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Research article

Fekete-Szegö inequality for a subclass of analytic functions defined by Komatu integral operator

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Abstract: In this paper, we introduce and study a new subclass of analytic functions defined by $\mathcal{D}^k\mathcal{L}_a^\delta f(z)$ differential operator in the unit disk. For this subclass, the Fekete–Szegö type coefficient inequalities are derived.

Keywords: analytic functions; starlike and convex functions; Sãlãgean differential operator; Komatu integral operator; Fekete-Szegö; inequaility
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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathcal{U}$. It is well-known that for $f \in \mathcal{S}$, $\left|a_3 - a_2^2\right| \leq 1$. A classical theorem of Fekete-Szegö [8] states that for $f \in \mathcal{S}$ given by (1.1)

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} 3 - 4\eta, & \text{if} \quad \eta \leq 0, \\ 1 + 2\exp(\frac{-2\eta}{1-\eta}), & \text{if} \quad 0 < \eta < 1, \\ 4\eta - 3, & \text{if} \quad \eta \geq 1. \end{cases}$$

The latter inequality is sharp in the sense that for each η there exists a function in $\mathcal{S}$ such that the equality holds. Later, Pfluger [24] has considered the complex values of η and provided the inequality

$$\left|a_3 - \eta a_2^2\right| \leq 1 + 2\left|\exp(\frac{-2\eta}{1-\eta})\right|.$$

Indeed, many authors have considered the Fekete-Szegö problem for various subclasses of $\mathcal{A}$, the upper bound for $\left|a_3 - \eta a_2^2\right|$ is investigated by various authors [1,5,6,13,16,17], see also recent investigations on this subject by [7,11,21–23].

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$ of starlike functions in $\mathcal{U}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \ \ (z \in \mathcal{U}).$$

On the other hand, a function $f \in \mathcal{A}$ is said to be in the class of convex functions in $\mathcal{U}$, denoted by C, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \ \ (z \in \mathcal{U}).$$

A function $f \in \mathcal{A}$ is said to be in the class of starlike functions of complex order $b(b \in \mathbb{C} - \{0\})$, denoted by $\mathcal{S}^*(b)$, provided that

$$\Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \ \ (z \in \mathcal{U}).$$

Furthermore, a function $f \in C(b)$ is convex functions of complex order $b(b \in \mathbb{C} - \{0\})$ if it satisfies the inequality

$$\Re\left\{1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0 \, (z \in \mathcal{U}).$$

Note that $\mathcal{S}^*(1) = \mathcal{S}^*$ and $C(1) = C$.

The class $\mathcal{S}^*(b)$ of starlike functions of complex order $b(b \in \mathbb{C} - \{0\})$ was introduced by Nasr and Aouf [19] while the class $C(b)$ of convex functions of complex order $b(b \in \mathbb{C} - \{0\})$ was presented earlier by Wiatrowski [28].

Sãlãgean [26] introduced the following differential operator for $f(z) \in \mathcal{A}$ which is called the Sãlãgean differential operator:

$$\begin{aligned}
\mathcal{D}^0 f(z) &= f(z) \\
\mathcal{D}^1 f(z) &= \mathcal{D} f(z) = zf'(z) \\
\mathcal{D}^k f(z) &= \mathcal{D}(\mathcal{D}^{k-1} f(z)) \quad (k \in \mathbb{N} = 1, 2, 3, ...).
\end{aligned}$$

We note that,

$$\mathcal{D}^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.2}$$

Recently, Komatu [14] introduced a certain integral operator $\mathcal{L}_a^\delta$ defined by

$$\mathcal{L}_a^\delta f(z) = \frac{a^\delta}{\Gamma(\delta)} \int_0^1 t^{a-2}\left(\log \frac{1}{t}\right)^{\delta-1} f(zt)dt \quad (a > 0, \ \delta \geq 0, \ f(z) \in \mathcal{A}, \ z \in \mathcal{U}). \tag{1.3}$$

Thus, if $f(z) \in \mathcal{A}$ is of the form (1.1), it is easily seen from (1.3) that [14]

$$\mathcal{L}_a^\delta f(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1} \right)^\delta a_n z^n \quad (a > 0,\ \delta \geq 0). \tag{1.4}$$

We note that:

- $\mathcal{L}_a^0 f(z) = f(z)$;
- $\mathcal{L}_1^1 f(z) = A[f](z)$ known as Alexander operator [2];
- $\mathcal{L}_2^1 f(z) = \mathcal{L}[f](z)$ known as Libera operator [15];
- $\mathcal{L}_{c+1}^1 f(z) = \mathcal{L}_c[f](z)$ called generalized Libera operator or Bernardi operator [3];
- For $a = 1$ and $\delta = k$ (k is any integer), the multiplier transformation $\mathcal{L}_1^k f(z) = \mathcal{I}^k f(z)$ was studied by Flett [9] and Sãlãgean [26];
- For $a = 1$ and $\delta = -k$ ($k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$), the differential operator $\mathcal{L}_1^{-k} f(z) = \mathcal{D}^k f(z)$ was studied by Sãlãgean [26];
- For $a = 2$ and $\delta = k$ (k is any integer), the operator $\mathcal{L}_2^k f(z) = \mathcal{L}^k f(z)$ was studied by Uralegaddi and Somanatha [27];
- For $a = 2$, the multiplier transformation $\mathcal{L}_2^\delta f(z) = \mathcal{I}^\delta f(z)$ was studied by Jung et al. [10].

For $\mathcal{D}^k f(z)$ given by (1.2) and $\mathcal{L}_a^\delta f(z)$ is given by (1.4), we define the differential operator $\mathcal{D}^k \mathcal{L}_a^\delta f(z)$ as follows:

$$\mathcal{D}^k \mathcal{L}_a^\delta f(z) = z + \sum_{n=2}^{\infty} n^k \left(\frac{a}{a+n-1} \right)^\delta a_n z^n. \tag{1.5}$$

Note that, by taking $\delta = 0$ and $k = 0$ in (1.5), the differential operator $\mathcal{D}^k \mathcal{L}_a^\delta f(z)$ reduces to Sãlãgean differential operator and Komatu integral operator, respectively.

Using the operator $\mathcal{D}^k \mathcal{L}_a^\delta f$, we now introduce a new subclass of analytic functions as follows:

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}_a^{k,\delta}(\lambda, b)$ if satisfies the inequality*

$$\Re \left(1 + \frac{1}{b} \left(\frac{z(\mathcal{D}^k \mathcal{L}_a^\delta f(z))' + \lambda z^2 (\mathcal{D}^k \mathcal{L}_a^\delta f(z))''}{(1-\lambda)\mathcal{D}^k \mathcal{L}_a^\delta f(z) + \lambda z (\mathcal{D}^k \mathcal{L}_a^\delta f(z))'} - 1 \right) \right) > 0$$

$$(a > 0,\ b \in \mathbb{C} - \{0\},\ \delta \geq 0,\ 0 \leq \lambda \leq 1,\ k \in \mathbb{N} = 1, 2, 3, ...,\ z \in \mathcal{U}).$$

Note that, $\mathcal{N}_a^{0,0}(0, b) = \mathcal{S}^*(b)$ and $\mathcal{N}_a^{0,0}(1, b) = C(b)$.

By giving specific values to the parameters and a, b, k, δ and λ, we obtain the following important subclasses studied by various authors in earlier works, for instance; $\mathcal{N}_a^{0,\delta}(0, b)$ and $\mathcal{N}_a^{0,\delta}(1, b)$ (Bulut [4]), $\mathcal{N}_a^{0,\delta}(\lambda, 1)$ (Mohapatra and Panigrahi [18]), $\mathcal{N}_a^{0,0}(0, b) = \mathcal{S}^*(b)$ (Nars and Aouf [19]), $\mathcal{N}_a^{0,0}(1, b) = C(b)$ (Wiatrowski [28], Nars and Aouf [20]).

In this paper, we find an upper bound for the functional $\left|a_3 - \eta a_2^2\right|$ for the functions f belongs to the class $\mathcal{N}_a^{k,\delta}(\lambda, b)$.

2. Main results

We denote by $\mathcal{P}$ a class of analytic function in $\mathcal{U}$ with $p(0) = 1$ and $\Re p(z) > 0$. In order to derive our main results, we have to recall here the following lemma [25].

Lemma 1. *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + ...$, then*

$$|c_n| \leq 2 \text{ for } n \geq 1.$$

If $|c_1| = 2$ then $p(z) \equiv p_1(z) = (1 + \gamma_1 z)/(1 - \gamma_1 z)$ with $\gamma_1 = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore, we have

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where $p_2(z) = \frac{1+z\frac{\gamma_2 z + \gamma_1}{1+\overline{\gamma}_1 \gamma_2 z}}{1-z\frac{\gamma_2 z+\gamma_1}{1+\overline{\gamma}_1\gamma_2 z}}$, and $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4-|c_1|^2}$. Conversely, if $p(z) \equiv p_2(z)$ for some $|\gamma_1| < 1$ and $|\gamma_2| = 1$ then $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2-c_1^2}{4-|c_1|^2}$ and $\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}$.

Now, consider the functional $\left| a_3 - \eta a_2^2 \right|$ for $b \in \mathbb{C} - \{0\}$ and $\eta \in \mathbb{C}$.

Theorem 1. *Let $b \in \mathbb{C} - \{0\}$ and $0 \leq \lambda \leq 1$, $\eta \in \mathbb{C}$, $a > 0$, $\delta \geq 0$. If f of the form (1.1) is in $\mathcal{N}_a^{k,\delta}(\lambda, b)$, then*

$$|a_2| \leq \frac{2\,|b|}{(\lambda + 1) A_1^\delta 2^k}, \tag{2.1}$$

$$|a_3| \leq \frac{|b|}{(2\lambda + 1) A_2^\delta 3^k} \max\{1, |1 + 2b|\} \tag{2.2}$$

and

$$\left| a_3 - \eta a_2^2 \right| \leq \frac{|b|}{(2\lambda + 1) A_2^\delta 3^k} \max\left\{1, \left| 1 + 2b - 4\eta b \frac{(2\lambda + 1) A_2^\delta 3^k}{(\lambda + 1)^2 A_1^{2\delta} 2^{2k}} \right| \right\} \tag{2.3}$$

where $A_1 = \left(\frac{a}{a+1}\right)$ and $A_2 = \left(\frac{a}{a+2}\right)$. Consider the functions

$$\frac{z\left(\mathcal{F}_{\lambda,a}^{k,\delta}(z)\right)'}{\mathcal{F}_{\lambda,a}^{k,\delta}(z)} = 1 + b\,[p_1(z) - 1] \tag{2.4}$$

$$\frac{z\left(\mathcal{F}_{\lambda,a}^{k,\delta}(z)\right)'}{\mathcal{F}_{\lambda,a}^{k,\delta}(z)} = 1 + b\,[p_2(z) - 1] \tag{2.5}$$

where p_1, p_2 are given in Lemma 1. Equality in (2.1) holds if (2.4); in (2.2) if (2.4) and (2.5); for each η in (2.3) if (2.4) and (2.5).

Proof. Denote $\mathcal{F}_{\lambda,a}^{k,\delta}(z) = (1 - \lambda)\mathcal{D}^k \mathcal{L}_a^\delta f(z) + \lambda z(\mathcal{D}^k \mathcal{L}_a^\delta f(z))' = z + \beta_2 z^2 + \beta_3 z^3 + ...$, then

$$\beta_2 = (\lambda + 1) A_1^\delta 2^k a_2, \quad \beta_3 = (2\lambda + 1) A_2^\delta 3^k a_3. \tag{2.6}$$

By the definition of the class $\mathcal{N}_a^{k,\delta}(\lambda, b)$, there exists $p \in \mathcal{P}$ such that $\frac{z\left(\mathcal{F}_{\lambda,a}^{k,\delta}(z)\right)'}{\mathcal{F}_{\lambda,a}^{k,\delta}(z)} = 1 + b(p(z) - 1)$, so that

$$\left(\frac{z(1 + 2\beta_2 z + 3\beta_3 z^2 + ...)}{z + \beta_2 z^2 + \beta_3 z^3 + ...} \right) = 1 - b + b(1 + c_1 z + c_2 z^2 + ...),$$

which implies the equality

$$z + 2\beta_2 z^2 + 3\beta_3 z^3 + ... = z + (bc_1 + \beta_2)z^2 + (bc_2 + \beta_2 bc_1 + \beta_3)z^3 + ...$$

Equating the coefficients of both sides of the latter, we have

$$\beta_2 = bc_1,\ \beta_3 = \frac{b^2c_1^2}{2} + \frac{bc_2}{2}, \tag{2.7}$$

so that, on account of (2.6) and (2.7)

$$a_2 = \frac{bc_1}{(\lambda+1)A_1^\delta 2^k}, \quad a_3 = \frac{b}{2(2\lambda+1)A_2^\delta 3^k}(bc_1^2 + c_2). \tag{2.8}$$

Taking into account (2.8) and Lemma 1, we obtain

$$|a_2| = \left|\frac{b}{(\lambda+1)A_1^\delta 2^k}c_1\right| \leq \frac{2\,|b|}{(\lambda+1)A_1^\delta 2^k} \tag{2.9}$$

and

$$\begin{aligned}
|a_3| &= \left|\frac{b}{2(2\lambda+1)A_2^\delta 3^k}\left[c_2 - \frac{c_1^2}{2} + \frac{1+2b}{2}c_1^2\right]\right| \\
&\leq \frac{|b|}{2(2\lambda+1)A_2^\delta 3^k}\left[2 - \frac{\left|c_1^2\right|}{2} + |1+2b|\frac{\left|c_1^2\right|}{2}\right] \\
&= \frac{|b|}{(2\lambda+1)A_2^\delta 3^k}\left[1 + |c_1|^2 + \frac{|1+2b|-1}{2}\right] \\
&\leq \frac{|b|}{(2\lambda+1)A_2^\delta 3^k}\max\{1, [1 + |1+2b| - 1]\}.
\end{aligned}$$

Thus, we have

$$|a_3| \leq \frac{|b|}{(2\lambda+1)A_2^\delta 3^k}\max\{1, |1+2b|\}.$$

Then, with the aid of Lemma 1, we obtain

$$\begin{aligned}
\left|a_3 - \eta a_2^2\right| &= \left|\frac{b}{2(2\lambda+1)A_2^\delta 3^k}(bc_1^2 + c_2) - \eta\frac{b^2c_1^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\right| \\
&\leq \frac{|b|}{2(2\lambda+1)A_2^\delta 3^k}\left[\left|c_2 - \frac{c_1^2}{2}\right| + \frac{\left|c_1^2\right|}{2}\left|1 + 2b - \frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\right|\right] \\
&\leq \frac{|b|}{2(2\lambda+1)A_2^\delta 3^k}\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{\left|c_1^2\right|}{2}\left|1 + 2b - \frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\right|\right] \\
&= \frac{|b|}{(2\lambda+1)A_2^\delta 3^k}\left[1 + \frac{\left|c_1^2\right|}{4}\left(\left|1 + 2b - \frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\right| - 1\right)\right] \\
&\leq \frac{|b|}{(2\lambda+1)A_2^\delta 3^k}\max\left\{1, \left|1 + 2b - \frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\right|\right\}.
\end{aligned} \tag{2.10}$$

We now obtain sharpness of the estimates in (2.1), (2.2) and (2.3).

Firstly, in (2.1) the equality holds if $c_1 = 2$. Equivalently, we have $p(z) \equiv p_1(z) = (1+z)/(1-z)$. Therefore, the extremal functions in $\mathcal{N}_a^{k,\delta}(\lambda, b)$ is given by

$$\frac{z\left(\mathcal{F}_{\lambda,a}^{k,\delta}(z)\right)'}{\mathcal{F}_{\lambda,a}^{k,\delta}(z)} = \frac{1+(2b-1)z}{1-z}. \tag{2.11}$$

Next, in (2.2), for first case, the equality holds if $c_1 = c_2 = 2$. Therefore, the extremal functions in $\mathcal{N}_a^{k,\delta}(\lambda, b)$ is given by (2.11) and for the second case, the equality holds if $c_1 = 0$, $c_2 = 2$. Equivalently, we have $p(z) \equiv p_2(z) = (1+z^2)/(1-z^2)$. Therefore, the extremal functions in $\mathcal{N}_a^{k,\delta}(\lambda, b)$ is given by

$$\frac{z\left(\mathcal{F}_{\lambda,a}^{k,\delta}(z)\right)'}{\mathcal{F}_{\lambda,a}^{k,\delta}(z)} = \frac{1+(2b-1)z^2}{1-z^2}. \tag{2.12}$$

Finally, in (2.3), the equality holds. Obtained extremal functions for (2.2) is also valid for (2.3).

Thus, the proof of Theorem 1 is completed. □

Taking $k = 0$ and $\lambda = 0$ in Theorem 1, we have

Corollary 1. *[4] Let $b \in \mathbb{C} - \{0\}$, $\eta \in \mathbb{C}$, $a > 0$ and $\delta \geq 0$. If f of the form (1.1), is in $\mathcal{N}_a^{0,\delta}(0, b)$, then*

$$|a_2| \leq \frac{2|b|}{A_1^\delta},$$

$$|a_3| \leq \frac{|b|}{A_2^\delta} \max\{1, |1+2b|\}$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|b|}{A_2^\delta} \max\left\{1, \left|1+2b-4\eta b\frac{A_2^\delta}{A_1^{2\delta}}\right|\right\}$$

where $A_1 = \left(\frac{a}{a+1}\right)$ and $A_2 = \left(\frac{a}{a+2}\right)$.

If we choose $k = 0$ and $\lambda = 1$ in Theorem 1, we get

Corollary 2. *[4] Let $b \in \mathbb{C} - \{0\}$, $\eta \in \mathbb{C}$, $a > 0$ and $\delta \geq 0$. If f of the form (1.1), is in $\mathcal{N}_a^{0,\delta}(1, b)$, then*

$$|a_2| \leq \frac{|b|}{A_1^\delta},$$

$$|a_3| \leq \frac{|b|}{3A_2^\delta} \max\{1, |1+2b|\}$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|b|}{3A_2^\delta} \max\left\{1, \left|1+2b-3\eta b\frac{A_2^\delta}{A_1^{2\delta}}\right|\right\}$$

where $A_1 = \left(\frac{a}{a+1}\right)$ and $A_2 = \left(\frac{a}{a+2}\right)$.

For $k = 0$, $\delta = 0$, $\lambda = 0$ and $b = 1$ in (2.3), we obtain

Corollary 3. *[12] Let $\eta \in \mathbb{C}$. If f of the form (1.1), is in $\mathcal{S}^*(1)$, then*

$$\left|a_3 - \eta a_2^2\right| \leq \max\{1, |4\eta - 3|\}.$$

Taking $k = 0$, $\delta = 0$, $\lambda = 1$ and $b = 1$ in (2.3), we have

Corollary 4. *[12] Let $\eta \in \mathbb{C}$. If f of the form (1.1), is in $C(1)$, then*

$$\left|a_3 - \eta a_2^2\right| \leq \max\left\{\frac{1}{3}, |\eta - 1|\right\}$$

We next consider the case, when η and b are real. In this case, the following theorem holds.

Theorem 2. *Let $b > 0$ and let $\mathcal{N}_a^{k,\delta}(\lambda, b)$. Then for $\eta \in \mathbb{R}$, we have*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{b}{(2\lambda+1)A_2^\delta 3^k}\left\{1 + 2b\left[1 - \frac{2\eta(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\right]\right\}, & \eta \leq M_1, \\ \frac{b}{(2\lambda+1)A_2^\delta 3^k}, & M_1 \leq \eta \leq M_2, \\ \frac{b}{(2\lambda+1)A_2^\delta 3^k}\left[\frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}} - 2b - 1\right], & \eta \geq M_2, \end{cases}$$

where $A_1 = \left(\frac{a}{a+1}\right)$, $A_2 = \left(\frac{a}{a+2}\right)$, $M_1 = \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{2(2\lambda+1)A_2^\delta 3^k}$ and $M_2 = \frac{(1+2b)(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{4b(2\lambda+1)A_2^\delta 3^k}$. For each η, the equality holds for the functions given in equations (2.4) and (2.5).

Proof. First, let $\eta \leq \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{2(2\lambda+1)A_2^\delta 3^k} \leq \frac{(1+2b)(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{4b(2\lambda+1)A_2^\delta 3^k}$. In this case it follows from (2.8) and Lemma 1 that

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leq \frac{b}{2(2\lambda+1)A_2^\delta 3^k}\left[2 - \frac{|c_1^2|}{2} + \frac{|c_1^2|}{2}\left(1 + 2b - \frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\right)\right] \\ &\leq \frac{b}{(2\lambda+1)A_2^\delta 3^k}\left[1 + 2b\left(1 - \frac{2\eta(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\right)\right]. \end{aligned}$$

Let, now $\frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{2(2\lambda+1)A_2^\delta 3^k} \leq \eta \leq \frac{(1+2b)(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{4b(2\lambda+1)A_2^\delta 3^k}$. Then, using the estimations obtained above we arrived

$$\left|a_3 - \eta a_2^2\right| \leq \frac{b}{(2\lambda+1)A_2^\delta 3^k}.$$

Finally, if $\eta \geq \frac{(1+2b)(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{4b(2\lambda+1)A_2^\delta 3^k}$, then

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leq \frac{b}{2(2\lambda+1)A_2^\delta 3^k}\left[2 - \frac{|c_1^2|}{2} + \frac{|c_1^2|}{2}\left(\frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}} - 1 - 2b\right)\right] \\ &= \frac{b}{2(2\lambda+1)A_2^\delta 3^k}\left[2 + \frac{|c_1^2|}{2}\left(\frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}} - 2 - 2b\right)\right] \\ &\leq \frac{b}{(2\lambda+1)A_2^\delta 3^k}\left[\frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}} - 2b - 1\right]. \end{aligned}$$

Thus, the proof of Theorem 2 is completed. □

Taking $k = 0$ and $\lambda = 0$ in Theorem 2, we have

Corollary 5. *[4] Let $b > 0$ and let $\mathcal{N}_a^{0,\delta}(0, b)$. Then for $\eta \in \mathbb{R}$, we have*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{b}{A_2^\delta}\left\{1 + 2b\left(1 - \frac{2\eta A_2^\delta}{A_1^{2\delta}}\right)\right\}, & \eta \leq \frac{A_1^{2\delta}}{2A_2^\delta}, \\ \frac{b}{A_2^\delta}, & \frac{A_1^{2\delta}}{2A_2^\delta} \leq \eta \leq \frac{(1+2b)A_1^{2\delta}}{4bA_2^\delta}, \\ \frac{b}{A_2^\delta}\left[\frac{4\eta b A_2^\delta}{A_1^{2\delta}} - 2b - 1\right], & \eta \geq \frac{(1+2b)A_1^{2\delta}}{4bA_2^\delta}, \end{cases}$$

where $A_1 = \left(\frac{a}{a+1}\right)$ and $A_2 = \left(\frac{a}{a+2}\right)$.

Finally, considering the case of $b \in \mathbb{C} - \{0\}$ and $\eta \in \mathbb{R}$, we obtain

Theorem 3. *Let $b \in \mathbb{C} - \{0\}$ and let $f \in \mathcal{N}_a^{k,\delta}(\lambda, b)$. For $\eta \in \mathbb{R}$, we have*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{4|b|^2}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\left[\Re(k_1) - \eta\right] + \frac{|b||\sin\theta|}{(2\lambda+1)A_2^\delta 3^k}, & \text{if} \quad \eta \leq T_1, \\ \frac{|b|}{(2\lambda+1)A_2^\delta 3^k}, & \text{if} \quad T_1 \leq \eta \leq T_2, \\ \frac{4|b|^2}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\left[\eta - \Re(k_1)\right] + \frac{|b||\sin\theta|}{(2\lambda+1)A_2^\delta 3^k}, & \text{if} \quad \eta \geq T_2, \end{cases}$$

where $A_1 = \left(\frac{a}{a+1}\right)$ and $A_2 = \left(\frac{a}{a+2}\right)$, $|b| = be^{i\theta}$, $k_1 = \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{2(2\lambda+1)A_2^\delta 3^k} + \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k} e^{i\theta}}{4|b|(2\lambda+1)A_2^\delta 3^k}$, $l_1 = \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{4|b|(2\lambda+1)A_2^\delta 3^k}$, $T_1 = \Re(k_1) - l_1(1 - |\sin\theta|)$ and $T_2 = \Re(k_1) + l_1(1 - |\sin\theta|)$. For each η there is a function in $\mathcal{N}_a^{k,\delta}(\lambda, b)$ such that the equality holds.

Proof. From inequality (2.10), we may write

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &= \frac{|b|}{2(2\lambda+1)A_2^\delta 3^k}\left[\left|c_2 - \frac{c_1^2}{2}\right| + \frac{\left|c_1^2\right|}{2}\left|1 + 2b - \frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\right|\right] \\ &\leq \frac{|b|}{2(2\lambda+1)A_2^\delta 3^k}\left[2 - \frac{\left|c_1^2\right|}{2} + \frac{\left|c_1^2\right|}{2}\left|1 + 2b - \frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\right|\right] \\ &= \frac{|b|}{2(2\lambda+1)A_2^\delta 3^k}\left[\frac{\left|c_1^2\right|}{2}\left(\left|1 + 2b - \frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\right| - 1\right) + 2\right] \\ &= \frac{|b|}{(2\lambda+1)A_2^\delta 3^k} + \frac{|b|}{4(2\lambda+1)A_2^\delta 3^k}\left[\left|\frac{4\eta b(2\lambda+1)A_2^\delta 3^k}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}} - 2b - 1\right| - 1\right]\left|c_1^2\right| \\ &= \frac{|b|}{(2\lambda+1)A_2^\delta 3^k} + \frac{|b|^2}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}} \\ &\quad \times\left[\left|\eta - \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{2(2\lambda+1)A_2^\delta 3^k} - \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{4b(2\lambda+1)A_2^\delta 3^k}\right| - \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{4\,|b|\,(2\lambda+1)A_2^\delta 3^k}\right]\left|c_1^2\right|. \end{aligned}$$

If we write $|b| = be^{i\theta}$ (or $b = |b|\, e^{-i\theta}$), $k_1 = \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{2(2\lambda+1)A_2^\delta 3^k} + \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k} e^{i\theta}}{4|b|(2\lambda+1)A_2^\delta 3^k}$ and $l_1 = \frac{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}{4|b|(2\lambda+1)A_2^\delta 3^k}$ in the last inequality, we get

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|b|}{(2\lambda+1)A_2^\delta 3^k} + \frac{|b|^2}{(\lambda+1)^2 A_1^{2\delta} 2^{2k}}\left[|\eta - k_1| - l_1\right]\left|c_1^2\right|$$

$$\begin{aligned} &\leq \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k} + \frac{|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\left[\left|\eta - \Re(k_1)\right| + l_1\left|\sin\theta\right| - l_1\right]\left|c_1^2\right| \\ &= \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k} + \frac{|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\left[\left|\eta - \Re(k_1)\right| - l_1(1-\left|\sin\theta\right|)\right]\left|c_1^2\right|. \end{aligned} \tag{2.13}$$

We consider the following cases for (2.13). Suppose $\eta \leq \Re(k_1)$. Then

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leq \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k} + \frac{|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\left[\Re(k_1) - l_1(1-|\sin\theta|) - \eta\right]\left|c_1^2\right| \\ &= \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k} + \frac{|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\left[T_1 - \eta\right]\left|c_1^2\right|. \end{aligned} \tag{2.14}$$

Let $\eta \leq T_1 = \Re(k_1) - l_1(1-|\sin\theta|)$. On using Lemma 1 and $l_1 = \frac{(\lambda+1)^2A_1^{2\delta}2^{2k}}{4|b|(2\lambda+1)A_2^{\delta}3^k}$ in inequality (2.14), we get

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leq \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k} + \frac{4\,|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\left(\Re(k_1) - \eta\right) - \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k}(1-|\sin\theta|) \\ &= \frac{4\,|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\left(\Re(k_1) - \eta\right) + \frac{|b|\,|\sin\theta|}{(2\lambda+1)A_2^{\delta}3^k}. \end{aligned}$$

If we take $T_1 = \Re(k_1) - l_1(1-|\sin\theta|) \leq \eta \leq \Re(k_1)$, then (2.14) gives

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k}.$$

Let $\eta \geq \Re(k_1)$. It then follows, from (2.13), that

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leq \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k} + \frac{|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\left[\eta - \Re(k_1) + l_1(1-|\sin\theta|)\right]\left|c_1^2\right| \\ &= \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k} + \frac{|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}\left[\eta - T_1\right]\left|c_1^2\right|. \end{aligned} \tag{2.15}$$

Let $\eta \leq T_2 = \Re(k_1) + l_1(1-|\sin\theta|)$. On using (2.15) we obtain

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k}.$$

Let $\eta \geq T_2 = \Re(k_1) + l_1(1-|\sin\theta|)$. Employing Lemma 1 together with $l_1 = \frac{(\lambda+1)^2A_1^{2\delta}2^{2k}}{4|b|(2\lambda+1)A_2^{\delta}3^k}$ in equality (2.15), we obtain

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &\leq \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k} + \frac{4\,|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}(\eta - \Re(k_1)) - \frac{|b|}{(2\lambda+1)A_2^{\delta}3^k}(1-|\sin\theta|) \\ &\leq \frac{4\,|b|^2}{(\lambda+1)^2A_1^{2\delta}2^{2k}}(\eta - \Re(k_1)) + \frac{|b|\,|\sin\theta|}{(2\lambda+1)A_2^{\delta}3^k}. \end{aligned}$$

Therefore, the proof is completed. □

Corollary 6. *If we take $a = 1$ in Theorems 1-3, we have the following results, respectively:*

1. Let $b \in \mathbb{C} - \{0\}$ and $f \in \mathcal{N}_1^{k,\delta}(\lambda, b)$. Then, for $\eta \in \mathbb{C}$, we have

$$|a_2| \leq \frac{|b|}{(\lambda+1)2^{k-\delta-1}},$$

$$|a_3| \leq \frac{|b|}{(2\lambda+1)3^{k-\delta}} \max\{1, |1+2b|\}$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|b|}{(2\lambda+1)3^{k-\delta}} \max\left\{1, \left|1 + 2b - 4\eta b \frac{(2\lambda+1)}{(\lambda+1)^2}\left(\frac{3}{4}\right)^{k-\delta}\right|\right\}.$$

Equality holds for the cases $a = 1$ of (2.4) and (2.5) in Theorem 1.

2. Let $b > 0$ and $f \in \mathcal{N}_1^{k,\delta}(\lambda, b)$.Then, for $\eta \in \mathbb{R}$, we have

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{b}{(2\lambda+1)3^{k-\delta}}\left\{1 + 2b\left[1 - \frac{2\eta(2\lambda+1)}{(\lambda+1)^2}\left(\frac{3}{4}\right)^{k-\delta}\right]\right\}, & if \quad \eta \leq Y_1, \\ \frac{b}{(2\lambda+1)3^{k-\delta}}, & if \quad Y_1 \leq \eta \leq Y_2, \\ \frac{b}{(2\lambda+1)3^{k-\delta}}\left[\frac{4\eta b(2\lambda+1)}{(\lambda+1)^2}\left(\frac{3}{4}\right)^{k-\delta} - 2b - 1\right], & if \quad \eta \geq Y_2, \end{cases}$$

where $Y_1 = \frac{(\lambda+1)^2}{2(2\lambda+1)}\left(\frac{4}{3}\right)^{k-\delta}$ and $Y_2 = \frac{(1+2b)(\lambda+1)^2}{4b(2\lambda+1)}\left(\frac{4}{3}\right)^{k-\delta}$. For each η, the equality holds for the cases $a = 1$ of (2.4) and (2.5).

3. Let $b \in \mathbb{C} - \{0\}$ and $f \in \mathcal{N}_1^{k,\delta}(\lambda, b)$. Then, for $\eta \in \mathbb{R}$, we have

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{|b|^2}{(\lambda+1)^2 4^{k-\delta-1}}\left[\Re(k_1) - \eta\right] + \frac{|b||\sin\theta|}{(2\lambda+1)3^{k-\delta}}, & if \quad \eta \leq T_1, \\ \frac{b}{(2\lambda+1)3^{k-\delta}}, & if \quad T_1 \leq \eta \leq T_2, \\ \frac{|b|^2}{(\lambda+1)^2 4^{k-\delta-1}}\left[\eta - \Re(k_1)\right] + \frac{|b||\sin\theta|}{(2\lambda+1)3^{k-\delta}}, & if \quad \eta \geq T_2, \end{cases}$$

where $|b| = be^{i\theta}$, $k_1 = \frac{(\lambda+1)^2}{2(2\lambda+1)}\left(\frac{4}{3}\right)^{k-\delta} - \left(\frac{4}{3}\right)^{k-\delta}\frac{(\lambda+1)^2 e^{i\theta}}{4|b|(2\lambda+1)}$, $l_1 = \left(\frac{4}{3}\right)^{k-\delta}\frac{(\lambda+1)^2}{4|b|(2\lambda+1)}$, $T_1 = \Re(k_1) - l_1(1 - |\sin\theta|)$ and $T_2 = \Re(k_1) + l_1(1 - |\sin\theta|)$. For each η there is a function in $\mathcal{N}_1^{k,\delta}(\lambda, b)$ such that the equality holds.

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Conflict of interest

All authors declare no conflicts of interest.

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