $y_6 = y_4 - y_2 = y_5 - y_1$. It follows that $\max(y_1, y_2) < y_3$ and $\min(y_4, y_5) > y_6$. Thus $\min_{1\le i\le 6}\{y_i\} = \min(y_1, y_2, y_6)$. So by the definition of d_{p^e}, we have

$$d_{p^e} = \min_{1\le i\le 6}\{y_i\} = \min(y_1, y_2, y_6) = \min(x_{2e} - x_{1e}, x_{3e} - x_{2e}, x_{1e} - x_{3e} + p^e)$$

as required.

Finally, since y_1, y_2 and y_6 are pairwise distinct and $y_1 + y_2 + y_6 = p^e$, we derive that

$$p^e \ge d_{p^e} + (d_{p^e} + 1) + (d_{p^e} + 2) = 3d_{p^e} + 3.$$

It then follows that

$$d_{p^e} = \min(y_1, y_2, y_6) \le \left\lfloor \frac{p^e - 3}{3} \right\rfloor = \frac{p^e - 4}{3}$$

as one desires. This finishes the proof of Lemma 4.4. □

Lemma 4.5. *Let $k \ge 1$ be an integer such that $v_p(k+1) < v_p(R_k)$. Let x_1, x_2 and x_3 be the three positive roots of $x^3 + 2 \equiv 0 \pmod{p^{l_p}}$ such that $x_1 < x_2 < x_3$ and $x_3 - x_1 \le p^{l_p} - 1$. If $\max(x_2 - x_1, x_3 - x_2) < x_1 + p^{l_p} - x_3$, then there exist positive integers n_1 and n_2 such that $n_1 \equiv n_2 \pmod{p^{l_p - 1}}$, $|S_k^{(l_p)}(n_1)| > |S_k^{(l_p)}(n_2)|$ and $|S_k^{(l_p)}(n_1)| \ge 2$.*

Proof. By Lemma 4.1 (i), there exist exactly 3 terms divisible by p^{l_p} in any p^{l_p} consecutive terms of the cubic progression $\{(n+i)^3 + 2\}_{i \in \mathbb{N}}$. Since $x_i^3 + 2 \equiv 0 \pmod{p^{l_p}}$ for $1 \le i \le 3$ with $x_1 < x_2 < x_3$ and $x_3 - x_1 \le p^{l_p} - 1$, we have $x_0^3 + 2 \not\equiv 0 \pmod{p^{l_p}}$ for any integer x_0 in the interval (x_1, x_3) with $x_0 \ne x_2$. By Lemma 4.1, we know that the terms divisible by p^{l_p} in the cubic progression $\{n^3 + 2\}_{n \in \mathbb{N}}$ must be of the form $(x_1 + tp^{l_p})^3 + 2$, or $(x_2 + tp^{l_p})^3 + 2$, or $(x_3 + tp^{l_p})^3 + 2$ with t being an integer. It then follows from Lemma 4.4 that

$$d_{p^{l_p}} \le \frac{p^{l_p} - 4}{3} < \frac{p+2}{3} p^{l_p - 1} \text{ and } d_{p^{l_p+1}} < \frac{p+2}{3} p^{l_p}. \tag{4.5}$$

So there exists a positive integer $u \in [1, \frac{p+2}{3}]$ such that

$$(u-1)p^{l_p - 1} \le d_{p^{l_p}} \le up^{l_p - 1} - 1. \tag{4.6}$$

Since $v_p(k+1) < l_p$, one has $p^{l_p} \nmid (k+1)$. Then we can write

$$k = vp^{l_p} + r \tag{4.7}$$

for some integers v and r with $0 \le v < \frac{p+2}{3}$ and $0 \le r < p^{l_p} - 1$. But $\max(x_2 - x_1, x_3 - x_2) < x_1 + p^{l_p} - x_3$ and $(x_2 - x_1) + (x_3 - x_2) + (x_1 + p^{l_p} - x_3) = p^{l_p}$. Hence

$$x_1 + p^{l_p} - x_3 \ge \left\lceil \frac{p^{l_p} + 3}{3} \right\rceil = \frac{p^{l_p} + 5}{3}. \tag{4.8}$$

Further, by Lemma 4.4 and the hypothesis one can write $d_{p^{l_p}} = x_{i_0+1} - x_{i_0}$ for $i_0 \in \{1, 2\}$. Let $x_4 := x_1 + p^{l_p}$. Then $x_3 < x_4$. Now we show Lemma 4.5 by considering the following five cases.

Case 1. $0 \le r < d_{p^{l_p}}$. Then by (4.2) and (4.7), we have $r + 1 \le d_{p^{l_p}} \le vp^{l_p} + r$, and so $v \ge 1$. Let $n_1 := x_3 - \min(p^{l_p - 1}, r+1) + 1$. Then $n_1 \le x_3$ and $n_1 + k = x_3 + vp^{l_p} + (r+1) - \min(p^{l_p - 1}, r+1) \ge x_3 + vp^{l_p}$. It follows that the set of all the terms divisible by p^{l_p} in the set $S_k(n_1)$ contains the set $\{x_3^3 + 2, (x_1 + p^{l_p})^3 + 2, (x_2 + p^{l_p})^3 + 2, (x_3 + p^{l_p})^3 + 2, \cdots, (x_1 + vp^{l_p})^3 + 2, (x_2 + vp^{l_p})^3 + 2, (x_3 + vp^{l_p})^3 + 2\}$. Thus $|S_k^{(l_p)}(n_1)| \ge 3v + 1 \ge 4$ since $v \ge 1$.

Now picking $n_2 := n_1 + p^{l_p - 1}$ gives us that $n_2 = x_3 + p^{l_p - 1} - \min(p^{l_p - 1}, r+1) + 1 > x_3$. But from (4.8) it follows that $x_1 + p^{l_p} - x_3 > p^{l_p - 1}$. This together with the assumption $x_1 + p^{l_p} - x_3 > d_{p^{l_p}} \ge r + 1$ implies that $x_1 + p^{l_p} - x_3 > \max(p^{l_p - 1}, r+1)$. Thus $x_3 + \max(p^{l_p - 1}, r+1) < x_1 + p^{l_p}$. We then deduce that $n_2 + k = x_3 + vp^{l_p} + p^{l_p - 1} + (r+1) - \min(p^{l_p - 1}, r+1) = x_3 + vp^{l_p} + \max(p^{l_p - 1}, r+1) < x_1 + (v+1)p^{l_p}$. Therefore the set of the terms divisible by p^{l_p} in the set $S_k(n_2)$ is contained in the set $\{(x_1 + p^{l_p})^3 + 2, (x_2 + p^{l_p})^3 + 2, (x_3 + p^{l_p})^3 + 2, \cdots, (x_1 + vp^{l_p})^3 + 2, (x_2 + vp^{l_p})^3 + 2, (x_3 + vp^{l_p})^3 + 2\}$. Thus $|S_k^{(l_p)}(n_2)| \le 3v$ which implies that $|S_k^{(l_p)}(n_1)| > |S_k^{(l_p)}(n_2)|$ as required. So Lemma 4.5 is true if $r \le d_{p^{l_p}}$.

Case 2. $d_{p^{l_p}} \le r < p^{l_p} - up^{l_p-1}$. Let $n_1 := x_{i_0}$. Since $r \ge d_{p^{l_p}}$ and $d_{p^{l_p}} = x_{i_0+1} - x_{i_0}$, one has $n_1 + k = x_{i_0} + r + vp^{l_p} \ge x_{i_0} + d_{p^{l_p}} + vp^{l_p} = x_{i_0+1} + vp^{l_p}$. It infers that the set of all the terms divisible by p^{l_p} in the set $S_k(n_1)$ contains the set $\{x_{i_0}^3 + 2, x_{i_0+1}^3 + 2, x_{i_0+2}^3 + 2, \cdots, (x_{i_0} + (v-1)p^{l_p})^3 + 2, (x_{i_0+1} + (v-1)p^{l_p})^3 + 2, (x_{i_0+2} + (v-1)p^{l_p})^3 + 2, (x_{i_0} + vp^{l_p})^3 + 2, (x_{i_0+1} + vp^{l_p})^3 + 2\}$. Hence $|S_k^{(l_p)}(n_1)| \ge 3v + 2 \ge 2$.

Now let $n_2 := x_{i_0} + up^{l_p-1}$. By (4.6), one has $n_2 > x_{i_0} + d_{p^{l_p}} = x_{i_0+1}$. However, it follows from $r < p^{l_p} - up^{l_p-1}$ that $n_2 + k = x_{i_0} + vp^{l_p} + up^{l_p-1} + r < x_{i_0} + vp^{l_p} + up^{l_p-1} + (p^{l_p} - up^{l_p-1}) = x_{i_0} + (v+1)p^{l_p}$. Thus the set of the terms divisible by p^{l_p} in the set $S_k(n_2)$ is contained in the set $\{x_{i_0+2}^3 + 2, (x_{i_0} + p^{l_p})^3 + 2, (x_{i_0+1} + p^{l_p})^3 + 2, (x_{i_0+2} + p^{l_p})^3 + 2, \cdots, (x_{i_0} + vp^{l_p})^3 + 2, (x_{i_0+1} + vp^{l_p})^3 + 2, (x_{i_0+2} + vp^{l_p})^3 + 2\}$. Thus $|S_k^{(l_p)}(n_2)| \le 3v + 1$. This implies that $|S_k^{(l_p)}(n_2)| < |S_k^{(l_p)}(n_1)|$. Lemma 4.5 holds in this case.

Case 3. $p^{l_p} - up^{l_p-1} \le r < p^{l_p} - d_{p^{l_p}} - 1$. Let $n_1 := x_{i_0+1} - up^{l_p-1} + 1$. Since $d_{p^{l_p}} = x_{i_0+1} - x_{i_0}$, by (4.6) one has $x_{i_0+1} - x_{i_0} \le up^{l_p-1} - 1$, i.e. $n_1 \le x_{i_0}$. But $1 \le u \le \frac{p+2}{3}$ together with $p > 4$ shows that $p^{l_p} > 2up^{l_p-1}$. Then it follows from $p^{l_p} - up^{l_p-1} \le r$ that $n_1 + k = x_{i_0+1} + vp^{l_p} - up^{l_p-1} + r + 1 \ge x_{i_0+1} + vp^{l_p} - up^{l_p-1} + (p^{l_p} - up^{l_p-1} + 1) = x_{i_0+1} + (v+1)p^{l_p} - 2up^{l_p-1} + 1 > x_{i_0+1} + vp^{l_p} + 1 > x_{i_0+1} + vp^{l_p}$. This concludes that the set of all the terms divisible by p^{l_p} in the set $S_k(n_1)$ contains the set $\{x_{i_0}^3 + 2, x_{i_0+1}^3 + 2, x_{i_0+2}^3 + 2, \cdots, (x_{i_0} + (v-1)p^{l_p})^3 + 2, (x_{i_0+1} + (v-1)p^{l_p})^3 + 2, (x_{i_0+2} + (v-1)p^{l_p})^3 + 2, (x_{i_0} + vp^{l_p})^3 + 2, (x_{i_0+1} + vp^{l_p})^3 + 2\}$. Thus $|S_k^{(l_p)}(n_1)| \ge 3v + 2 \ge 2$.

On the other hand, let $n_2 := x_{i_0+1} + 1$. Then $n_2 > x_{i_0+1}$. Since $r < p^{l_p} - d_{p^{l_p}} - 1 = p^{l_p} - (x_{i_0+1} - x_{i_0}) - 1$, one has $n_2 + k = x_{i_0+1} + vp^{l_p} + r + 1 < x_{i_0+1} + vp^{l_p} + (p^{l_p} - (x_{i_0+1} - x_{i_0})) = x_{i_0} + (v+1)p^{l_p}$. Hence the set of the terms divisible by p^{l_p} in the set $S_k(n_2)$ is contained in the set $\{x_{i_0+2}^3 + 2, (x_{i_0} + p^{l_p})^3 + 2, (x_{i_0+1} + p^{l_p})^3 + 2, (x_{i_0+2} + p^{l_p})^3 + 2, \cdots, (x_{i_0} + vp^{l_p})^3 + 2, (x_{i_0+1} + vp^{l_p})^3 + 2, (x_{i_0+2} + vp^{l_p})^3 + 2\}$. So $|S_k^{(l_p)}(n_2)| \le 3v + 1$. Thus Lemma 4.5 is proved in this case.

Case 4. $p^{l_p} - d_{p^{l_p}} - 1 \le r < p^{l_p} - p^{l_p-1}$. Let $n_1 := x_{i_0}$. We assert that $x_{i_0+2} - x_{i_0} < p^{l_p} - d_{p^{l_p}}$. In fact, if $i_0 = 1$, then by Lemma 4.4, one has $d_{p^{l_p}} = \min(x_2 - x_1, x_3 - x_2)$ since $\max(x_2 - x_1, x_3 - x_2) < x_1 + p^{l_p} - x_3$. So $d_{p^{l_p}} < x_1 + p^{l_p} - x_3$ and the assertion follows immediately. If $i_0 = 2$, then $d_{p^{l_p}} = x_3 - x_2 \le x_2 - x_1$. But the proof of Lemma 4.4 tells us that $x_3 - x_2 \not\equiv x_2 - x_1 \pmod{p^{l_p}}$. Thus $x_3 - x_2 < x_2 - x_1$ that implies that $d_{p^{l_p}} < x_2 - x_1$. It then follows from $x_4 = x_1 + p^{l_p}$ that $x_4 - x_2 < p^{l_p} - d_{p^{l_p}}$. The claim is proved.

Now by the claim and the hypothesis that $p^{l_p} - d_{p^{l_p}} - 1 \le r$, we derive that $x_{i_0+2} - x_{i_0} + 1 \le p^{l_p} - d_{p^{l_p}} \le r + 1$. Then $n_1 + k = x_{i_0} + vp^{l_p} + r \ge x_{i_0} + vp^{l_p} + (x_{i_0+2} - x_{i_0}) = x_{i_0+2} + vp^{l_p}$. It implies that the set of all the terms divisible by p^{l_p} in the set $S_k(n_1)$ contains the set $\{x_{i_0}^3 + 2, x_{i_0+1}^3 + 2, x_{i_0+2}^3 + 2, (x_{i_0} + p^{l_p})^3 + 2, \cdots, (x_{i_0} + vp^{l_p})^3 + 2, (x_{i_0+1} + vp^{l_p})^3 + 2, (x_{i_0+2} + vp^{l_p})^3 + 2\}$. Thus $|S_k^{(l_p)}(n_1)| \ge 3v + 3$.

Now let $n_2 := x_{i_0} + p^{l_p-1}$. Then $n_2 > x_{i_0}$. Since $r < p^{l_p} - p^{l_p-1}$, one has $n_2 + k = x_{i_0} + vp^{l_p} + p^{l_p-1} + r < x_{i_0} + vp^{l_p} + p^{l_p-1} + (p^{l_p} - p^{l_p-1}) = x_{i_0} + (v+1)p^{l_p}$. Hence the set of the terms divisible by p^{l_p} in the set $S_k(n_2)$ is contained in the set $\{x_{i_0+1}^3 + 2, x_{i_0+2}^3 + 2, (x_{i_0} + p^{l_p})^3 + 2, (x_{i_0+1} + p^{l_p})^3 + 2, (x_{i_0+2} + p^{l_p})^3 + 2, \cdots, (x_{i_0} + vp^{l_p})^3 + 2, (x_{i_0+1} + vp^{l_p})^3 + 2, (x_{i_0+2} + vp^{l_p})^3 + 2\}$. Therefore $|S_k^{(l_p)}(n_2)| \le 3v + 2$ that infers that the desired result $|S_k^{(l_p)}(n_2)| < |S_k^{(l_p)}(n_1)|$ is true. Hence Lemma 4.5 is proved in this case.

Case 5. $p^{l_p} - p^{l_p-1} \le r < p^{l_p} - 1$. Let $n_1 := x_3 + p^{l_p-1} + 1$. By (4.8), one has $x_1 + p^{l_p} - x_3 > p^{l_p-1} + 1$. Then $x_1 + p^{l_p} - n_1 = (x_1 + p^{l_p} - x_3) - (p^{l_p-1} + 1) > 0$, that is, $n_1 < x_1 + p^{l_p}$. Moreover, the assumption $p^{l_p} - p^{l_p-1} \le r$ implies that $n_1 + k = x_3 + p^{l_p-1} + vp^{l_p} + r + 1 > x_3 + p^{l_p-1} + vp^{l_p} + (p^{l_p} - p^{l_p-1}) = x_3 + (v+1)p^{l_p}$. This infers that the set of all the terms divisible by p^{l_p} in the set $S_k(n_1)$ contains the set $\{(x_1 + p^{l_p})^3 + 2, (x_2 + p^{l_p})^3 + 2, (x_3 + p^{l_p})^3 + 2, \cdots, (x_1 + (v+1)p^{l_p})^3 + 2, (x_2 + (v+1)p^{l_p})^3 + 2, (x_3 + (v+1)p^{l_p})^3 + 2\}$. Hence $|S_k^{(l_p)}(n_1)| \ge 3v + 3$.

Let $n_2 := x_3 + 1$. Then $n_2 > x_3$. Since $r < p^{l_p} - 1$, we deduce that $n_2 + k = x_3 + vp^{l_p} + r + 1 <$

$x_3 + vp^{l_p} + (p^{l_p} - 1) + 1 = x_3 + (v+1)p^{l_p}$. Then the set of the terms divisible by p^{l_p} in the set $S_k(n_2)$ is contained in the set $\{(x_1+p^{l_p})^3+2, (x_2+p^{l_p})^3+2, (x_3+p^{l_p})^3+2, \cdots, (x_1+(v+1)p^{l_p})^3+2, (x_2+(v+1)p^{l_p})^3+2\}$. Thus $|S_k^{(l_p)}(n_2)| \le 3v+2$. So $|S_k^{(l_p)}(n_2)| < |S_k^{(l_p)}(n_1)|$ as required.

This completes the proof of Lemma 4.5. □

Lemma 4.6. *Let p be a prime number such that $p \equiv 1 \pmod 6$, $p|R_k$ and $2^{\frac{p-1}{3}} \equiv 1 \pmod p$. Then $P_{p,k} = p^{v_p(R_k)}$ except that $v_p(k+1) \ge v_p(R_k) \ge 1$, in which case one has $P_{p,k} = 1$.*

Proof. First of all, we let $v_p(k+1) \ge v_p(R_k) := l_p$. Then $p^{l_p}|(k+1)$. But $k+1 \le p^{l_p+1}$. So we can write $k+1 = vp^{l_p}$ for some positive integer v with $1 \le v \le p$. For all positive integers n and e with $1 \le e \le l_p$, by Lemma 4.1 one deduces $|S_k^{(e)}(n)| = 3vp^{l_p-e}$. Thus $|S_k^{(e)}(n)| = |S_k^{(e)}(n+1)|$. By (2.15), we deduce that $f_e(n) = f_e(n+1)$. That is, $P_{p,k} = 1$ if $v_p(k+1) \ge v_p(R_k)$. So Lemma 4.6 is true if $v_p(k+1) \ge v_p(R_k)$.

In what follows, we let $v_p(k+1) < l_p$. By Lemma 2.2, p^{l_p} is a period of $\mathcal{G}_{p,k}$. So $P_{p,k} = p^{l_p}$ if and only if p^{l_p-1} is not a period of $\mathcal{G}_{p,k}$. In the following, we show that p^{l_p-1} is not a period of $\mathcal{G}_{p,k}$.

Let x_1, x_2 and x_3 be the three positive roots of $x^3 + 2 \equiv 0 \pmod{p^{l_p}}$ such that $x_1 < x_2 < x_3$ and $x_3 - x_1 \le p^{l_p} - 1$. Then $x_1 + p^{l_p}$ and $x_2 + p^{l_p}$ are the roots of $x^3 + 2 \equiv 0 \pmod{p^{l_p}}$ and $x_2 < x_3 < x_1 + p^{l_p} < x_2 + p^{l_p}$. By the proof of Lemma 4.4, one knows that $x_2 - x_1, x_3 - x_2$ and $x_1 + p^{l_p} - x_3$ are distinct roots of the congruence $x^6 + 108 \equiv 0 \pmod{p^{l_p}}$. Then exactly one of the following three cases happens:

(i). $\max(x_2 - x_1, x_3 - x_2) < x_1 + p^{l_p} - x_3$,

(ii). $\max(x_3 - x_2, x_1 + p^{l_p} - x_3) < x_2 - x_1$, which is equivalent to $\max(x_3 - x_2, (x_1 + p^{l_p}) - x_3) < x_2 + p^{l_p} - (x_1 + p^{l_p})$,

(iii). $\max(x_2 - x_1, x_1 + p^{l_p} - x_3) < x_3 - x_2$, which is equivalent to $\max((x_1 + p^{l_p}) - x_3, (x_2 + p^{l_p}) - (x_1 + p^{l_p})) < x_3 + p^{l_p} - (x_2 + p^{l_p})$.

So with Lemma 4.5 applied to the three roots x_1, x_2 and x_3 of $x^3 + 2 \equiv 0 \pmod{p^{l_p}}$ for case (i), and with Lemma 4.5 applied to the three roots x_2, x_3 and $x_1 + p^{l_p}$ of $x^3 + 2 \equiv 0 \pmod{p^{l_p}}$ for case (ii), and with Lemma 4.5 applied to the three roots $x_3, x_1 + p^{l_p}$ and $x_2 + p^{l_p}$ of $x^3 + 2 \equiv 0 \pmod{p^{l_p}}$ for case (iii), Lemma 4.5 tells us that there exist positive integers n_1 and n_2 such that $n_1 \equiv n_2 \pmod{p^{l_p-1}}$, $|S_k^{(l_p)}(n_1)| > |S_k^{(l_p)}(n_2)|$ and $|S_k^{(l_p)}(n_1)| \ge 2$. Then one can deduce that

$$f_{l_p}(n_1) > f_{l_p}(n_2). \tag{4.9}$$

On the other hand, one has $n_1^3 + 2 \equiv n_2^3 + 2 \pmod{p^e}$ for any integer e with $1 \le e \le l_p - 1$. Hence $|S_k^{(e)}(n_1)| = |S_k^{(e)}(n_2)|$. It then follows that

$$\sum_{e=1}^{l_p-1} f_e(n_1) = \sum_{e=1}^{l_p-1} f_e(n_2). \tag{4.10}$$

By Lemma 4.3, $|S_k^{(l_p+1)}(n)| \le 1$ for any positive integer n. Then we can use (2.14) and put (4.9) and (4.10) together to obtain that

$$\mathcal{G}_{p,k}(n_1) = \sum_{e=1}^{l_p} f_e(n_1) > \sum_{e=1}^{l_p} f_e(n_2) = \mathcal{G}_{p,k}(n_2).$$

Thus p^{l_p-1} is not a period of $\mathcal{G}_{p,k}$. It concludes that $P_{p,k} = p^{l_p}$ if $v_p(k+1) < v_p(R_k) = l_p$.

So Lemma 4.6 is proved. □

5. Proofs of Theorems 1.1 and 1.2 and examples

Using the lemmas presented in previous sections, we are now in a position to prove Theorems 1.1 and 1.2. We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. First of all, by Lemma 2.1, we know that $\mathcal{G}_k$ is periodic.

Consequently, since $R_1 = 109$ is a prime number with $109 \equiv 1 \pmod 6$,

$$2^{\frac{109-1}{3}} \equiv 1 \pmod{109}$$

and

$$0 = v_{109}(2) < v_{109}(109) = 1,$$

one derives from Lemma 4.6 that $P_{109,1} = 109$. It then follows from Lemma 2.2 that $P_1 = 109$.

Now let $k \geq 2$. By Lemmas 2.3 and 2.4, one has

$$P_{2,k} = 2^{\frac{(-1)^k+1}{2}}$$

and

$$P_{3,k} = 3^{\lceil\{\frac{k+1}{3}\}\rceil}.$$

Further, if $p \equiv 1 \pmod 6$ and $2^{\frac{p-1}{3}} \not\equiv 1 \pmod p$, Lemma 2.6 tells us that $P_{p,k} = 1$.

By Lemma 2.8, we know that there is at most one prime $p_0 \geq 5$ such that $v_{p_0}(k+1) \geq v_{p_0}(R_k) \geq 1$. If $p_0 \equiv 5 \pmod 6$, then by Lemma 3.4 one has $P_{p_0,k} = 1$. If $p_0 \equiv 1 \pmod 6$, then we can deduce from Lemmas 2.6 and 4.6 that $P_{p_0,k} = 1$. For all other primes $q \geq 5$ with $q|R_k$, one derives from Lemmas 3.4 and 4.6 that $P_{q,k} = q^{v_q(R_k)}$.

Finally, by Lemma 2.2 one then derives that the smallest period P_k of $\mathcal{G}_k$ is equal to

$$Q_k := 2^{\frac{(-1)^k+1}{2}} 3^{\lceil\{\frac{k+1}{3}\}\rceil} \cdot \frac{R_k}{2^{v_2(R_k)} 3^{v_3(R_k)} \prod\limits_{\substack{p|R_k \\ p\equiv 1 (\mathrm{mod} 6) \\ 2^{\frac{p-1}{3}} \not\equiv 1 (\mathrm{mod} p)}} p^{v_p(R_k)}}$$

except that $v_p(k+1) \geq v_p(Q_k) \geq 1$ for at most one prime $p \geq 5$, in which case its smallest period P_k equals $\frac{Q_k}{p^{v_p(Q_k)}}$.

This finishes the proof of Theorem 1.1. □

Consequently, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. At first, $\mathcal{G}_k$ is periodic by Theorem 1.1. Then for any positive integer n, one has $\mathcal{G}_k(n) \leq M := \max_{1\leq m\leq P_k}\{\mathcal{G}_k(m)\}$. So one deduces that

$$\begin{aligned}\log\Big(\prod_{i=0}^{k}((n+i)^3+2)\Big) - \log M &\leq \log \mathrm{lcm}_{0\leq i\leq k}\{(n+i)^3+2\} \\ &\leq \log\Big(\prod_{i=0}^{k}((n+i)^3+2)\Big).\end{aligned}$$

However, one has

$$\begin{aligned}&\log(\prod_{i=0}^{k}((n+i)^3+2))-\log M\\=&3(k+1)\log n+\sum_{i=0}^{k}\log\Big(1+\frac{3i}{n}+\frac{3i^2}{n^2}+\frac{i^3+2}{n^3}\Big)-\log M.\end{aligned}$$

It implies that

$$\lim_{n\to\infty}\frac{\log(\prod_{i=0}^{k}((n+i)^3+2))-\log M}{3(k+1)\log n}=1.$$

On the other hand, we have

$$\lim_{n\to\infty}\frac{\log(\prod_{i=0}^{k}((n+i)^3+2))}{3(k+1)\log n}=\lim_{n\to\infty}\Big(1+\sum_{i=0}^{k}\frac{\log(1+\frac{3i}{n}+\frac{3i^2}{n^2}+\frac{i^3+2}{n^3})}{3(k+1)\log n}\Big)=1.$$

It then follows that

$$\lim_{n\to\infty}\frac{\log \operatorname{lcm}_{0\le i\le k}\{(n+i)^3+2\}}{3(k+1)\log n}=1$$

as one desires. The proof of Theorem 1.2 is complete. □

By Theorem 1.1, we can easily find infinitely many positive integers k such that $P_k = Q_k$ as the following example shows.

Example 5.1. *If $k+1$ has no prime factors congruent to 5 modulo 6, and $k+1$ has no prime factors p such that $p \equiv 1 \pmod 6$ with $2^{\frac{p-1}{3}} \equiv 1 \pmod p$, then $P_k = Q_k$ by Theorem 1.1. For instance, if $k+1$ equals 6^r with r being a positive integer, then $P_k = Q_k$.*

On the other hand, there are also infinitely many positive integers k such that P_k equals Q_k divided by a power of one prime p. Moreover, by Theorem 1.1 we can present the following proposition that gives us such an example.

Proposition 5.2. *If $k+1$ is equal to tp^e for any positive integer e, where p is a prime number with $p \equiv 5 \pmod 6$ and $t \in \{2, \cdots, p-1\}$, then $P_k = Q_k/p^e$.*

Remark 5.3. Let k be a positive integer and $f(x)$ be a polynomial with integer coefficients. Let the arithmetic function $\mathcal{G}_{k,f}$ be defined as in the Introduction section. If $f(x)$ is linear, then it was proved in 2011 by Hong and Qian [12] that $\mathcal{G}_{k,f}$ is periodic with determination of its smallest period. In 2015, Hong and Qian [13] characterized the quadratic polynomial $f(x)$ such that $\mathcal{G}_{k,f}$ is almost periodic and an explicit and complicated formula for the smallest period of $\mathcal{G}_{k,f}$ is obtained too. Now let $\deg f(x) \ge 3$. If $f(x) = x^3+2$, then by Theorem 1.1 of this paper, we know that $\mathcal{G}_{k,f}$ is a periodic function. Furthermore, Theorem 1.1 gives us an explicit formula for its smallest period. By developing the methods presented in [12], [13] and [24] and in this paper, we can show that the arithmetic function $\mathcal{G}_{k,f}$ is periodic when $f(x)$ is irreducible. One can also find reducible polynomials $f_1(x)$ and $f_2(x)$ such that $\mathcal{G}_{k,f_1}$ is almost periodic and $\mathcal{G}_{k,f_2}$ is not almost periodic. For which reducible polynomials $f(x)$, the arithmetic function $\mathcal{G}_{k,f}$ is almost periodic? What is the smallest period of $\mathcal{G}_{k,f}$ if $\mathcal{G}_{k,f}$ is almost periodic? They were answered in [12], [13] and in this paper when $\deg f(x) \in \{1,2\}$ and $f(x) = x^3+2$. However, these problems are kept widely open when $f(x) \ne x^3+2$ and $\deg f(x) \ge 3$.

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Conflict of interest

We declare that we have no conflict of interest.

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