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Research article

Decay estimate and non-extinction of solutions of p-Laplacian nonlocal heat equations

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This presented work is in memory of third author's father (1910–1999) Mr Mahmoud ben Mouha Boulaaras.

Abstract: The main goal of this work is to study the initial boundary value problem of a nonlocal heat equations with logarithmic nonlinearity in a bounded domain. By using the logarithmic Sobolev inequality and potential wells method, we obtain the decay, blow-up and non-extinction of solutions under some conditions, and the results extend the results of a recent paper Lijun Yan and Zuodong Yang (2018).

Keywords: global existence; decay estimates; potential well; blow-up; non-extinction; nonlocal heat equations
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1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)-$growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [1–9]).

In this paper, we consider the Neumann problem to the following initial parabolic equation with logarithmic source:

$$\begin{cases} u_t - div\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u\log|u| - \oint_\Omega |u|^{p-2}u\log|u|\,dx, & x\in\Omega, t>0,\\ \dfrac{\partial u(x,t)}{\partial\eta} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p\in(2,+\infty)$, $\oint_\Omega u_0dx = \frac{1}{|\Omega|}\int_\Omega u_0dx = 0$ with $u_0\neq 0$.

Problem (1.1) has been studied by many other authors in a more general form

$$\begin{cases} u_t - \Delta u = f(u) - \oint_\Omega f(u)dx, & x\in\Omega, t>0,\\ \dfrac{\partial u(x,t)}{\partial\eta} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0, & x\in\Omega, t>0, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N\geq 1)$ with $|\Omega|$ denoting its Lebesgue measure, N is the outer normal vector of $\partial\Omega$, and the function $f(u)$ is usually taken to be a power of u.

Wang. M, Wang,.Y in [10], studied the properties of positive solutions when $f(u) = |u|^p$. The authors showed global existence and exponential decay in the case where $|\Omega|\leq k$ and they obtained a blow-up result under the assumption that the initial data is bigger than some "Gaussian function" in the case where $|\Omega|>k$.. When $f(u) = u|u|^p$ and $\int_\Omega udx>0$, non-global existence result is discussed by [11].

C. Qu, X. Bai, S. Zheng [12] considered the nonlocal p-Laplace equation

$$\begin{cases} u_t - div\left(|\nabla u|^{p-2}\nabla u\right) = u^q - \oint_\Omega u^qdx, & x\in\Omega, t>0,\\ \frac{\partial u(x,t)}{\partial\eta} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0, & x\in\Omega, t>0, \end{cases} \tag{1.3}$$

where a critical blow-up solution is determined by q and the sign of the initial energy.

More recently, L. Yan, Z. Yong [13] established a blow-up and non-extinction of solutions under appropriate conditions for (1.1) in the case $p=2$.

Apart the aforesaid attention given to polynomial nonlinear terms, logarithmic nonlinearity has also received a great deal of interest from both physicists and mathematicians (see for example [14–18]). This type of nonlinearity was introduced in the nonrelativistic wave equations describing spinning particles moving in an external electromagnetic field and also in the relativistic wave equation for spinless particles [19]. Moreover, the logarithmic nonlinearity appears in several branches of physics such as inflationary cosmology [20], nuclear physics [21], optics [22] and geophysics [23]. With all those specific underlying meaning in physics, the global-in-time well-posedness of solution to the problem of evolution equation with such logarithmic type nonlinearity captures lots of attention. Birula and Mycielski ([24, 25]) studied the following problem:

$$\begin{cases} u_{tt} - u_{xx} + u - \varepsilon u\ln|u|^2 = 0 \text{ in } [a,b]\times(0,T),\\ u(a,t) = u(b,t) = 0, (0,T),\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } [a,b], \end{cases} \tag{1.4}$$

which is a relativistic version of logarithmic quantum mechanics and can also be obtained by taking the limit p goes to 1 for the p-adic string equation ([26]). In [27], Cazenave and Haraux considered

$$u_{tt} - \Delta u = u \ln |u|^k \text{ in } \mathbb{R}^3 \tag{1.5}$$

and established the existence and uniqueness of the solution for the Cauchy problem. Gorka [28] used some compactness arguments and obtained the global existence of weak solutions, for all

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2([a, b]),$$

to the initial-boundary value problem (1.4) in the one-dimensional case. Bartkowski and Gorka, [29] proved the existence of classical solutions and investigated the weak solutions for the corresponding one-dimensional Cauchy problem for Equation (1.5). Hiramatsu et al. [30] introduced the following equation

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u| \tag{1.6}$$

to study the dynamics of Q-ball in theoretical physics and presented a numerical study. However, there was no theoretical analysis for the problem. In [31], Han proved the global existence of weak solutions, for all

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega), \tag{1.7}$$

to the initial boundary value problem (1.6) in $\mathbb{R}^3$.

Motivated by the above studies, in this paper we investigate a blow up, non existence and decay of solutions of problem (1.1).

It is necessary to note that the presence of the logarithmic nonlinearity causes some difficulties in deploying the potential well method. In order to handle this situation we need the following logarithmic Sobolev inequality which was introduced in [10].

Lemma 1. *Let $p > 1, \mu > 0$, and $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$. Then we have*

$$p \int_{\mathbb{R}^n} |u(x)|^p \log\left(\frac{|u(x)|}{\|u(x)\|_{L^p(\mathbb{R}^n)}}\right) dx + \frac{n}{p} \log\left(\frac{p\mu e}{n\mathcal{L}_p}\right) \int_{\mathbb{R}^n} |u(x)|^p \, dx$$
$$\leq \mu \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx,$$

where

$$\mathcal{L}_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p-1}{p}+1\right)}\right]^{\frac{p}{n}}.$$

2. Preliminaries

2.1. Notations and some inequalities

We begin this section by introducing some notations that will be used throughout the paper

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \quad \|u\|_{1,\Omega} = \|u\|_{W_0^{1,p}} = \|\nabla u\|_p,$$

for $1 < p < +\infty$.We also we define $X_0 = W_0^{1,p}(\Omega) \setminus \{0\}$.

Lemma 2. *Let ϱ be a positive number. Then the following inequality holds*

$$\log s \leq \frac{e^{-1}}{\varrho} s^{\varrho}, \text{ for all } s \in [1, +\infty].$$

Lemma 3. *(a) For any function $u \in W_0^{1,p}(\Omega)$, we have the inequality*

$$\|u\|_q \leq B_{q,p} \|\nabla u\|_p ,$$

for all $q \in [1, \infty)$ if $n \leq p$, and $1 \leq q \leq \frac{np}{n-p}$ if $n > p$. Then the best constant depends $B_{q,p}$ only on Ω, n, p and q.

We will denote the constant $B_{p,p}$ by B_p.

(b) Let $2 \leq p < q < p^*$. For any $u \in W_0^{1,p}(\Omega)$ we have

$$\|u\|_q \leq C \|\nabla u\|_p^{\alpha} \|u\|_p^{1-\alpha} ,$$

where C is a positive constant and

$$\alpha = \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{p}\right)^{-1} .$$

Remark 1. *It follows from Lemma 2 that*

$$s^p \log s \leq \frac{e^{-1}}{\varrho} s^{p+\varrho}, \text{ for all } \varrho > 0 \text{ and } s \in [1, +\infty).$$

Now we considering the functional J and I defined on X_0 as follows

$$J(u) = \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} \int_{\Omega} |u|^p \ln |u| \, dx + \frac{1}{p^2} \|u\|_p^p . \tag{2.1}$$

$$I(u) = \|\nabla u\|_p^p - \int_{\Omega} |u|^p \ln |u| \, dx. \tag{2.2}$$

The functions I and J are continuous (they are defined as in [32] with some modifications). Moreover, we have

$$J(u) = \frac{1}{p} I(u) + \frac{1}{p^2} \|u\|_p^p . \tag{2.3}$$

Then it is obvious that

$\phi = \left\{u \in X_0 : I(u) = 0, \ \|u\|_p^p \neq 0\right\}$.

$d = \inf_{u \in \phi} J(u)$.

$M = \frac{R^p}{p^2}$.

From [33], we know $d \geq M$.

$\mathcal{N}_\delta = \{u \in X_0 : I_\delta(u) = 0\}$.

Theorem 1. (*Local existence*) *Let $u_0 \in X_0$. Then there exists a positive constant T_0 such that the problem (1.1) has a weak solution $u(x,t)$ on $\Omega\times (0, T_0)$. Furthermore, $u(x,t)$ satisfies the energy inequality*

$$\int_0^t \|u_s(s)\|_2^2 \, ds + J(u(t)) \leq J(u_0), \quad \forall t \in [t, T_0) . \tag{2.4}$$

Lemma 4. *Suppose that $\theta > 0$, $\alpha > 0$, $\beta > 0$ and $h(t)$ is a nonnegative and absolutely continuous function satisfying $h'(t) + \alpha h^{\theta}(t) \geq \beta$, then for $0 < t < \infty$, it holds*

$$h(t) \geq \min\left\{h(0), \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\theta}}\right\}.$$

Lemma 5. *If $0 < J(u_0) < E_1 = \frac{1}{p^2}e^{\frac{b}{p}}$, where $b = n\log\left(\frac{p^2 e}{n\mathcal{L}_p}\right)$, , then there exists a positive constant $\alpha_2 > \alpha_1$ such that*

$$\|u\|_p \geq \alpha_2. \tag{2.5}$$

Proof. Using the logarithmic Sobolev inequality in Lemma 1 and $\mu = p$, we have

$$\begin{aligned} J(u) &= \frac{1}{p}\|\nabla u\|^p - \frac{1}{p}\int_\Omega |u|^p \log|u|\,dx + \frac{1}{p^2}\|u\|_p^p \\ &\geq \left[\frac{n}{p^3}\log\left(\frac{p^2 e}{n\mathcal{L}_p}\right) - \frac{1}{p}\log\|u\|_p + \frac{1}{p^2}\right]\|u\|_p^p, \end{aligned} \tag{2.6}$$

Denote $\alpha = \|u\|_p$, $b = n\log\left(\frac{p^2 e}{n\mathcal{L}_p}\right)$, we have

$$h(\alpha) = \left[\frac{b}{p^3} - \frac{1}{p}\ln\alpha + \frac{1}{p^2}\right]\alpha^p. \tag{2.7}$$

Let $h'(\alpha_1) = 0$, $E_1 = h(\alpha_1) = \frac{1}{p^2}e^{\frac{b}{p^2}}$

$$\begin{aligned} h'(\alpha_1) &= 0 \Rightarrow \left[\frac{b}{p^3}\alpha_1^p - \frac{1}{p}\alpha_1^p\log\alpha + \alpha_1^p\frac{1}{p^2}\right]' = 0 \\ &\Rightarrow \frac{b}{p^2} - \log\alpha_1 = 0 \Rightarrow \alpha_1 = e^{\frac{b}{p^2}}. \end{aligned}$$

Furthermore,we get $h(\alpha)$ is increasing in $(0, \alpha_1)$ and decreasing in (α_1, ∞). Since $J(u_0) < E_1$, there exists a positive constant $\alpha_2 > \alpha_1$ such that $J(u_0) = h(\alpha_2)$. Let $\alpha_0 = \|u_0\|_2$, from (2.6) and (2.7), we have

$$h(\alpha_0) \leq J(u_0).$$

Since $\alpha_0, \alpha_2 \geq \alpha_1$, we get $\alpha_0 \geq \alpha_2$, so (2.5) holds for $t = 0$.

To prove (2.5) for $t > 0$, we assume the contrary that $\|u(., t)\|_2 < \alpha_2$ for some $t_0 > 0$. By the continuity of $\|u(., t)\|_2$ and $\alpha_1 < \alpha_2$, we may choose t_0 such that $\|u(., t_0)\|_2 > \alpha_1$, then it follows from (2.6)

$$J(u_0) = h(\alpha_2) < h\left(\|u(., t_0)\|_2\right) \leq J(u)(t_0),$$

which contradicts the fact that $J(u)$ is nonincreasing in t by (2.4), so (2.5) is true. □

Lemma 6. *Let $H(u) = E_1 - J(u)$, $J(u_0) < E_1$, then $H(u)$ satisfies the following estimates*

$$0 < H(u_0) \leq H(u).$$

Proof. It is obvious that $H(u)$is nondecreasing in t, by (2.4), then it follows from $J(u_0) < E_1$that

$$H(u) \geq H(u_0) = E_1 - J(u_0) > 0.$$

□

2.2. Potential well

Let $u \in X_0$ and consider the real function $j : \lambda \to J(\lambda u)$ for $\lambda > 0$,
The following Lemma shows that $j(\lambda)$ has a unique positive critical point $\lambda^* = \lambda^*(u)$ see [3] .

Lemma 7. *Let $u \in X_0$.Then it holds*
(1) $\lim_{\lambda \to 0^+} j(\lambda) = 0$ *and* $\lim_{\lambda \to +\infty} j(\lambda) = -\infty$,
(2) there is a unique $\lambda^* = \lambda^*(u) > 0$ *such that* $j'(\lambda^*) = 0$,
(3) $j(\lambda)$ *is increasing on* $(0, \lambda^*)$, *decreasing on* $(\lambda^*, +\infty)$ *and attains the maximum at* λ^*,
(4) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < +\infty$ *and* $I(\lambda^* u) = 0$.

Proof. For $u \in X_0$, by the definition of j , It is clear that (1) holds due to $\|u\|_p \neq 0$, and by derivation of j, we have

$$\frac{d}{d\lambda} j(\lambda) = \lambda^{p-1} \int_\Omega [|\nabla u|^p - |u|^p \log |u| \, dx - |u|^p \log \lambda] \, dx$$

$$\frac{d}{d\lambda} j(\lambda^*) = 0$$

which implies that

$$\lambda^* = \exp \frac{\int_\Omega [|\nabla u|^p \, dx - |u|^p \log |u|] \, dx}{\int_\Omega |u|^p \, dx}.$$

the statements of (2) and (3) can be shown easily. The last property, (4), is only a simple corollary of the fact that

$$\begin{aligned} I(\lambda^*) &= \lambda^* \left[\lambda^{*p-1} \int_\Omega (|\nabla u|^p - |u|^p \log |u| \, dx - |u|^p \log \lambda^*) \, dx \right] \\ &= \lambda^* j(\lambda^*) \\ &= 0. \end{aligned}$$

The proof of lemma is complete. □

Next we denote

$$R = \left(\frac{p^2 e}{n \mathcal{L}_p} \right)^{\frac{n}{p^2}},$$

Lemma 8. *(1) if* $I(u) > 0$ *then* $0 < \|u\|_p < R$,
(2) if $I(u) < 0$ *then* $\|u\|_p > R$,
(3) if $I(u) = 0$ *then* $\|u\|_p \geq R$.

Proof. By the definition of $I(u)$, we have

$$\begin{aligned} I(u) &\geq \|\nabla u\|_p^p - \int_\Omega |u|^p \log |u| \, dx \\ &= \left(1 - \frac{\mu}{p}\right) \|\nabla u\|_p^p + \left[\frac{\mu}{p} \|\nabla u\|_p^p - \int_\Omega |u|^p \log \left(\frac{|u|}{\|u\|_p} \right) dx \right] - \int_\Omega |u|^p \log \|u\|_p \, dx, \end{aligned}$$

Choosing $\mu = p$, and we apply the logarithmic Sobolev inequality (Lemma 1), we obtain

$$I(u) \geq \left(\frac{n}{p^2}\log\frac{p^2 e}{n\mathcal{L}_p} - \log\|u\|_p\right)\|u\|_p^p,$$

if $I(u) > 0$, then

$$\log\|u\|_p < \log\left(\frac{p^2 e}{n\mathcal{L}_p}\right)^{\frac{n}{p^2}},$$

that's mean

$$\|u\|_p < \left(\frac{p^2 e}{n\mathcal{L}_p}\right)^{\frac{n}{p^2}} = R,$$

and if $I(u) < 0$, we obtain

$$\|u\|_p > \left(\frac{p^2 e}{n\mathcal{L}_p}\right)^{\frac{n}{p^2}} = R.$$

property (3) we can argue similarly the proof of (2).
The proof of lemma is complete. □

3. Global existence and decay estimates

Lemma 9. *(see [34]) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and σ is a nonnegative constant such that*

$$\int_t^{+\infty} f^{1+\sigma}(s)ds \leq \frac{1}{\omega} f^{\sigma}(0)f(t). \quad \forall t \geq 0.$$

Then we have
(a) $f(t) \leq f(0)e^{1-\omega t}$, for all $t \geq 0$, whenever $\sigma = 0$,
(b) $f(t) \leq f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, for all $t \geq 0$, whenever $\sigma > 0$.

Remark 2. *As in [33], we introduce the following set:*

$$\begin{aligned} W_1^+ &= \{u \in X_0, I(u) > 0\}. \\ W_1^- &= \{u \in X_0, I(u) < 0\}. \end{aligned}$$

Theorem 2. *if $u_0 \in W_1^+, 0 < J(u_0) < M' = \frac{R}{p^2}$, Then the solution $u(x,t)$ of problem (1.1) admits a global weak solution such that*

$$u(t) \in \overline{W_1^+}, \text{ for } 0 \leq t < \infty,$$

satisfying the energy estimate

$$\int_0^t \|u_s(s)\|_2^2\, ds + J(u(t)) \leq J(u_0), \quad \forall t \in [t, T_0).$$

Moreover, the solution decays polynomially, namely

$$\|u\|_2 \leq C_s\left(\frac{p}{2\left(1+\zeta_s\,(p-2)\,t\right)}\right)^{\frac{1}{p-2}}, \quad t \geq 0, , \ t \geq 0,$$

where C_s and ζ_s are positives constants .

Proof. **Existence of global weak solutions**

It suffices to show that $\|\nabla u\|_p^p$ and $\|u\|_p^p$ are bounded independent of t.

In the space $W_0^{1,p}(\Omega)$, we take a basis $\{w_j\}_{j=1}^{\infty}$ and define the finite dimensional space

$$V_m = span\{w_1, w_2, ...w_m, \}.$$

Let u_{0m} be an element of V_m such that

$$u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to u_0 \text{ strongly in } W_0^{1,p}(\Omega). \tag{3.1}$$

as $m \to +\infty$, We find the approximate solution $u_m(x,t)$ of the problem (1.1) in the form

$$u_m(x,t) = \sum_{j=1}^{m} \alpha_{mj}(t) w_j(x).$$

where the coefficients $\alpha_{mj}(1 \le j \le m)$ where $(\alpha_{mj}(0) = a_{m,j})$, satisfy the system of ordinary differential equations

$$\begin{aligned} & (u_{mt}, w_i)_2 + \left(\left(|\nabla u_m|^{p-2}\nabla u_m\right), \nabla w_i\right)_2 \\ = \; & (|u_m|^p u_m \log|u_m|, w_i)_2 - \left(\oint |u_m|^p u_m \log|u_m|, w_i\right)_2 . \end{aligned} \tag{3.2}$$

We multiply both sides of (3.2) by $\alpha'_{mi}(t)$,and we take the sum, we get

$$\begin{aligned} & \int_\Omega \alpha'_{mm}(t) u_{mt}(t) w_m(x) dx + \int_\Omega \alpha'_{mm}(t) |\nabla u_m(t)|^{p-2} \nabla u(t) \nabla w_m(x) dx \\ = \; & \int_\Omega \alpha'_{mm}(t) |u_m|^{p-2}(t) u_m(t) \log|u_m(t)| \, w_m(x) dx, \end{aligned}$$

that's mean

$$\int_\Omega |u_{mt}(t)|^2 dx + \int_\Omega |\nabla u_m(t)|^{p-2} \nabla u_m(t) \nabla u_{mt}(t) dx = \int_\Omega |u_m|^{p-2}(t) u_m(t) u_{mt}(t) \log|u_m(t)| \, dx,$$

this implies that

$$\|u_{mt}(t)\|^2 + \frac{d}{dt}\left[\frac{1}{p}\|\nabla u_m(t)\|^p + \frac{1}{p^2}\|u_m(t)\|^p - \frac{1}{p}\int_\Omega |u_m|^p(t) \log|u_m(t)| \, dx\right] = 0,$$

we deduce

$$\|u_{mt}(t)\|^2 + \frac{d}{dt} J(u_m(t)) = 0, \tag{3.3}$$

by integrating (3.3) with respect to t on $[0,t]$, we obtain the following equality

$$\int_0^t \|u_{mt}(s)\|^2 ds + J(u_m(t)) = J(u_m(0)), \quad 0 \le t \le T_m, \tag{3.4}$$

where T_m is the maximal existence time of solution $u_{mt}(x,t)$.

It follows from (3.1), (3.4), and the continuity of J that

$$J(u_m(0)) \to J(u_0),\ où\ \ m \to +\infty,$$

with $J(u_0) < d$ and

$$\int_0^t \|u_{mt}(s)\|^2\, ds + J(u_m(t)) < d,\ 0 \le t \le T_m, \tag{3.5}$$

for m large sufficiently large m, We will show that

$$u_m(t) \in W_1^+,\ \forall t \ge 0, \tag{*}$$

and for sufficiently large m, and assume that (*) does not hold and let t_* be the smallest time for which $u_m(t_*) \notin W_1^+$. Then, by the continuity of $u_m(t_*) \in \partial W_1^+$, we have

$$J(u_m(t_*)) = d,\ \text{ and } I(u_m(t_*)) = 0, \tag{3.6}$$

Nevertheless, it is clear that $(3.6)_1$ could not occur by (3.5) while if $(3.6)_2$ holds then, by the definition of d, we have

$$J(u_m(t_*)) \ge \inf_{u \in \mathcal{N}_\delta} J(u) = d, \tag{**}$$

which also contradicts with (3.5). Thus, (*) is true.

On the other hand, since $u_m(t) \in W_1^+$, and

$$J(u_m(t)) = \frac{1}{p} I(u_m(t)) + \frac{1}{p^2} \|u_m(t)\|_p^p,\ \forall t \in [0, T_m),$$

we deduces from (3.5) that

$$\|u_m(t)\|_p^p < p^2 d,\ \text{ and } \int_0^t \|u_{ms}(s)\|_2^2\, ds < d, \tag{3.7}$$

for sufficiently large m and $t \in [0, T_m)$. Further, by the logarithmic inequality, we have

$$\begin{aligned}
&\|\nabla u_m(t)\|_p^p = pJ(u_m(t)) + \textstyle\int_\Omega |u_m|^p(t) \log|u_m(t)|\, dx - \frac{1}{p}\|u_m(t)\|_p^p \\
&\le pJ(u_m(0)) + \textstyle\int_\Omega |u_m|^p(t) \left(\frac{\log|u_m(t)| dx}{\|u_m(t)\|_p} + \log\|u_m(t)\|_p \right) dx \\
&\le pJ(u_m(0)) + \tfrac{\mu}{p}\|\nabla u_m(t)\|_p^p - \tfrac{n}{p^2}\log\left(\tfrac{p\mu e}{nL_p}\right)\|u_m(t)\|_p^p \\
&+ \|u_m(t)\|_p^p \log\|u_m(t)\|_p,
\end{aligned}$$

This implies that

$$\left(\frac{p-\mu}{p}\right)\|\nabla u_m(t)\|_p^p \le pJ(u_m(0)) + \|u_m(t)\|_p^p \log\|u_m(t)\|_p - \frac{n}{p^2}\log\left(\frac{p\mu e}{nL_p}\right)\|u_m(t)\|_p^p$$

Taking $\mu < p$, we deduce that

$$\|\nabla u_m(t)\|_p^p \le C_d,\ \ \forall t \in [0, T_m).$$

Decay estimates

We define

$$M(t) = \frac{1}{2}\|u\|_2^2 .$$

$$\begin{aligned} &M^{'}(t) = \int_\Omega u_t u dx \\ &= \left(\left(|u|^{p-2} u \log|u|, u\right)_2 - \left(\oint_\Omega |u|^{p-2} u \log|u|\, dx, u\right)_2 + \|\nabla u\|_p^p\right) \\ &= -I(u) \end{aligned} \tag{3.8}$$

for $u(t) \in W_1^+$ by using (2.3) and the energy inequality that, we know

$$\|u\|_p^p \leq p^2 J(u) \leq p^2 J(u_0). \tag{3.9}$$

By using the logarithmic Sobolev inequality in Lemma 1 and we put $\mu = p$, we have

$$\begin{aligned} I(u) &\geq \left(\frac{n}{p^2}\log\left(\frac{p^2 e}{nL_p}\right) - \log\|u\|_p\right)\|u\|_p^p \\ &\geq \left(\log\left(\frac{p^2}{nL_p}\right)^{\frac{n}{p^2}} - \log p^2 J(u_0)\right)\|u\|_p^p \\ &= \left(\log\frac{\left(\frac{p^2}{nL_p}\right)^{\frac{n}{p^2}}}{p^2 J(u_0)}\right)\|u\|_p^p \\ &\geq \frac{1}{l_{2,p}}\left(\log\frac{\left(\frac{p^2}{nL_p}\right)^{\frac{n}{p^2}}}{p^2 J(u_0)}\right)\|u\|_2^p \\ &= \zeta\|u\|_2^p , \end{aligned} \tag{3.10}$$

where $l_{2,p}$ is a constant in the embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$, $p > 2$ and $\zeta = \frac{1}{l_{2,p}} \log \frac{M}{J(u_0)}$.

From (3.8), we have

$$\begin{aligned} \int_t^T I(u(s))ds &= -\int_t^T \int_\Omega u_s(s)u(s)dxds \\ &= -\frac{1}{2}\|u(T)\|_2^2 + \frac{1}{2}\|u(t)\|_2^2 \\ &\leq \frac{1}{2}\|u\|_2^2 . \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), it follows that

$$\int_t^T \|u\|_2^p \leq \frac{1}{2\zeta}\|u\|_2^2 ,\ \forall t \in [0, T] . \tag{3.12}$$

Let $T \to +\infty$ and apply Lemma 9, such that $f(t) = \|u(t)\|_2^2 , \sigma = \frac{p-2}{2}, f(0) = 1, \omega = \frac{1}{2\varsigma}$ we obtain the following decay estimate

$$\|u\|_2 \leq C_s\left(\frac{p}{2\left(1 + \zeta_s (p-2)\, t\right)}\right)^{\frac{1}{p-2}},\quad t \geq 0, \tag{3.13}$$

where C_s is positive constant and $\zeta_s = \frac{1}{4\varsigma}$. □

4. Blow up of weak solutions

4.1. Blow up at $+\infty$

Definition 1. ***(Blow-up at*** $+\infty$*) Let $u(x,t)$ be a weak solution of (1.1). We call $u(x,t)$ blow-up at $+\infty$ if the maximal existence time $T = +\infty$ and*

$$\lim_{t\to+\infty} \|u(.,t)\|_2 = +\infty$$

Theorem 3. *Assume $J(u_0) < 0$, then the solution $u(x,t)$ of problem (1.1) is blow-up at $+\infty$. Moreover, if $\|u_0\|_2 \le \left(\frac{-pJ(u_0)}{l_{2,p}}\right)^p$, the lower bound for blow-up rate can be estimated by*

$$\|u\|_2^2 \ge \|u_0\|_2^2, \tag{4.1}$$

which is independent of t.

Proof. By the definition of $J(u)$ and (2.4), $M(t)$ satisfies

$$\begin{aligned} M'(t) &= \int_\Omega u_t, u dx \\ &= -\|\nabla u\|_p^p + \int_\Omega \left(|u|^p \log|u| - \oint |u|^p u \log|u|\right) dx \\ &= -\|\nabla u\|_p^p + \int_\Omega |u|^p \log|u|\, dx \\ &= -pJ(u) + \frac{1}{p}\|u\|_p^p \\ &\ge -pJ(u), \end{aligned} \tag{*}$$

by using (2.4) defined in theorem 1 and the condition $J(u_0) < 0$, we have

$$-pJ(u) \ge p\int_0^t \|u_s(s)\|_2^2\, ds, \tag{**}$$

so by (*) and (**), we get

$$M'(t) \ge p\int_0^t \|u_s(s)\|_2^2\, ds, \tag{4.2}$$

And by the definition of weak solution, we know that $u \in W^{1,p}(\Omega)$ For any $t_0 > 0$, we claim that

$$\int_0^{t_0} \|u_s\|_2^2\, ds > 0, \tag{4.3}$$

Otherwise, there exists $t_0 > 0$ such that $\int_0^{t_0} \|u_s\|_2^2\, ds = 0$, and hence $u_t = 0$ for a.e., $(x,t) \in \Omega\times(0,t_0)$. Then it follows from (4.2) that

$$-|\nabla u|^p + |u|^p \log|u| = 0,$$

for a.e., $t \in (0,t_0)$, and then we get from (2.1)

$$J(u) = \frac{1}{p^2}\int_\Omega |u|^p\, dx.$$

Combining it with $J(u) \le J(u_0) \le 0$, we obtain$\| u\|_p = 0$ for all $t \in [0,t_0]$, which contradicts the definition of u. Then (4.3) follows.

Fix $t_0 > 0$ and let $\delta = \int_0^t \| u\|_2^2 \, ds$, then we know that δ is a positive constant. Integrating (4.2) over (t_0, t), we obtain

$$M(t) \geq M(t_0) + p \int_{t_0}^{t} \int_0^t \| u(s)\|_2^2 \, dsd\tau \geq M(t_0) + p \int_{t_0}^{t} \delta d\tau \geq \delta (t - t_0) . \tag{4.4}$$

We have

$$\begin{aligned} H^{'}(t) &= -J^{'}(t) \\ &= \|u_t\|_2^2 , \end{aligned}$$

where $H(t)$ defined in lemma 6

Hence

$$\lim_{t\to\infty} H^{'}(t) = \lim_{t\to\infty} M(t) = \infty. \tag{4.5}$$

And from (4.2), we know

$$\begin{aligned} M^{'}(t) &= -pJ(u) + \frac{1}{p} \|u\|_p^p \\ &\geq -pJ(u_0) + \frac{1}{p} \|u\|_p^p , \end{aligned} \tag{4.6}$$

we have

$$\begin{aligned} M^{'}(t) + l_{2,p} M^{\frac{p}{2}}(t) &\geq -\frac{1}{p} \|u\|_p^p - pJ(u_0) + l_{2,p} M^{\frac{p}{2}}(t) \\ &\geq -\frac{l_{2,p}}{p} \|u\|_2^p - pJ(u_0) + l_{2,p} \|u\|_2^p \\ &\geq \frac{l_{2,p} p - l_{2,p}}{p} \|u\|_2^p - pJ(u_0) \\ &\geq -pJ(u_0), \end{aligned} \tag{4.7}$$

where $l_{2,p}$ is a constant in the embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$, $p > 2$.

By using Lemma 2.1, $J(u_0) < 0$, and $\|u_0\|_2^2 \leq \left(\frac{-pJ(u0)}{l_{2,p}}\right)^{\frac{2}{p}}$, we have

$$\begin{aligned} M(t) &\geq \min \left\{ \|u_0\|_2^2 , \left(\frac{-pJ(u0)}{l_{2,p}} \right)^{\frac{2}{p}} \right\} \\ &\geq \|u_0\|_2^2 , \end{aligned}$$

which means (4.1) is true. □

4.2. *Non-extinct in finite time*

Definition 2. ***(Finite time blow-up)*** *Let $u(x,t)$ be a weak solution of (1.1).We call $u(x,t)$ blow-up in finite time if the maximal existence time T is finite and*

$$\lim_{t\to T^-} \|u(.,t)\|_2 = +\infty.$$

Lemma 10. *Let ϕ be a positive, twice differentiable function satisfying the following conditions*

$$\phi(\bar{t}) > 0, \text{ and } \phi^{'}(\bar{t}) > 0,$$

for some $\bar{t} \in [0, T)$, and the inequality

$$\phi(t)\phi^{''}(t) - \alpha(\phi^{'}(t))^2 \geq 0, \quad \forall t \in \left[\bar{t}, T\right], \tag{4.8}$$

where $\alpha > 1$. Then we have

$$\phi(t) \geq \left(\frac{1}{\phi^{1-\alpha}(\bar{t}) - \sigma(t - \bar{t})}\right), \quad t \in \left[\bar{t}, T^*\right).$$

with σ is a positive constant, and

$$T^* = \bar{t} + \frac{\phi(t)}{(\alpha - 1)\phi'(\bar{t})}.$$

This implies

$$\lim_{t \to T^*} \phi(t) = \infty.$$

Theorem 4. *Assume $0 < J(u_0) < M$ and $u \in W_1^-$, then the solution $u(x,t)$ of problem (1.1) is non-extinct in finite time, defined by*

$$T^* = \bar{t} + \frac{\int_0^t \|u(s)\|_2^2 \, ds}{(\frac{p-2}{2}) \left\|u(\bar{t})\right\|_2^2}, \quad s \in \left[\bar{t}, T^*\right).$$

Proof. we define the functional

$$\Gamma(t) = \int_0^t \|u(s)\|_2^2 \, ds. \tag{4.9}$$

Then one has

$$\Gamma^{'}(t) = \|u(t)\|_2^2, \tag{4.10}$$

and

$$\begin{aligned} \Gamma^{''}(t) &= 2\int_\Omega u_t u dx \\ &= -2\|\nabla u(t)\|_p^p + 2\int_\Omega |u|^p \log|u| \, dx \\ &= -2pJ(u) + \frac{2}{p}\|u(t)\|_p^p, \end{aligned} \tag{4.11}$$

by using (2.4) in theorem 1, we have

$$-2pJ(u) \geq -2pJ(u_0) + 2p\int_0^t \|u_s(s)\|_2^2 \, ds, \tag{4.12}$$

by lemma 8 for $I(u) < 0$, which implies

$$\|u(t)\|_p^p \geq R, \tag{4.13}$$

by (4.13) and (4.14), we get

$$\begin{aligned}\Gamma^{''}(t) &\geq -2pJ(u_0) + 2p\int_0^t \|u_s(s)\|_2^2\, ds + \frac{2}{p}\|u(t)\|_p^p \\ &= 2p\left(\frac{1}{p^2}\|u(t)\|_p^p - J(u_0)\right) + 2p\int_0^t \|u_s(s)\|_2^2\, ds \\ &\geq 2p\,(M - J(u_0)) + 2p\int_0^t \|u_s(s)\|_2^2\, ds, \qquad (4.14)\end{aligned}$$

where $M = \frac{R}{p^2}$.

In other hand we have

$$\Gamma^{'}(t) = \Gamma^{'}(t) + \int_0^t \Gamma^{''}(s)ds \geq 2p\,(M - J(u_0))\, t \geq 0, \quad t \in [0, t], \qquad (4.15)$$

also, we have

$$\begin{aligned}\frac{1}{4}\left(\Gamma^{'}(t)\right)^2 &\leq \left(\int_0^t \int_\Omega u_s(s)u(s)dxds\right)^2 \\ &\leq \int_0^t \|u(s)\|_2^2\, ds \int_0^t \|u_s(s)\|_2^2\, ds, \qquad (4.16)\end{aligned}$$

Now, multiplying (4.15) by $\Gamma(t)$, we get

$$\begin{aligned}\Gamma^{''}(t)\Gamma(t) &\geq 2p\,(M - J(u_0))\,\Gamma(t) + 2p\int_0^t \|u_s(s)\|_2^2\, ds\Gamma(t) \\ &= 2p\,(M - J(u_0))\,\Gamma(t) + 2p\int_0^t \|u_s(s)\|_2^2\, ds \int_0^t \|u(s)\|_2^2\, ds, \qquad (4.17)\end{aligned}$$

by using (4.17) in (4.18), we obtain

$$\Gamma^{''}(t)\Gamma(t) \geq 2p\,(M - J(u_0))\,\Gamma(t) + \frac{p}{2}\left(\Gamma^{'}(t)\right)^2, \; for\ all\ t \in [0, T]. \qquad (4.18)$$

This follows

$$\Gamma^{''}(t)\Gamma(t) - \frac{p}{2}\left(\Gamma^{'}(t)\right)^2 \geq 2p\,(M - J(u_0))\,\Gamma(t), \quad for\ all\ t \in [0, T]. \qquad (4.19)$$

By virtue of lemma 10, where $\alpha = \frac{p}{2} > 1$, and $\phi(t) = \Gamma(t)$, we get
there exists $T_* > 0$ such that

$$\lim_{t \to T_*^-} \Gamma(t) = +\infty,$$

which implies

$$\lim_{t \to T_*^-} \int_0^t \|u(s)\|_2^2\, ds = +\infty,$$

therefore, we get

$$\lim_{t \to T_*^-} \|u(t)\|_2^2 = +\infty.$$

This ends the proof.

□

5. Conclusions

In this work, by using the logarithmic Sobolev inequality and potential wells method, we study the initial boundary value problem of a nonlocal heat equations with logarithmic nonlinearity in a bounded domain, where we obtain the decay, blow-up and non-extinction of solutions under some conditions, and the results extend the results of a recent paper Lijun Yan and Zuodong Yang [13]. In our next study, we will try to apply an alternative approach using the variational principle that has been presented in previous studies [35].

Conflict of interest

The authors declare no conflict of interest.

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