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Research article

A new improvement of Hölder inequality via isotonic linear functionals

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Abstract: In this paper, a new improvement of celebrated Hölder inequality using isotonic linear functionals is established. An important feature of the new inequality obtained here is that many existing inequalities related to the Hölder inequality can be improved which we also illustrate with an application.

Keywords: Hölder inequality; Young inequality; integral inequalities; Hermite-Hadamard type inequality
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1. Introduction

The famous Young's inequality, as a classical result, state that: if $a, b > 0$ and $t \in [0,1]$, then

$$a^t b^{1-t} \leq ta + (1-t)b \tag{1.1}$$

with equality if and only if $a = b$. Let $p, q > 1$ such that $1/p + 1/q = 1$. The inequality (1.1) can be written as

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{1.2}$$

for any $a, b \geq 0$. In this form, the inequality (1.2) was used to prove the celebrated Hölder inequality. One of the most important inequalities of analysis is Hölder's inequality. It contributes wide area of pure and applied mathematics and plays a key role in resolving many problems in social science and cultural science as well as in natural science.

Theorem 1 (Hölder inequality for integrals [11])**.** *Let $p > 1$ and $1/p + 1/q = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$ then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} \left(\int_a^b |g(x)|^q\,dx\right)^{1/q}, \tag{1.3}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

Theorem 2 (Hölder inequality for sums [11])**.** *Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ be two positive n-tuples and $p, q > 1$ such that $1/p + 1/q = 1$. Then we have*

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}. \tag{1.4}$$

Equality hold in (1.4) if and only if a^p and b^q are proportional.

In [10], İşcan gave new improvements for integral ans sum forms of the Hölder inequality as follow:

Theorem 3. *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \;\leq\; & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \end{aligned} \tag{1.5}$$

Theorem 4. *Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ be two positive n-tuples and $p, q > 1$ such that $1/p + 1/q = 1$. Then*

$$\sum_{k=1}^{n} a_k b_k \leq \frac{1}{n}\left\{\left(\sum_{k=1}^{n} k a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} k b_k^q\right)^{1/q} + \left(\sum_{k=1}^{n} (n-k)\, a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} (n-k)\, b_k^q\right)^{1/q}\right\}. \tag{1.6}$$

2. Hölder's inequality for positive functionals

Let E be a nonempty set and L be a linear class of real valued functions on E having the following properties

$L1$: If $f, g \in L$ then $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

$L2$: $1 \in L$, that is if $f(t) = 1, t \in E$, then $f \in L$;

We also consider positive isotonic linear functionals $A : L \to \mathbb{R}$ is a functional satisfying the following properties:

$A1$: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;

$A2$: If $f \in L$, $f(t) \geq 0$ on E then $A(f) \geq 0$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Functional versions of well-known inequalities and related results could be found in [1–9, 11, 12].

Example 1. ***i.)*** *If $E = [a, b] \subseteq \mathbb{R}$ and $L = L[a, b]$, then*

$$A(f) = \int_a^b f(t)dt$$

is an isotonic linear functional.

ii.) *If* $E = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ *and* $L = L([a,b] \times [c,d])$, *then*

$$A(f) = \int_a^b \int_c^d f(x,y)dxdy$$

is an isotonic linear functional.

iii.) *If* (E, Σ, μ) *is a measure space with* μ *positive measure on* E *and* $L = L(\mu)$ *then*

$$A(f) = \int_E f d\mu$$

is an isotonic linear functional.

iv.) *If* E *is a subset of the natural numbers* $\mathbb{N}$ *with all* $p_k \geq 0$, *then* $A(f) = \sum_{k\in E} p_k f_k$ *is an isotonic linear functional. For example; If* $E = \{1, 2, ..., n\}$ *and* $f : E \to \mathbb{R}, f(k) = a_k$, *then* $A(f) = \sum_{k=1}^n a_k$ *is an isotonic linear functional. If* $E = \{1, 2, ..., n\} \times \{1, 2, ..., m\}$ *and* $f : E \to \mathbb{R}, f(k,l) = a_{k,l}$, *then* $A(f) = \sum_{k=1}^n \sum_{l=1}^m a_{k,l}$ *is an isotonic linear functional.*

Theorem 5 (Hölder's inequality for isotonic functionals [13])**.** *Let* L *satisfy conditions* $L1$, $L2$, *and* A *satisfy conditions* $A1$, $A2$ *on a base set* E. *Let* $p > 1$ *and* $p^{-1} + q^{-1} = 1$. *If* $w, f, g \geq 0$ *on* E *and* $wf^p, wg^q, wfg \in L$ *then we have*

$$A(wfg) \leq A^{1/p}(wf^p) A^{1/q}(wg^q). \tag{2.1}$$

In the case $0 < p < 1$ *and* $A(wg^q) > 0$ *(or* $p < 0$ *and* $A(wf^p) > 0$*), the inequality in (2.1) is reversed.*

Remark 1. *i.) If we choose* $E = [a,b] \subseteq \mathbb{R}$, $L = L[a,b]$, $w = 1$ *on* E *and* $A(f) = \int_a^b |f(t)| dt$ *in the Theorem 5, then the inequality (2.1) reduce the inequality (1.3).*

ii.) If we choose $E = \{1, 2, ..., n\}$, $w = 1$ *on* E, $f : E \to [0, \infty)$, $f(k) = a_k$, *and* $A(f) = \sum_{k=1}^n a_k$ *in the Theorem 5, then the inequality (2.1) reduce the inequality (1.4).*

iii.) If we choose $E = [a,b] \times [c,d]$, $L = L(E)$, $w = 1$ *on* E *and* $A(f) = \int_a^b \int_c^d |f(x,y)| dxdy$ *in the Theorem 5, then the inequality (2.1) reduce the following inequality for double integrals:*

$$\int_a^b \int_c^d |f(x,y)| |g(x,y)| dxdy \leq \left(\int_a^b \int_c^d |f(x,y)|^p dx \right)^{1/p} \left(\int_a^b \int_c^d |g(x,y)|^q dx \right)^{1/q}.$$

The aim of this paper is to give a new general improvement of Hölder inequality for isotonic linear functional. As applications, this new inequality will be rewritten for several important particular cases of isotonic linear functionals. Also, we give an application to show that improvement is hold for double integrals.

3. Main results

Theorem 6. *Let* L *satisfy conditions* $L1$, $L2$, *and* A *satisfy conditions* $A1$, $A2$ *on a base set* E. *Let* $p > 1$ *and* $p^{-1} + q^{-1} = 1$. *If* $\alpha, \beta, w, f, g \geq 0$ *on* E, $\alpha wfg, \beta wfg, \alpha wf^p, \alpha wg^q, \beta wf^p, \beta wg^q, wfg \in L$ *and* $\alpha + \beta = 1$ *on* E, *then we have* *i.)*

$$A\left(wfg\right)\leq A^{1/p}\left(\alpha wf^{p}\right)A^{1/q}\left(\alpha wg^{q}\right)+A^{1/p}\left(\beta wf^{q}\right)A^{1/q}\left(\beta wg^{q}\right) \tag{3.1}$$

ii.)

$$A^{1/p}\left(\alpha wf^{p}\right)A^{1/q}\left(\alpha wg^{q}\right)+A^{1/p}\left(\beta wf^{p}\right)A^{1/q}\left(\beta wg^{q}\right)\leq A^{1/p}\left(wf^{p}\right)A^{1/q}\left(wg^{q}\right). \tag{3.2}$$

Proof. i.) By using of Hölder inequality for isotonic functionals in (2.1) and linearity of A, it is easily seen that

$$\begin{aligned} A\left(wfg\right) &= A\left(\alpha wfg+\beta wfg\right)=A\left(\alpha wfg\right)+A\left(\beta wfg\right)\\ &\leq A^{1/p}\left(\alpha wf^{p}\right)A^{1/q}\left(\alpha wg^{q}\right)+A^{1/p}\left(\beta wf^{p}\right)A^{1/q}\left(\beta wg^{q}\right). \end{aligned}$$

ii.) Firstly, we assume that $A^{1/p}\left(wf^{p}\right)A^{1/q}\left(wg^{q}\right)\neq 0$. then

$$\begin{aligned} &\frac{A^{1/p}\left(\alpha wf^{p}\right)A^{1/q}\left(\alpha wg^{q}\right)+A^{1/p}\left(\beta wf^{p}\right)A^{1/q}\left(\beta wg^{q}\right)}{A^{1/p}\left(wf^{p}\right)A^{1/q}\left(wg^{q}\right)}\\ &= \left(\frac{A\left(\alpha wf^{p}\right)}{A\left(wf^{p}\right)}\right)^{1/p}\left(\frac{A\left(\alpha wg^{q}\right)}{A\left(wg^{q}\right)}\right)^{1/q}+\left(\frac{A\left(\beta wf^{p}\right)}{A\left(wf^{p}\right)}\right)^{1/p}\left(\frac{A\left(\beta wg^{q}\right)}{A\left(wg^{q}\right)}\right)^{1/q}, \end{aligned}$$

By the inequality (1.1) and linearity of A, we have

$$\begin{aligned} &\frac{A^{1/p}\left(\alpha wf^{p}\right)A^{1/q}\left(\alpha wg^{q}\right)+A^{1/p}\left(\beta wf^{p}\right)A^{1/q}\left(\beta wg^{q}\right)}{A^{1/p}\left(wf^{p}\right)A^{1/q}\left(wg^{q}\right)}\\ &\leq \frac{1}{p}\left[\frac{A\left(\alpha wf^{p}\right)}{A\left(wf^{p}\right)}+\frac{A\left(\beta wf^{p}\right)}{A\left(wf^{p}\right)}\right]+\frac{1}{q}\left[\frac{A\left(\alpha wg^{q}\right)}{A\left(wg^{q}\right)}+\frac{A\left(\beta wg^{q}\right)}{A\left(wg^{q}\right)}\right]\\ &= 1. \end{aligned}$$

Finally, suppose that $A^{1/p}\left(wf^{p}\right)A^{1/q}\left(wg^{q}\right)=0$. Then $A^{1/p}\left(wf^{p}\right)=0$ or $A^{1/q}\left(wg^{q}\right)=0$, i.e. $A\left(wf^{p}\right)=0$ or $A\left(wg^{q}\right)=0$. We assume that $A\left(wf^{p}\right)=0$. Then by using linearity of A we have,

$$0=A\left(wf^{p}\right)=A\left(\alpha wf^{p}+\beta wf^{p}\right)=A\left(\alpha wf^{p}\right)+A\left(\beta wf^{p}\right).$$

Since $A\left(\alpha wf\right),A\left(\beta wf\right)\geq 0$, we get $A\left(\alpha wf^{p}\right)=0$ and $A\left(\beta wf^{p}\right)=0$. From here, it follows that

$$A^{1/p}\left(\alpha wf^{p}\right)A^{1/q}\left(\alpha wg^{q}\right)+A^{1/p}\left(\beta wf^{p}\right)A^{1/q}\left(\beta wg^{q}\right)=0\leq 0=A^{1/p}\left(wf^{p}\right)A^{1/q}\left(wg^{q}\right).$$

In case of $A\left(wg^{q}\right)=0$, the proof is done similarly. This completes the proof. □

Remark 2. *The inequality (3.2) shows that the inequality (3.1) is better than the inequality (2.1).*

If we take $w=1$ on E in the Theorem 6, then we can give the following corollary:

Corollary 1. *Let L satisfy conditions $L1$, $L2$, and A satisfy conditions $A1$, $A2$ on a base set E. Let $p>1$ and $p^{-1}+q^{-1}=1$. If $\alpha,\beta,f,g\geq 0$ on E, $\alpha fg,\beta fg,\alpha f^{p},\alpha g^{q},\beta f^{p},\beta g^{q},fg\in L$ and $\alpha+\beta=1$ on E, then we have*

i.)

$$A\left(fg\right)\leq A^{1/p}\left(\alpha f^{p}\right)A^{1/q}\left(\alpha g^{q}\right)+A^{1/p}\left(\beta f^{q}\right)A^{1/q}\left(\beta g^{q}\right) \tag{3.3}$$

ii.)

$$A^{1/p}\left(\alpha f^{p}\right)A^{1/q}\left(\alpha g^{q}\right)+A^{1/p}\left(\beta f^{p}\right)A^{1/q}\left(\beta g^{q}\right)\leq A^{1/p}\left(f^{p}\right)A^{1/q}\left(g^{q}\right).$$

Remark 3. *i.) If we choose $E = [a,b] \subseteq \mathbb{R}$, $L = L[a,b]$, $\alpha(t) = \frac{b-t}{b-a}, \beta(t) = \frac{t-a}{b-a}$ on E and $A(f) = \int_a^b |f(t)|\, dt$ in the Corollary 1, then the inequality (3.3) reduce the inequality (1.5).*

ii.) If we choose $E = \{1, 2, ..., n\}$, $\alpha(k) = \frac{k}{n}, \beta(k) = \frac{n-k}{n}$ on E, $f : E \to [0, \infty)$, $f(k) = a_k$, and $A(f) = \sum_{k=1}^{n} a_k$ in the Theorem1, then the inequality (3.3) reduce the inequality (1.6).

We can give more general form of the Theorem 6 as follows:

Theorem 7. *Let L satisfy conditions $L1$, $L2$, and A satisfy conditions $A1$, $A2$ on a base set E. Let $p > 1$ and $p^{-1} + q^{-1} = 1$. If $\alpha_i, w, f, g \geq 0$ on E, $\alpha_i wfg, \alpha_i wf^p, \alpha_i wg^q, wfg \in L, i = 1, 2, ..., m$, and $\sum_{i=1}^{m} \alpha_i = 1$ on E, then we have*

i.)

$$A(wfg) \leq \sum_{i=1}^{m} A^{1/p}(\alpha_i wf^p) A^{1/q}(\alpha_i wg^q)$$

ii.)

$$\sum_{i=1}^{m} A^{1/p}(\alpha_i wf^p) A^{1/q}(\alpha_i wg^q) \leq A^{1/p}(wf^p) A^{1/q}(wg^q).$$

Proof. The proof can be easily done similarly to the proof of Theorem 6. □

If we take $w = 1$ on E in the Theorem 6, then we can give the following corollary:

Corollary 2. *Let L satisfy conditions $L1$, $L2$, and A satisfy conditions $A1$, $A2$ on a base set E. Let $p > 1$ and $p^{-1} + q^{-1} = 1$. If $\alpha_i, f, g \geq 0$ on E, $\alpha_i fg, \alpha_i f^p, \alpha_i g^q, fg \in L, i = 1, 2, ..., m$, and $\sum_{i=1}^{m} \alpha_i = 1$ on E, then we have*

i.)

$$A(fg) \leq \sum_{i=1}^{m} A^{1/p}(\alpha_i f^p) A^{1/q}(\alpha_i g^q) \tag{3.4}$$

ii.)

$$\sum_{i=1}^{m} A^{1/p}(\alpha_i f^p) A^{1/q}(\alpha_i g^q) \leq A^{1/p}(f^p) A^{1/q}(g^q).$$

Corollary 3 (Improvement of Hölder inequality for double integrals)**.** *Let $p, q > 1$ and $1/p + 1/q = 1$. If f and g are real functions defined on $E = [a,b] \times [c,d]$ and if $|f|^p, |g|^q \in L(E)$ then*

$$\int_a^b \int_c^d |f(x,y)|\, |g(x,y)|\, dxdy \leq \sum_{i=1}^{4} \left(\int_a^b \int_c^d \alpha_i(x,y)\, |f(x,y)|^p\, dx \right)^{1/p} \left(\int_a^b \int_c^d \alpha_i(x,y)\, |g(x,y)|^q\, dx \right)^{1/q}, \tag{3.5}$$

where $\alpha_1(x,y) = \frac{(b-x)(d-y)}{(b-a)(d-c)}, \alpha_2(x,y) = \frac{(b-x)(y-c)}{(b-a)(d-c)}, \alpha_3(x,y) = \frac{(x-a)(y-c)}{(b-a)(d-c)}, , \alpha_4(x,y) = \frac{(x-a)(d-y)}{(b-a)(d-c)}$ on E

Proof. If we choose $E = [a,b] \times [c,d] \subseteq \mathbb{R}^2$, $L = L(E)$, $\alpha_1(x,y) = \frac{(b-x)(d-y)}{(b-a)(d-c)}, \alpha_2(x,y) = \frac{(b-x)(y-c)}{(b-a)(d-c)}, \alpha_3(x,y) = \frac{(x-a)(y-c)}{(b-a)(d-c)}, \alpha_4(x,y) = \frac{(x-a)(d-y)}{(b-a)(d-c)}$ on E and $A(f) = \int_a^b \int_c^d |f(x,y)|\, dxdy$ in the Corollary 1, then we get the inequality (3.5). □

Corollary 4. *Let $(a_{k,l})$ and $(b_{k,l})$ be two tuples of positive numbers and $p,q>1$ such that $1/p+1/q=1$. Then we have*

$$\sum_{k=1}^{n}\sum_{l=1}^{m}a_{k,l}b_{k,l}\leq\sum_{i=1}^{4}\left(\sum_{k=1}^{n}\sum_{l=1}^{m}\alpha_i(k,l)a_{k,l}^{p}\right)^{1/p}\left(\sum_{k=1}^{n}\sum_{l=1}^{m}\alpha_i(k,l)b_{k,l}^{q}\right)^{1/q}, \tag{3.6}$$

where $\alpha_1(k,l)=\frac{kl}{nm}, \alpha_2(k,l)=\frac{(n-k)l}{nm}, \alpha_3(k,l)=\frac{(n-k)(m-l)}{nm}, \alpha_4(k,l)=\frac{k(m-l)}{nm}$ on E.

Proof. If we choose $E=\{1,2,...,n\}\times\{1,2,...,m\}$, $\alpha_1(k,l)=\frac{kl}{nm}, \alpha_2(k,l)=\frac{(n-k)l}{nm}, \alpha_3(k,l)=\frac{(n-k)(m-l)}{nm}, \alpha_4(k,l)=\frac{k(m-l)}{nm}$ on E, $f:E\to[0,\infty)$, $f(k,l)=a_{k,l}$, and $A(f)=\sum_{k=1}^{n}\sum_{l=1}^{m}a_{k,l}$ in the Theorem1, then we get the inequality (3.6). □

4. An application for double integrals

In [14], Sarıkaya et al. gave the following lemma for obtain main results.

Lemma 1. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$with $a<b$ and $c<d$. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then the following equality holds:*

$$\begin{aligned}&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\&-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right]\\&=\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1(1-2t)(1-2s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\,dtds.\end{aligned}$$

By using this equality and Hölder integral inequality for double integrals, Sarıkaya et al. obtained the following inequality:

Theorem 8. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q, q>1$, is convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right| \\&\leq\frac{(b-a)(d-c)}{4(p+1)^{2/p}}\left[\frac{|f_{st}(a,c)|^q+|f_{st}(a,d)|^q+|f_{st}(b,c)|^q+|f_{st}(b,d)|^q}{4}\right]^{1/q},\end{aligned}\tag{4.1}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right],$$

$1/p+1/q=1$ *and* $f_{st}=\frac{\partial^2 f}{\partial t\partial s}$.

If Theorem 8 are resulted again by using the inequality (3.5), then we get the following result:

Theorem 9. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q, q > 1$, is convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right| \tag{4.2}$$

$$\leq \frac{(b-a)(d-c)}{4^{1+1/p}(p+1)^{2/p}}\left\{\left[\frac{4|f_{st}(a,c)|^q+2|f_{st}(a,d)|^q+2|f_{st}(b,c)|^q+|f_{st}(b,d)|^q}{36}\right]^{1/q}\right.$$

$$+\left[\frac{2|f_{st}(a,c)|^q+|f_{st}(a,d)|^q+4|f_{st}(b,c)|^q+2|f_{st}(b,d)|^q}{36}\right]^{1/q}$$

$$+\left[\frac{2|f_{st}(a,c)|^q+4|f_{st}(a,d)|^q+|f_{st}(b,c)|^q+2|f_{st}(b,d)|^q}{36}\right]^{1/q}$$

$$\left.+\left[\frac{|f_{st}(a,c)|^q+2|f_{st}(a,d)|^q+2|f_{st}(b,c)|^q+4|f_{st}(b,d)|^q}{36}\right]^{1/q}\right\},$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right],$$

$1/p + 1/q = 1$ *and* $f_{st} = \frac{\partial^2 f}{\partial t \partial s}$.

Proof. Using Lemma 1 and the inequality (3.5), we find

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right| \tag{4.3}$$

$$\leq \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1|1-2t||1-2s||f_{st}(ta+(1-t)b,sc+(1-s))|\,dtds$$

$$\leq \frac{(b-a)(d-c)}{4}\left\{\left(\int_0^1\int_0^1 ts|1-2t|^p|1-2s|^p\,dtds\right)^{1/p}\right.$$

$$\times\left(\int_0^1\int_0^1 ts|f_{st}(ta+(1-t)b,sc+(1-s))|^q\,dtds\right)^{1/q}$$

$$+\left(\int_0^1\int_0^1 t(1-s)|1-2t|^p|1-2s|^p\,dtds\right)^{1/p}$$

$$\times\left(\int_0^1\int_0^1 t(1-s)|f_{st}(ta+(1-t)b,sc+(1-s))|^q\,dtds\right)^{1/q}$$

$$+\left(\int_0^1\int_0^1(1-t)s|1-2t|^p|1-2s|^p\,dtds\right)^{1/p}$$

$$\times\left(\int_0^1\int_0^1(1-t)s|f_{st}(ta+(1-t)b,sc+(1-s))|^q\,dtds\right)^{1/q}$$

$$+\left(\int_0^1\int_0^1(1-t)(1-s)|1-2t|^p|1-2s|^p\,dtds\right)^{1/p}$$

$$\times\left(\int_0^1\int_0^1(1-t)(1-s)\left|f_{st}\left(ta+(1-t)b,sc+(1-s)\right)\right|^q dtds\right)^{1/q}\Bigg\}.$$

Since $|f_{st}|^q$ is convex function on the co-ordinates on Δ, we have for all $t,s\in[0,1]$

$$\begin{aligned}&\left|f_{st}\left(ta+(1-t)b,sc+(1-s)\right)\right|^q \quad (4.4)\\ \leq\ & ts\left|f_{st}(a,c)\right|^q+t(1-s)\left|f_{st}(a,d)\right|^q+(1-t)s\left|f_{st}(a,c)\right|^q+(1-t)(1-s)\left|f_{st}(a,c)\right|^q\end{aligned}$$

for all $t,s\in[0,1]$. Further since

$$\begin{aligned}\int_0^1\int_0^1 ts\,|1-2t|^p\,|1-2s|^p\,dtds &= \int_0^1\int_0^1 t(1-s)\,|1-2t|^p\,|1-2s|^p\,dtds\\ &= \int_0^1\int_0^1(1-t)s\,|1-2t|^p\,|1-2s|^p\,dtds \quad (4.5)\\ &= \int_0^1\int_0^1(1-t)(1-s)\,|1-2t|^p\,|1-2s|^p\,dtds\\ &= \frac{1}{4\,(p+1)^2}, \quad (4.6)\end{aligned}$$

a combination of (4.3) - (4.5) immediately gives the required inequality (4.2). □

Remark 4. *Since $\eta:[0,\infty)\to\mathbb{R},\eta(x)=x^s,0<s\leq1,$ is a concave function, for all $u,v\geq0$ we have*

$$\eta\left(\frac{u+v}{2}\right)=\left(\frac{u+v}{2}\right)^s\geq\frac{\eta(u)+\eta(v)}{2}=\frac{u^s+v^s}{2}.$$

From here, we get

$$\begin{aligned}I &= \left\{\left[\frac{4\,|f_{st}(a,c)|^q+2\,|f_{st}(a,d)|^q+2\,|f_{st}(b,c)|^q+|f_{st}(b,d)|^q}{36}\right]^{1/q}\right.\\ &\quad+\left[\frac{2\,|f_{st}(a,c)|^q+|f_{st}(a,d)|^q+4\,|f_{st}(b,c)|^q+2\,|f_{st}(b,d)|^q}{36}\right]^{1/q}\\ &\quad+\left[\frac{2\,|f_{st}(a,c)|^q+4\,|f_{st}(a,d)|^q+|f_{st}(b,c)|^q+2\,|f_{st}(b,d)|^q}{36}\right]^{1/q}\\ &\quad\left.+\left[\frac{|f_{st}(a,c)|^q+2\,|f_{st}(a,d)|^q+2\,|f_{st}(b,c)|^q+4\,|f_{st}(b,d)|^q}{36}\right]^{1/q}\right\}\\ &\leq 2\left\{\left[\frac{6\,|f_{st}(a,c)|^q+3\,|f_{st}(a,d)|^q+6\,|f_{st}(b,c)|^q+3\,|f_{st}(b,d)|^q}{72}\right]^{1/q}\right.\\ &\quad\left.+\left[\frac{3\,|f_{st}(a,c)|^q+6\,|f_{st}(a,d)|^q+3\,|f_{st}(b,c)|^q+6\,|f_{st}(b,d)|^q}{72}\right]^{1/q}\right\}\\ &\leq 4\left\{\left[\frac{|f_{st}(a,c)|^q+|f_{st}(a,d)|^q+|f_{st}(b,c)|^q+|f_{st}(b,d)|^q}{16}\right]^{1/q}\right.\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\frac{(b-a)(d-c)}{4^{1+1/p}(p+1)^{2/p}} I \\
\leq\ &\frac{(b-a)(d-c)}{4^{1+1/p}(p+1)^{2/p}} 4\left\{\left[\frac{|f_{st}(a,c)|^q+|f_{st}(a,d)|^q+|f_{st}(b,c)|^q+|f_{st}(b,d)|^q}{16}\right]^{1/q}\right\} \\
\leq\ &\frac{(b-a)(d-c)}{4(p+1)^{2/p}}\left\{\left[\frac{|f_{st}(a,c)|^q+|f_{st}(a,d)|^q+|f_{st}(b,c)|^q+|f_{st}(b,d)|^q}{4}\right]^{1/q}\right\}.
\end{aligned}$$

This shows that the inequality (4.2) is better than the inequality (4.1).

5. Conclusions

The aim of this paper is to give a new general improvement of Hölder inequality via isotonic linear functional. An important feature of the new inequality obtained here is that many existing inequalities related to the Hölder inequality can be improved. As applications, this new inequality will be rewritten for several important particular cases of isotonic linear functionals. Also, we give an application to show that improvement is hold for double integrals. Similar method can be applied to the different type of convex functions.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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