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Research article

Random attractors for non-autonomous stochastic plate equations with multiplicative noise and nonlinear damping

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Abstract: Based on the abstract theory of pullback attractors of non-autonomous non-compact dynamical systems by differential equations with both dependent-time deterministic and stochastic forcing terms, which introduced by B. Wang, we investigate existence of pullback attractors for the non-autonomous stochastic plate equations with multiplicative noise defined in the entire space $\mathbb{R}^n$.

Keywords: pullback attractors; plate equation; unbounded domains; multiplicative white noise
Mathematics Subject Classification: 35B40, 35B41

1. Introduction

In this paper, we study the asymptotic behavior of solutions for the following non-autonomous stochastic plate equation with multiplicative noise and nonlinear damping defined on the unbounded domain $\mathbb{R}^n$:

$$u_{tt} + \Delta^2 u + h(u_t) + \lambda u + f(x, u) = g(x, t) + \varepsilon u \circ \frac{dw}{dt} \tag{1.1}$$

with the initial value conditions

$$u(x, \tau) = u_0(x), \quad u_t(x, \tau) = u_1(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, $\lambda > 0$ and ε are constants, $h(u_t)$ is a nonlinear damping term, f is a given interaction term, g is a given function satisfying $g \in L^2_{loc}(\mathbb{R}, H^1(\mathbb{R}^n))$, and w is a two-sided real-valued Wiener process on a probability space. The stochastic Eq. (1.1) is understood in the sense of Stratonovich's integration.

Plate equations like (1.1), especially when $h(u_t) = \alpha u_t$, have been investigated for many years due to their importance in some physical areas such as vibration and elasticity theory of solid mechanics.

The study of the long-time dynamics of plate equations has become an outstanding topic in the field of the infinite dimensional dynamical system.

As we know, the attractor is regarded as a proper notation describing the long-time dynamics of solutions, and many classical literatures and monographs have been appeared for both the deterministic and stochastic dynamical systems over the last decades years, see [1, 5, 6, 8, 9, 12, 12–15, 22, 26, 27, 40] and references therein. However, in reality, a system is always affected by some random factors such as external noise. In order to study the large-time behavior and characterization of solution for the stochastic partial differential equations driven by noise, H. Crauel & Franco Flandoi [8, 9], Franco Flandoi & B. Schmalfuss [12] and B. Schmalfuss [22] introduced the concept of pullback attractors, and established some abstract results for existence of such attractors about compact dynamical system [1, 9, 12, 15, 18]. Since these methods required the compactness of pullback absorbing set for systems, it could not be used to deal with the stochastic PDEs on unbounded domains. P. W. Bates, H. Lisei & K. Lu [4] presented the concept of asymptotic compactness for random dynamical systems, they proved the existence of random attractors for reaction-diffusion equations on unbounded domain using these abstract results in [3]. B. Wang in [27] further extended the concept of asymptotic compactness to the case of partial differential equations with both the random and the time-dependent forcing terms; moreover, he applied this criteria to the stochastic reaction-diffusion equation with additive noise on $\mathbb{R}^n$, and obtained existence of an unique pullback attractor. Most of works on stochastic PDEs, please refer to [10, 25, 28–30, 32, 36] and references therein.

Just for problem (1.1)–(1.2) and the corresponding plate equations, in the deterministic case (i.e., $\varepsilon = 0$), existence of global attractors has been studied by several authors, see for instance [2, 15–17, 34, 35, 38, 39, 41]. As far as the stochastic case driven by additive noise, when the deterministic forcing term g is independent of time, that is, $g(x,\ t) \equiv g(x)$, existence of random pullback attractor on bounded domain was obtained in [20, 23, 24]. Recently, on the unbounded domain, the authors investigated existence and upper semi-continuity of random attractors for stochastic plate equation with rotational inertia and Kelvin-Voigt dissipative term as well as time dependent terms see [37] for details. To the best of our knowledge, it is not considered by any predecessors for the stochastic plate equation with multiplicative noise on unbounded domain. It is well known that multiplicative noise makes the problem more complex and interesting even to the case of bounded domain. Based the theory and applications of B. Wang in [27, 31, 33], we decide to study existence of pullback attractors for problem (1.1)–(1.2).

Notice that (1.1) is a non-autonomous stochastic equation in the sense that the external term g is time-dependent. In this case, like in [27], we need to introduce two parametric spaces to describe its dynamics: one is responsible for the deterministic non-autonomous perturbations and the other for the stochastic perturbations. In addition, since Sobolev embeddings are not compact on unbounded domain, we can not get the desired asymptotic compactness directly from the regularity of solutions. We here overcome this difficulty by using the uniform estimates on the tails of solutions outside a bounded ball in $\mathbb{R}^n$ and the splitting technique [28], as well as the compactness methods introduced in [19].

In comparison with the results recently published in [37], the novelty and the difficulties of this work are as follows: (i) The nonlinear damping $h(u_t)$ in Eq. (1.1) and its treatment; (ii) Using a new Ornstein-Uhlenbeck process which does not depends on the damping coefficients but depends on an adjustable parameter δ, which is substantially different from [37].

The rest of this paper is organized as follows. In the next section, we recall some basic concepts related to random attractor for general random dynamical systems. In section 3, we provide some basic settings about Eq. (1.1) and show that it generates a continuous cocycle. Then we derive all necessary uniform estimates of solutions in section 4, and prove the existence of random attractors in sections 5. In section 6, we give conclusion as well as some comments on possible applications for these results.

Throughout the paper, we use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. The norms of $L^p(\mathbb{R}^n)$ and a Banach space X are generally written as $\|\cdot\|_p$ and $\|\cdot\|_X$, respectively. The letters c and c_i $(i = 1, 2, \ldots)$ are generic positive constants which may change their values from line to line or even in the same line and do not depend on ε.

2. Materials and method

2.1. Preliminaries

In this section, we recall some definitions and known results regarding pullback attractors of non-autonomous random dynamical systems from [7,27], which they are useful to our problem.

In the sequel, we use $(\Omega, \mathcal{F}, \mathcal{P})$ and (X, d) to denote a probability space and a complete separable metric space, respectively. If A and B are two nonempty subsets of X, then we use $d(A, B)$ to denote their Hausdorff semi-distance.

Definition 2.1.1 Let $\theta : \mathbb{R} \times \Omega \to \Omega$ be a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable mapping. We say $(\Omega, \mathcal{F}, \mathcal{P}, \theta)$ is a parametric dynamical system if $\theta(0, \cdot)$ is the identity on Ω, $\theta(s + t, \cdot) = \theta(t, \cdot) \circ \theta(s, \cdot)$ for all $t, s \in \mathbb{R}$, and $P\theta(t, \cdot) = P$ for all $t \in \mathbb{R}$.

Definition 2.1.2 Let $K : \mathbb{R} \times \Omega \to 2^X$ be a set-valued mapping with closed nonempty images. We say K is measurable with respect to $\mathcal{F}$ in Ω if the mapping $\omega \in \Omega \to d(x, K(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$.

Definition 2.1.3 A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (1)–(4) are satisfied:

(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;

(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;

(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Hereafter, we assume Φ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$, and $\mathcal{D}$ is the collection of some families of nonempty bounded subsets of X parameterized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$:

$$\mathcal{D} = \{D = \{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\}\}.$$

Definition 2.1.4 Let $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty subsets of X. For every $\tau \in \mathbb{R}, \omega \in \Omega$, let

$$\Omega(B, \tau, \omega) = \bigcap_{r\geq 0} \overline{\bigcup_{t\geq r} \Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))}.$$

Then the family $\{\Omega(B, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called the Ω-limit set of B and is denoted by $\Omega(B)$.

Definition 2.1.5 Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then K is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $B \in \mathcal{D}$, there exists $T = T(B, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \text{ for all } t \geq T.$$

If, in addition, $K(\tau,\omega)$ is closed in X and is measurable in ω with respect to $\mathcal{F}$, then K is called a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.1.6 Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$, the sequence

$$\{\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X$$

whenever $t_n\to\infty$, and $x_n\in B(\tau-t_n,\theta_{-t_n}\omega)$ with $\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$.

Definition 2.1.7 Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (1)–(3) are fulfilled: for all $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

(1) $\mathcal{A}(\tau,\omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$.

(2) $\mathcal{A}$ is invariant, that is,

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega))=\mathcal{A}(\tau+t,\theta_t\omega).$$

(3) For every $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$,

$$\lim_{t\to\infty} d(\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)),\mathcal{A}(\tau,\omega))=0.$$

Proposition 2.1.8 *Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathcal{P},\{\theta_t\}_{t\in\mathbb{R}})$. If Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$, then Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ which is given by, for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$,*

$$\mathcal{A}(\tau,\omega)=\Omega(K,\tau,\omega)=\bigcup_{D\in\mathcal{D}}\Omega(B,\tau,\omega)$$

2.2. Cocycles for stochastic plate equation

In this section, we outline some basic settings about (1.1)–(1.2) and show that it generates a continuous cocycle in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.

Let $\xi=u_t+\delta u$, where δ is a small positive constant whose value will be determined later. Substituting $u_t=\xi-\delta u$ into (1.1) we find

$$\frac{du}{dt}+\delta u=\xi, \tag{2.2.1}$$

$$\frac{d\xi}{dt}-\delta\xi+(\delta^2+\lambda)u+\Delta^2u+h(\xi-\delta u)+f(x,u)=g(x,t)+\varepsilon u\circ\frac{dw}{dt} \tag{2.2.2}$$

with the initial value conditions

$$u(x,\tau)=u_0(x),\qquad \xi(x,\tau)=z_0(x), \tag{2.2.3}$$

where $\xi_0(x)=u_1(x)+\delta u_0(x),\ x\in\mathbb{R}^n$.

Assumption I. Assume that the functions $h,\ f$ satisfy the following conditions:

(1) Let $F(x,u) = \int_0^u f(x,s)ds$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$, there exist positive constants $c_i (i = 1,2,3,4)$, such that

$$|f(x,u)| \le c_1|u|^\gamma + \phi_1(x), \quad \phi_1 \in L^2(\mathbb{R}^n), \tag{2.2.4}$$
$$f(x,u)u - c_2F(x,u) \ge \phi_2(x), \quad \phi_2 \in L^1(\mathbb{R}^n), \tag{2.2.5}$$
$$F(x,u) \ge c_3|u|^{\gamma+1} - \phi_3(x), \quad \phi_3 \in L^1(\mathbb{R}^n), \tag{2.2.6}$$
$$|\frac{\partial f}{\partial u}(x,u)| \le \beta, \quad |\frac{\partial f}{\partial x}(x,u)| \le \phi_4(x), \quad \phi_4 \in L^2(\mathbb{R}^n), \tag{2.2.7}$$

where $\beta > 0,\ 1 \le \gamma \le \frac{n+4}{n-4}$. Note that (2.2.4) and (2.2.5) imply

$$F(x,u) \le c(|u|^2 + |u|^{\gamma+1} + \phi_1^2 + \phi_2). \tag{2.2.8}$$

(2) There exist two constants β_1, β_2 such that

$$h(0) = 0, \quad 0 < \beta_1 \le h'(v) \le \beta_2 < \infty. \tag{2.2.9}$$

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space, where $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathcal{P})$ which is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega,\ t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system.

It is convenient to convert the problem (2.2.1)–(2.2.3) into a deterministic system with a random parameter, and then show that it generates a cocycle over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Consider Ornstein-Uhlenbeck equation $dz + \delta z dt = d\omega$, $z(-\infty) = 0$, and Ornstein-Uhlenbeck process

$$z(\theta_t\omega) = z(t,\omega) = -\delta \int_{-\infty}^0 e^{\delta s}(\theta_t\omega)(s)ds. \tag{2.2.10}$$

From [1, 11, 18], it is known that the random variable $|z(\omega)|$ is a stationary, ergodic and tempered stochastic process, and there is a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full $\mathcal{P}$ measure such that $z(\theta_t\omega)$ is continuous in t for every $\omega \in \widetilde{\Omega}$. For convenience, we shall simply write $\widetilde{\Omega}$ as Ω.

Now, let $v(x,t) = \xi(x,t) - \varepsilon u(x,t)z(\theta_t\omega)$, we obtain the equivalent system of (2.2.1)–(2.2.3),

$$\frac{du}{dt} + \delta u - v = \varepsilon uz(\theta_t\omega), \tag{2.2.11}$$
$$\frac{dv}{dt} - \delta v + (\delta^2 + \lambda + A)u + f(x,u) = g(x,t) - h(v + \varepsilon uz(\theta_t\omega) - \delta u) - \varepsilon(v - 3\delta u + \varepsilon uz(\theta_t\omega))z(\theta_t\omega) \tag{2.2.12}$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \qquad v(x,\tau) = v_0(x), \tag{2.2.13}$$

where A is defined below and $v_0(x) = \xi_0(x) - \varepsilon z(\theta_\tau\omega)u_0,\ x \in \mathbb{R}^n$.

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $A = \Delta^2$ with the domain $D(A) = H^4(\mathbb{R}^n)$. We can also define the powers A^ν of A for $\nu \in \mathbb{R}$. The space $V_\nu = D(A^{\frac{\nu}{4}})$ is a Hilbert space with the following inner product and norm

$$(u,v)_\nu = (A^{\frac{\nu}{4}}u, A^{\frac{\nu}{4}}v), \qquad \|\cdot\|_\nu = \|A^{\frac{\nu}{4}}\cdot\|.$$

For brevity, the notation $(\cdot,\cdot)$ for L^2-inner product will also be used for the notation of duality pairing between dual spaces, $\|\cdot\|$ denotes the L^2-norm.

Let $E = H^2 \times L^2$, with the Sobolev norm

$$\|y\|_{H^2\times L^2} = (\|v\|^2 + \|u\|^2 + \|\Delta u\|^2)^{\frac{1}{2}}, \text{ for } y = (u,v)^\top \in E. \tag{2.2.14}$$

We shall drop the transpose superscript for all column vectors of u and v. The well-posedness of local weak solutions for the problem of the random PDE (2.2.11)–(2.2.13) in $E = H^2(\mathbb{R}^n) \times H(\mathbb{R}^n)$ can be shown by Galerkin approximation and compactness method as in [5, 21, 26, 37]. Under conditions (2.2.4)–(2.2.7) and (2.2.9), for every $\omega \in \Omega, \tau \in \mathbb{R}$ and $(u_0, v_0) \in E$, we can prove the problem (2.2.11)–(2.2.13) has a unique global solution $(u(\cdot,\tau,\omega,u_0), v(\cdot,\tau,\omega,v_0)) \in C([\tau,\infty), E)$. Moreover, for $t \geq \tau$, $(u(t,\tau,\omega,u_0), v(t,\tau,\omega,v_0))$ is $(\mathcal{F},\ \mathcal{B}(H^2(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in ω and continuous in (u_0, v_0) with respect to the E-norm.

Thus the solution mapping can be used to define a continuous cocycle for (2.2.1)–(2.2.3). Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ be a mapping given by

$$\Phi(t,\tau,\omega,(u_0,v_0)) = (u(t+\tau,\tau,\theta_{-\tau}\omega,u_0), v(t+\tau,\tau,\theta_{-\tau}\omega,v_0)), \tag{2.2.15}$$

where $v(t+\tau,\tau,\theta_{-\tau}\omega,v_0) = \xi(t+\tau,\tau,\theta_{-\tau}\omega,\xi_0) - \varepsilon z(\theta_t\omega)u(t+\tau,\tau,\theta_{-\tau}\omega,u_0)$ with $v_0 = \xi_0 - \varepsilon z(\omega)u_0$. Then Φ is a continuous cocycle over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ on E. For each $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$,

$$\begin{aligned}&\Phi(t,\tau-t,\theta_{-t}\omega,(u_0,v_0)) = (u(\tau,\tau-t,\theta_{-\tau}\omega,u_0), v(\tau,\tau-t,\theta_{-\tau}\omega,v_0))\\ =&(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0), \xi(\tau,\tau-t,\theta_{-\tau}\omega,\xi_0) + \varepsilon z(\omega)u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)).\end{aligned} \tag{2.2.16}$$

This identity is useful when proving pullback asymptotic compactness of Φ. Next we make another assumption.

Assumption II. We assume that $\sigma, \delta, \varepsilon$ and $g(x,t)$ satisfy the following conditions:

$$\sigma = \frac{1}{2}\min\{\delta, \delta c_2\}, \tag{2.2.17}$$

$$\delta > 0 \text{ satisfies } \lambda + \delta^2 - \beta_2\delta > 0,\ \beta_1 > 5\delta + \frac{\beta^2}{\delta(\lambda+\delta^2-\beta_2\delta)}, \tag{2.2.18}$$

$$\begin{aligned}|\varepsilon| < \min\Big\{&\frac{-2\sqrt{\delta}(\gamma_2\gamma_3+\gamma_1) + \sqrt{4\delta(\gamma_2\gamma_3+\gamma_1)^2 + \pi\delta\gamma_2\sigma}}{\gamma_2\sqrt{\pi}},\\ &\frac{-2\sqrt{\delta}(\gamma_2\gamma_4+1) + \sqrt{4\delta(\gamma_2\gamma_4+1)^2 + \pi\delta\gamma_2\sigma}}{\gamma_2\sqrt{\pi}}\Big\},\end{aligned} \tag{2.2.19}$$

where $\gamma_1 = \max\{1, \frac{c_1c_3^{-1}}{2}\}$, $\gamma_2 = 1 + \frac{1}{\lambda+\delta^2-\beta_2\delta}, \gamma_3 = \frac{3}{2}\delta + \frac{1}{2}\beta_2 + 2\beta_2\delta + 1$, $\gamma_4 = \frac{3}{2}\delta + \frac{1}{2}\beta_2 + 2\beta_2\delta$.
Moreover,

$$\int_{-\infty}^{0} e^{\sigma s}\|g(\cdot,\tau+s)\|_1^2 ds < \infty, \quad \forall\, \tau \in \mathbb{R}, \tag{2.2.20}$$

and

$$\lim_{k\to\infty}\int_{-\infty}^{0} e^{\sigma s}\int_{|x|\geq k}|g(x,\tau+s)|^2 dxds = 0,\ \forall\,\tau\in\mathbb{R}, \tag{2.2.21}$$

where $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$.

Given a bounded nonempty subset B of E, we write $\|B\| = \sup_{\phi\in B}\|\phi\|_E$. Let $D = \{D(\tau,\omega) : \tau\in\mathbb{R}, \omega\in\Omega\}$ be a family of bounded nonempty subsets of E such that for every $\tau\in\mathbb{R}, \omega\in\Omega$,

$$\lim_{s\to-\infty} e^{\sigma s}\|D(\tau+s,\theta_s\omega)\|_E^2 = 0. \tag{2.2.22}$$

Let $\mathcal{D}$ be the collection of all such families, that is,

$$\mathcal{D} = \{D = \{D(\tau,\omega) : \tau\in\mathbb{R}, \omega\in\Omega\} :\ D \text{ satisfies (2.2.22)}\}. \tag{2.2.23}$$

2.3. Uniform estimates of solutions

In this section, we conduct uniform estimates on the weak solutions of the stochastic plate Eqs. (2.2.1)–(2.2.3) defined on $\mathbb{R}^n$, through the converted random Eq. (2.2.11)–(2.2.13), for the purposes of showing the existence of a pullback absorbing sets and the pullback asymptotic compactness of the cocycle.

We define a new norm $\|\cdot\|_E$ by

$$\|Y\|_E = (\|v\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u\|^2 + \|\Delta u\|^2)^{\frac{1}{2}},\quad \text{for}\ \ Y=(u,v)\in E. \tag{2.3.1}$$

It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^2\times L^2}$ in (2.2.14). First we show that the cocycle Φ has a pullback $\mathcal{D}$-absorbing set in $\mathcal{D}$.

Lemma 2.3.1 *Under Assumptions* I *and* II*, for every* $\tau\in\mathbb{R}, \omega\in\Omega$*,* $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$*, there exists* $T=T(\tau,\omega,D)>0$ *such that for all* $t\geq T$ *the solution of problem* (2.2.11)–(2.2.13) *satisfies*

$$\|Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\|_E^2 \leq R_1(\tau,\omega),$$

and $R_1(\tau,\omega)$ *is given by*

$$R_1(\tau,\omega) = M+M\int_{-\infty}^{0} e^{2\int_0^s\left[\sigma-\gamma_1|\varepsilon||z(\theta_r\omega)|-\gamma_2\left(\frac{1}{2}\varepsilon^2|z(\theta_r\omega)|^2+\gamma_3|\varepsilon||z(\theta_t\omega)|\right)\right]dr} \cdot(\|g(\cdot,s+\tau)\|^2+|\varepsilon||z(\theta_s\omega)|)ds \tag{2.3.2}$$

where M *is a positive constant independent of* τ,ω,D *and* ε*.*

Proof. Taking the inner product of (2.2.12) with v in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2-(\delta-\varepsilon z(\theta_t\omega))\|v\|^2+(\lambda+\delta^2)(u,v)+(Au,v)+(f(x,u),v)\\&=\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))(u,v)-(h(v+\varepsilon uz(\theta_t\omega)-\delta u),v)+(g(x,t),v).\end{aligned} \tag{2.3.3}$$

By the first equation of (2.2.11), we have

$$v = u_t - \varepsilon uz(\theta_t\omega)+\delta u. \tag{2.3.4}$$

By (2.2.9) and Lagrange's mean value theorem, we have

$$
\begin{aligned}
& -(h(v+\varepsilon uz(\theta_t\omega)-\delta u),v)\\
=& -(h(v+\varepsilon uz(\theta_t\omega)-\delta u)-h(0),v)\\
=& -(h'(\vartheta)(v+\varepsilon uz(\theta_t\omega)-\delta u),v)\\
\le& -\beta_1\|v\|^2-(h'(\vartheta)(\varepsilon uz(\theta_t\omega)-\delta u),v)\\
\le& -\beta_1\|v\|^2+\beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\|+h'(\vartheta)\delta(u,v),
\end{aligned}
\tag{2.3.5}
$$

where ϑ is between 0 and $v+\varepsilon uz(\theta_t\omega)-\delta u$.
By (2.2.9) and (2.3.4), we get

$$
\begin{aligned}
& h'(\vartheta)\delta(u,v)\\
=& h'(\vartheta)\delta(u,u_t-\varepsilon uz(\theta_t\omega)+\delta u)\\
\le& \beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2+\beta_2\delta^2\|u\|^2+\beta_2\delta|\varepsilon||z(\theta_t\omega)|\|u\|^2.
\end{aligned}
\tag{2.3.6}
$$

Substituting (2.3.4) into the third and fourth terms on the left-hand side of (2.3.3), we find that

$$
\begin{aligned}
& (u,v)\\
=& (u,u_t-\varepsilon uz(\theta_t\omega)+\delta u)\\
\ge& \frac{1}{2}\frac{d}{dt}\|u\|^2+\delta\|u\|^2-|\varepsilon||z(\theta_t\omega)|\|u\|^2,
\end{aligned}
\tag{2.3.7}
$$

and

$$
\begin{aligned}
(Au,v)=(\Delta u,\Delta v)&=(\Delta u,\Delta u_t-\varepsilon z(\theta_t\omega)\Delta u+\delta\Delta u)\\
&\ge \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2+\delta\|\Delta u\|^2-|\varepsilon||z(\theta_t\omega)|\|\Delta u\|^2.
\end{aligned}
\tag{2.3.8}
$$

For the first term on the right-hand side of (2.3.3), by (2.3.5), using the Cauchy-Schwarz inequality and Young's inequality, we have

$$
\begin{aligned}
& \varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))(u,v)+\beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\|\\
=& (3\delta\varepsilon z(\theta_t\omega)-\varepsilon^2z^2(\theta_t\omega))(u,v)+\beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\|\\
\le& (3\delta|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\|u\|\|v\|+\beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\|\\
=& ((3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\|u\|\|v\|\\
\le& (\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2),
\end{aligned}
\tag{2.3.9}
$$

and for the last term on the right-hand side of (2.3.3),

$$
(g,v)\le\|g\|\|v\|\le\frac{\|g\|^2}{2(\beta_1-\delta)}+\frac{\beta_1-\delta}{2}\|v\|^2.
\tag{2.3.10}
$$

Let $\widetilde{F}(x,u)=\int_{\mathbb{R}^n}F(x,u)dx$. Then for the last term on the left-hand side of (2.3.3) we have

$$
(f(x,u),v)=(f(x,u),u_t-\varepsilon z(\theta_t\omega)u+\delta u)
$$

$$= \frac{d}{dt}\widetilde{F}(x,u) + \delta(f(x,u),u) - \varepsilon z(\theta_t\omega)(f(x,u),u). \tag{2.3.11}$$

By condition (2.2.4) and (2.2.6), we have

$$\begin{aligned}
&\varepsilon z(\theta_t\omega)(f(x,u),u)\\
\le& c_1|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}|u|^{\gamma+1}dx + |\varepsilon||z(\theta_t\omega)|\|\phi_1\|^2 + |\varepsilon||z(\theta_t\omega)|\|u\|^2\\
\le& c_1c_3^{-1}|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}(F(x,u)+\phi_3)dx + |\varepsilon||z(\theta_t\omega)|\|\phi_1\|^2 + |\varepsilon||z(\theta_t\omega)|\|u\|^2\\
\le& c_1c_3^{-1}|\varepsilon||z(\theta_t\omega)|\widetilde{F}(x,u) + c|\varepsilon||z(\theta_t\omega)| + |\varepsilon||z(\theta_t\omega)|\|u\|^2.
\end{aligned} \tag{2.3.12}$$

Substitute (2.3.5)–(2.3.12) into (2.3.3) and together with (2.2.5) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + 2\widetilde{F}(x,u))\\
&+\delta(\|v\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u\|^2 + \|\Delta u\|^2) + \delta c_2\widetilde{F}(x,u)\\
\le& c + \Big(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\Big)(\|u\|^2+\|v\|^2)\\
&+|\varepsilon||z(\theta_t\omega)|(\|v\|^2 + (\lambda+\delta^2+\beta_2\delta)\|u\|^2 + \|\Delta u\|^2) + |\varepsilon||z(\theta_t\omega)|\|u\|^2\\
&+\frac{3\delta-\beta_1}{2}\|v\|^2 + \frac{\|g\|^2}{2(\beta_1-\delta)} + c_1c_3^{-1}|\varepsilon||z(\theta_t\omega)|\widetilde{F}(x,u) + c|\varepsilon||z(\theta_t\omega)|\\
\le&\Big(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\Big)(\|u\|^2+\|v\|^2)\\
&+\gamma_1|\varepsilon||z(\theta_t\omega)|(\|v\|^2 + (\lambda+\delta^2+\beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + 2\widetilde{F}(x,u))\\
&+|\varepsilon||z(\theta_t\omega)|\|u\|^2 + c(1+\|g\|^2+|\varepsilon||z(\theta_t\omega)|),
\end{aligned} \tag{2.3.13}$$

where $\gamma_1 = \max\{1, \frac{c_1c_3^{-1}}{2}\}$.
Let $\sigma = \frac{1}{2}\min\{\delta, \delta c_2\}$, then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + 2\widetilde{F}(x,u))\\
\le& -[\sigma - \gamma_1|\varepsilon||z(\theta_t\omega)| - \gamma_2(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2 + \gamma_3|\varepsilon||z(\theta_t\omega)|)]\\
&\cdot(\|v\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + 2\widetilde{F}(x,u))\\
&+c(1+\|g\|^2+|\varepsilon||z(\theta_t\omega)|),
\end{aligned} \tag{2.3.14}$$

where $\gamma_2 = 1 + \frac{1}{\lambda+\delta^2-\beta_2\delta}$, $\gamma_3 = \frac{3}{2}\delta + \frac{1}{2}\beta_2 + 2\beta_2\delta + 1$.
Let us denote

$$\varrho(\tau,\omega) = \sigma - \gamma_1|\varepsilon||z(\theta_t\omega)| - \gamma_2(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2 + \gamma_3|\varepsilon||z(\theta_t\omega)|). \tag{2.3.15}$$

Using the Gronwall inequality to integrate (2.3.14) over $(\tau-t,\tau)$ with $t\ge 0$, we get

$$\|v(\tau,\tau-t,\omega,v_0)\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\omega,u_0)\|^2$$

$$+\|\Delta u(\tau,\tau-t,\omega,u_0)\|^2+2\widetilde{F}(x,u(\tau,\tau-t,\omega,u_0))$$
$$\le(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+2\widetilde{F}(x,u_0))e^{2\int_\tau^{\tau-t}\varrho(s,\omega)ds}$$
$$+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(r,\omega)dr}(1+\|g(\cdot,s)\|^2+|\varepsilon||z(\theta_s\omega)|)ds. \tag{2.3.16}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above we obtain, for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R},$ and $\omega\in\Omega$,

$$\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2$$
$$+\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+2\widetilde{F}(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))$$
$$\le(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+2\widetilde{F}(x,u_0))e^{2\int_\tau^{\tau-t}\varrho(s-\tau,\omega)ds}$$
$$+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(r-\tau,\omega)dr}(1+\|g(\cdot,s)\|^2+|\varepsilon||z(\theta_{s-\tau}\omega)|)ds, \tag{2.3.17}$$

then

$$\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2$$
$$+\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+2\widetilde{F}(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))$$
$$\le(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+2\widetilde{F}(x,u_0))e^{2\int_0^{-t}\varrho(s,\omega)ds}$$
$$+c\int_{-t}^{0}e^{2\int_0^s\varrho(r,\omega)dr}(1+\|g(\cdot,s+\tau)\|^2+|\varepsilon||z(\theta_s\omega)|)ds. \tag{2.3.18}$$

Since $|z(\theta_t\omega)|$ is stationary and ergodic, from (2.2.10) and the ergodic theorem we can get

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_r\omega)|dr=\mathbf{E}(|z(\theta_r\omega)|)=\frac{1}{\sqrt{\pi\delta}}, \tag{2.3.19}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_r\omega)|^2dr=\mathbf{E}(|z(\theta_r\omega)|^2)=\frac{1}{2\delta}. \tag{2.3.20}$$

By (2.3.19)–(2.3.20), there exists $T_1(\omega)>0$ such that for all $t\ge T_1(\omega)$,

$$\int_{-t}^{0}|z(\theta_r\omega)|dr<\frac{2}{\sqrt{\pi\delta}}\,t,\quad \int_{-t}^{0}|z(\theta_r\omega)|^2dr<\frac{1}{\delta}\,t. \tag{2.3.21}$$

Next we show that for any $s\le -T_1$

$$e^{2\int_0^s\varrho(r,\omega)dr}\le e^{\sigma s}. \tag{2.3.22}$$

By using the two inequalities in (2.3.21), we have

$$\int_0^s\Big[\sigma-\gamma_1|\varepsilon||z(\theta_r\omega)|-\gamma_2(\frac{1}{2}\varepsilon^2|z(\theta_r\omega)|^2+\gamma_3|\varepsilon||z(\theta_r\omega)|)\Big]dr$$
$$>\sigma s-|\varepsilon|\frac{2\gamma_1}{\sqrt{\pi\delta}}s-\gamma_2[\frac{1}{2}\varepsilon^2\frac{1}{\delta}+\gamma_3|\varepsilon|\frac{2}{\sqrt{\pi\delta}}]s$$
$$=-\frac{\gamma_2}{2\delta}\varepsilon^2s-\frac{2}{\sqrt{\pi\delta}}[\gamma_3\gamma_2+\gamma_1]|\varepsilon|s+\sigma s. \tag{2.3.23}$$

In order to have the inequality in (2.3.22) valid, we need

$$\int_0^s \Big[\sigma - \gamma_1|\varepsilon||z(\theta_r\omega)| - \gamma_2(\frac{1}{2}\varepsilon^2|z(\theta_r\omega)|^2 + \gamma_3|\varepsilon||z(\theta_r\omega)|)\Big]dr \le \frac{\sigma}{2}s.$$

Since $s \le -T_1$, then it requires that

$$\frac{\gamma_2}{2\delta}\varepsilon^2 + \frac{2}{\sqrt{\pi\delta}}[\gamma_3\gamma_2 + \gamma_1]|\varepsilon| - \frac{\sigma}{2} < 0.$$

Solving this quadratic inequality, ε needs to satisfy (2.2.19) as we have assumed in Assumption II.

Since $|z(\theta_t\omega)|$ is tempered, by (2.2.20) and (2.3.22), we see that the following integral is convergent,

$$R_2^2(\tau,\omega) = c\int_{-\infty}^0 e^{2\int_0^s \varrho(r,\omega)dr}(1 + \|g(\cdot, s+\tau)\|^2 + |\varepsilon||z(\theta_s\omega)|)ds. \tag{2.3.24}$$

Note that (2.2.8) implies

$$\int_{\mathbb{R}^n} F(x,u_0)dx \le c(1 + \|u_0\|^2 + \|u_0\|_{H^2}^{\gamma+1}). \tag{2.3.25}$$

Since $D \in \mathcal{D}$ and $(u_0, v_0) \in D(\tau - t, \theta_{-t}\omega)$, for all $t \ge T_1$, we get from (2.3.24) and (2.3.25) that

$$\begin{aligned}
&(\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + 2\widetilde{F}(x,u_0))e^{2\int_0^{-t}\varrho(s,\omega)ds}\\
\le& ce^{-\sigma t}(1 + \|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^2}^{\gamma+1})\\
\le& ce^{-\sigma t}(1 + \|D(\tau - t, \theta_{-t}\omega)\|^2 + \|D(\tau - t, \theta_{-t}\omega)\|^{\gamma+1}) \to 0, \text{ as } t \to +\infty.
\end{aligned} \tag{2.3.26}$$

From (2.3.1), (2.3.18), (2.3.24) and (2.3.26), there exists $T_2 = T_2(\tau, \omega, D) \ge T_1$ such for all that $t \ge T_2$,

$$\|Y(\tau, \tau - t, \theta_{-\tau}\omega, Y_0(\theta_{-\tau}\omega))\|_E^2 \le c(1 + R_2^2(\tau,\omega)),$$

thus the proof is completed. □

The following lemmas will be used to show the uniform estimates of solutions as well as to establish pullback asymptotic compactness.

Lemma 2.3.2 *Under Assumptions* I *and* II, *for every* $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, D) > 0$ *such that for all* $t \ge T$, $s \in [-t, 0]$, *the solution of problem* (2.2.11)–(2.2.13) *satisfies*

$$\|Y(\tau + s, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega))\|_E^2 \le M + R_3(\tau,\omega)e^{2\int_s^0 \varrho(r,\omega)dr},$$

where $(u_0, v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$, *M is a positive constant independent of* τ, ω, D *and* ε, *and* $R_3(\tau,\omega)$ *is a specific random variable.*

Proof. Similar to (2.3.18), integrating (2.3.14) over $(\tau - t, \tau + s)$ with $t \ge 0$ and $s \in [-t, 0]$, we can obtain

$$\begin{aligned}
&\|v(\tau + s, \tau - t, \omega, v_0)\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u(\tau + s, \tau - t, \omega, u_0)\|^2\\
&+ \|\Delta u(\tau + s, \tau - t, \omega, u_0)\|^2 + 2\widetilde{F}(x, u(\tau + s, \tau - t, \omega, u_0))\\
\le& (\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + 2\widetilde{F}(x,u_0))e^{2\int_{\tau+s}^{\tau-t}\varrho(r-t,\omega)dr}
\end{aligned}$$

$$
\begin{aligned}
&+c\int_{\tau-t}^{\tau+s} e^{2\int_{\tau+s}^{\zeta}\varrho(r-\tau,\omega)dr}(1+\|g(\cdot,\zeta)\|^2+|\varepsilon||z(\theta_{\zeta-\tau}\omega)|)d\zeta\\
\le&(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+2\widetilde{F}(x,u_0))e^{2\int_s^{-t}\varrho(r,\omega)dr}\\
&+c\int_{-t}^{s} e^{2\int_s^{\zeta}\varrho(r,\omega)dr}(1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta.
\end{aligned}\tag{2.3.27}
$$

Moreover we have the following estimates for the last integral term on the right-hand side of (2.3.27):

$$
\begin{aligned}
&c\int_{-t}^{s} e^{2\int_s^{\zeta}\varrho(r-t,\omega)dr}(1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta\\
=&c\Big[\int_{-t}^{-T_1} e^{2\int_s^{\zeta}\varrho(r,\omega)dr}+\int_{-T_1}^{s} e^{2\int_s^{\zeta}\varrho(r,\omega)dr}\Big](1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta\\
\le&ce^{2\int_s^{0}\varrho(r,\omega)dr}\int_{-t}^{-T_1} e^{2\int_0^{\zeta}\varrho(r,\omega)dr}(1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta\\
&+ce^{2\int_s^{0}\varrho(r,\omega)dr}\int_{-T_1}^{0} e^{2\int_0^{\zeta}\varrho(r,\omega)dr}(1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta\\
\le&ce^{2\int_s^{0}\varrho(r,\omega)dr}\int_{-t}^{-T_1} e^{\sigma\zeta}(1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta\\
&+ce^{2\int_s^{0}\varrho(r,\omega)dr}\int_{-T_1}^{0} e^{2\int_0^{\zeta}\varrho(r,\omega)dr}(1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta\\
\le&e^{2\int_s^{0}\varrho(r,\omega)dr}R_4(\tau,\omega),
\end{aligned}\tag{2.3.28}
$$

where

$$
\begin{aligned}
R_4(\tau,\omega)=&c\int_{-\infty}^{0} e^{\sigma\zeta}(1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta\\
&+c\int_{-T_1}^{0} e^{2\int_0^{\zeta}\varrho(r,\omega)dr}(1+\|g(\cdot,\zeta+\tau)\|^2+|\varepsilon||z(\theta_{\zeta}\omega)|)d\zeta.
\end{aligned}
$$

Note that $R_4(\tau,\omega)$ is well defined by (2.2.20) and that $z(\theta_t\omega)$ is tempered. On the other hand, as in (2.3.26), we find that there exists $T_3 = T_3(\tau,\omega,D) \ge T_1$ such that for all $t \ge T_3$,

$$
\begin{aligned}
&(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+2\widetilde{F}(x,u_0))e^{2\int_s^{-t}\varrho(r,\omega)dr}\\
\le&ce^{2\int_s^{0}\varrho(r,\omega)dr}e^{2\int_0^{-t}\varrho(r,\omega)dr}(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+2\widetilde{F}(x,u_0))\\
\le&e^{2\int_s^{0}\varrho(r,\omega)dr}R_4(\tau,\omega).
\end{aligned}\tag{2.3.29}
$$

It follows from (2.3.27)–(2.3.29) and (2.3.25) that, for all $t \ge T_3$, $s \in [-t, 0]$, and ε satisfying (2.2.16),

$$
\begin{aligned}
&\|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\Delta u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\le 2e^{2\int_s^{0}\varrho(r,\omega)dr}R_4(\tau,\omega).
\end{aligned}\tag{2.3.30}
$$

The proof is completed. □

Lemma 2.3.3 *Under Assumptions* I *and* II, *for every* $\tau\in\mathbb{R}$, $\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T=T(\tau,\omega,D)>0$ *such that for all* $t\ge T$ *the solution of problem* (2.2.11)–(2.2.13) *satisfies*

$$
\|A^{\frac{1}{4}}Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\|_E^2\le R_5(\tau,\omega),
$$

and $R_5(\tau,\omega)$ is given by

$$R_5(\tau,\omega) = R_6^2(\tau,\omega) + ce^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2), \tag{2.3.31}$$

where $(u_0,v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$, c is a positive constant independent of τ,ω, D and ε, and $R_6(\tau,\omega)$ is a specific random variable.

Proof. Taking the inner product of (2.2.12) with $A^{\frac{1}{2}}v$ in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}v\|^2 - (\delta - \varepsilon z(\theta_t\omega))\|A^{\frac{1}{4}}v\|^2 + (\lambda+\delta^2)(u, A^{\frac{1}{2}}v) + (Au, A^{\frac{1}{2}}v) + (f(x,u), A^{\frac{1}{2}}v)\\ &= \varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(u, A^{\frac{1}{2}}v) - (h(v + \varepsilon u z(\theta_t\omega) - \delta u), A^{\frac{1}{2}}v) + (g(x,t), A^{\frac{1}{2}}v).\end{aligned} \tag{2.3.32}$$

Similar to the proof of Lemma 2.3.1, we have the following estimates:

$$\begin{aligned}&-\Big(h(v + \varepsilon u z(\theta_t\omega) - \delta u), A^{\frac{1}{2}}v\Big)\\ =&-\Big(h(v + \varepsilon u z(\theta_t\omega) - \delta u) - h(0), A^{\frac{1}{2}}v\Big)\\ =&-\Big(h'(\vartheta)(v + \varepsilon u z(\theta_t\omega) - \delta u), A^{\frac{1}{2}}v\Big)\\ \le&-\beta_1\|A^{\frac{1}{4}}v\|^2 - \Big(h'(\vartheta)(\varepsilon u z(\theta_t\omega) - \delta u), A^{\frac{1}{2}}v\Big)\\ \le&-\beta_1\|A^{\frac{1}{4}}v\|^2 + \beta_2|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\| + h'(\vartheta)\delta(u, A^{\frac{1}{2}}v),\end{aligned} \tag{2.3.33}$$

$$\begin{aligned}&h'(\vartheta)\delta(u, A^{\frac{1}{2}}v)\\ =&h'(\vartheta)\delta(u, A^{\frac{1}{2}}u_t - \varepsilon z(\theta_t\omega)A^{\frac{1}{2}}u) + \delta A^{\frac{1}{2}}u)\\ \le&\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}u\|^2 + \beta_2\delta^2\|A^{\frac{1}{4}}u\|^2 + \beta_2\delta|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|^2,\end{aligned} \tag{2.3.34}$$

$$\begin{aligned}&(u, A^{\frac{1}{2}}v)\\ =&(u, A^{\frac{1}{2}}u_t - \varepsilon z(\theta_t\omega)A^{\frac{1}{2}}u + \delta A^{\frac{1}{2}}u)\\ \ge&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}u\|^2 + \delta\|A^{\frac{1}{4}}u\|^2 - |\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|^2,\end{aligned} \tag{2.3.35}$$

$$\begin{aligned}&(Au, A^{\frac{1}{2}}v)\\ =&(Au, A^{\frac{1}{2}}u_t - \varepsilon z(\theta_t\omega)A^{\frac{1}{2}}u + \delta A^{\frac{1}{2}}u)\\ \ge&\frac{1}{2}\frac{d}{dt}\|A^{\frac{3}{4}}u\|^2 + \delta\|A^{\frac{3}{4}}u\|^2 - |\varepsilon||z(\theta_t\omega)|\|A^{\frac{3}{4}}u\|^2,\end{aligned} \tag{2.3.36}$$

$$\begin{aligned}&\varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(u, A^{\frac{1}{2}}v) + \beta_2|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\ =&(3\delta\varepsilon z(\theta_t\omega) - \varepsilon^2 z^2(\theta_t\omega))(u, A^{\frac{1}{2}}v) + \beta_2|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\ \le&(3\delta|\varepsilon||z(\theta_t\omega)| + \varepsilon^2|z(\theta_t\omega)|^2)\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\| + \beta_2|\varepsilon||z(\theta_t\omega)|\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\ =&\Big((3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)| + \varepsilon^2|z(\theta_t\omega)|^2\Big)\|u\|\|v\|\\ \le&\Big(\frac{1}{2}(3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\Big)(\|A^{\frac{1}{4}}u\|^2 + \|A^{\frac{1}{4}}v\|^2),\end{aligned} \tag{2.3.37}$$

$$(g, A^{\frac{1}{2}}v) \le \|g\|_1\|A^{\frac{1}{4}}v\| \le \frac{\|g\|_1^2}{2(\beta_1-\delta)} + \frac{\beta_1-\delta}{2}\|A^{\frac{1}{4}}v\|^2. \tag{2.3.38}$$

For the last term on the left-hand side of (2.3.32), by (2.2.7), we have

$$\begin{aligned}
&-(f(x,u), A^{\frac{1}{2}}v)\\
=&-\int_{\mathbb{R}^n}\frac{\partial}{\partial x}f(x,u)\cdot A^{\frac{1}{4}}vdx-\int_{\mathbb{R}^n}\frac{\partial}{\partial u}f(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}vdx\\
\le&\int_{\mathbb{R}^n}|\frac{\partial}{\partial x}f(x,u)|\cdot|A^{\frac{1}{4}}v|dx+\beta\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}v|dx\\
\le&\int_{\mathbb{R}^n}|\eta_4|\cdot|A^{\frac{1}{4}}v|dx+\beta\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}v|dx\\
\le&\|\eta_4\|\|A^{\frac{1}{4}}v\|+\beta\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\
\le&c_{12}+\Big(\delta+\frac{\beta^2}{2\delta(\lambda+\delta^2-\beta_2\delta)}\Big)\|A^{\frac{1}{4}}v\|^2+\frac{1}{2}\delta(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2.
\end{aligned} \tag{2.3.39}$$

Substitute (2.3.33)–(2.3.39) into (2.3.32) and together with (2.2.18) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2)\\
&+\sigma(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2)\\
\le&\Big(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\Big)(\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{1}{4}}v\|^2)\\
&+|\varepsilon||z(\theta_t\omega)|(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2+\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2)+\frac{\|g\|_1^2}{2(\beta_1-\delta)}.
\end{aligned} \tag{2.3.40}$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2)\\
\le&-[\sigma-|\varepsilon||z(\theta_t\omega)|-\gamma_2(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+\gamma_4|\varepsilon||z(\theta_t\omega)|)]\\
&\cdot(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2)+\frac{\|g\|_1^2}{2(\beta_1-\delta)},
\end{aligned} \tag{2.3.41}$$

where $\gamma_4=\frac{3}{2}\delta+\frac{1}{2}\beta_2+2\beta_2\delta$.
Let us denote

$$\varrho_1(\tau,\omega)=\sigma-|\varepsilon||z(\theta_t\omega)|-\gamma_2(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+\gamma_4|\varepsilon||z(\theta_t\omega)|). \tag{2.3.42}$$

Using the Gronwall inequality to integrate (2.3.42) over $(\tau-t,\tau)$ with $t\ge 0$, we get

$$\begin{aligned}
&\|A^{\frac{1}{4}}v(\tau,\tau-t,\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u(\tau,\tau-t,\omega,u_0)\|^2+\|A^{\frac{3}{4}}u(\tau,\tau-t,\omega,u_0)\|^2\\
\le&(\|A^{\frac{1}{4}}v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u_0\|^2+\|A^{\frac{3}{4}}u_0\|^2)e^{2\int_\tau^{\tau-t}\varrho_1(s,\omega)ds}\\
&+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho_1(r,\omega)dr}\|g(\cdot,s)\|_1^2ds.
\end{aligned} \tag{2.3.43}$$

Replacing ω by $\theta_{-\tau}\omega$ in (2.3.43), for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R},$ and $\omega\in\Omega$,

$$\begin{aligned}&\|A^{\frac14}v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|A^{\frac34}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\ \le&(\|A^{\frac14}v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u_0\|^2+\|A^{\frac34}u_0\|^2)e^{2\int_\tau^{\tau-t}\varrho_1(s-\tau,\omega)ds}\\ &+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho_1(r-\tau,\omega)dr}\|g(\cdot,s)\|_1^2ds,\end{aligned}\tag{2.3.44}$$

then

$$\begin{aligned}&\|A^{\frac14}v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\ &+\|A^{\frac34}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\ \le&(\|A^{\frac14}v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u_0\|^2+\|A^{\frac34}u_0\|^2)e^{2\int_0^{-t}\varrho_1(s,\omega)ds}\\ &+c_{13}\int_{-t}^{0}e^{2\int_0^s\varrho_1(r,\omega)dr}\|g(\cdot,s+\tau)\|_1^2ds.\end{aligned}\tag{2.3.45}$$

Next we show that for any $s\le -T_1$

$$e^{2\int_0^s\varrho_1(r,\omega)dr}\le e^{\sigma s}.\tag{2.3.46}$$

In fact, using the two inequalities in (2.3.21), we have

$$\begin{aligned}&\int_0^s\Big[\sigma-|\varepsilon||z(\theta_r\omega)|-\gamma_2(\frac12\varepsilon^2|z(\theta_r\omega)|^2+\gamma_4|\varepsilon||z(\theta_r\omega)|)\Big]dr\\ >&\sigma s-|\varepsilon|\frac{2}{\sqrt{\pi\delta}}s-\gamma_2[\frac12\varepsilon^2\frac1\delta+\gamma_4|\varepsilon|\frac{2}{\sqrt{\pi\delta}}]s\\ =&-\frac{\gamma_2}{2\delta}\varepsilon^2s-\frac{2}{\sqrt{\pi\delta}}[\gamma_4\gamma_2+1]|\varepsilon|s+\delta s.\end{aligned}$$

In order to have the inequality in (2.3.46) valid, we need

$$\int_0^s\Big[\sigma-|\varepsilon||z(\theta_r\omega)|-\gamma_2(\frac12\varepsilon^2|z(\theta_r\omega)|^2+\gamma_4|\varepsilon||z(\theta_r\omega)|)\Big]dr\le\frac{\sigma}{2}s.$$

Since $s\le -T_1$, then it requires that

$$\frac{\gamma_2}{2\delta}\varepsilon^2+\frac{2}{\sqrt{\pi\delta}}[\gamma_4\gamma_2+1]|\varepsilon|-\frac{\sigma}{2}<0.$$

Solving this quadratic inequality, ε needs to satisfy (2.2.19).

By (2.2.20) and (2.3.46), we see that the following integral is convergent,

$$R_6^2(\tau,\omega)=c\int_{-\infty}^{0}e^{2\int_0^s\varrho_1(r,\omega)dr}\|g(\cdot,s+\tau)\|_1^2ds.\tag{2.3.47}$$

For all $t\ge T_1$, we get from (2.3.46) that

$$\begin{aligned}&(\|A^{\frac14}v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u_0\|^2+\|A^{\frac34}u_0\|^2)e^{2\int_0^{-t}\Gamma_1(s,\omega)ds}\\ \le&ce^{-\sigma t}(\|A^{\frac14}v_0\|^2+\|A^{\frac14}u_0\|^2+\|A^{\frac34}u_0\|^2).\end{aligned}\tag{2.3.48}$$

From (2.3.1), (2.3.45), (2.3.47) and (2.3.48), there exists $T_4 = T_4(\tau, \omega, D) \geq T_1$ such for all that $t \geq T_4$,

$$\|A^{\frac{1}{4}}Y(\tau, \tau - t, \theta_{-\tau}\omega, Y_0(\theta_{-\tau}\omega))\|_E^2 \leq R_6^2(\tau, \omega) + ce^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2). \tag{2.3.49}$$

Thus the proof is completed. □

Next we conduct uniform estimates on the tail parts of the solutions for large space variables when time is sufficiently large in order to prove the pullback asymptotic compactness of the cocycle associated with Eqs.(2.2.11)–(2.2.13) on the unbounded domain $\mathbb{R}^n$.

Lemma 2.3.4 *Under Assumptions* I *and* II*, for every* $\eta > 0, \tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, there exists* $T = T(\tau, \omega, D, \eta) > 0, K = K(\tau, \omega, \eta) \geq 1$ *such that for all* $t \geq T$, $k \geq K$*, the solution of problem (2.2.11)–(2.2.13) satisfies*

$$\|Y(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega))\|_{E(\mathbb{R}^n \setminus \mathbb{B}_k)}^2 \leq \eta, \tag{2.3.50}$$

where for $k \geq 1$, $\mathbb{B}_k = \{x \in \mathbb{R}^n : |x| \leq k\}$ *and* $\mathbb{R}^n \setminus \mathbb{B}_k$ *is the complement of* $\mathbb{B}_k$.

Proof. Choose a smooth function ρ, such that $0 \leq \rho \leq 1$ for $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & \text{if } 0 \leq |s| \leq 1, \\ 1, & \text{if } |s| \geq 2, \end{cases} \tag{2.3.51}$$

and there exist constants $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ such that $|\rho'(s)| \leq \mu_1,\ |\rho''(s)| \leq \mu_2,\ |\rho'''(s)| \leq \mu_3,\ |\rho''''(s)| \leq \mu_4$ for $s \in \mathbb{R}$. Taking the inner product of (2.2.10) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx - (\delta - \varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx \\
&\quad + (\lambda + \delta^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx + \int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)vdx \\
=&\varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx \\
&\quad - \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v + \varepsilon uz(\theta_t\omega) - \delta u)vdx + \int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g(x,t)vdx.
\end{aligned} \tag{2.3.52}$$

First, by (2.2.9), similar to (2.3.5), we have

$$\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v + \varepsilon uz(\theta_t\omega) - \delta u)vdx \\
=&-\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v + \varepsilon uz(\theta_t\omega) - \delta u) - h(0))vdx \\
\leq&-\beta_1\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx + h'(\vartheta)\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx \\
&+\beta_2|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u||v|dx.
\end{aligned} \tag{2.3.53}$$

Taking (2.3.53) into (2.3.52), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx - (\delta - \varepsilon z(\theta_t\omega) - \beta_1)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx$$

$$
\begin{aligned}
&+(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx+\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)vdx\\
\leq&\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g(x,t)vdx\\
&+\beta_2|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u||v|dx. \qquad (2.3.54)
\end{aligned}
$$

For the third term on the left-hand side of (2.3.54), we have

$$
\begin{aligned}
&(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx\\
=&(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})u(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega))dx\\
=&(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Big(\frac{1}{2}\frac{d}{dt}u^2+(\delta-\varepsilon z(\theta_t\omega))u^2\Big)dx\\
\geq&(\lambda+\delta^2-\beta_2\delta)\Big(\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx\Big)\\
&-(\lambda+\delta^2+\beta_2\delta)|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx. \qquad (2.3.55)
\end{aligned}
$$

For the fourth term on the left-hand side of (2.3.54), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx\\
=&\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega))dx\\
=&\int_{\mathbb{R}^n}(\Delta^2u)\rho(\frac{|x|^2}{k^2})(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u)dx\\
=&\int_{\mathbb{R}^n}(\Delta u)\Delta\Big(\rho(\frac{|x|^2}{k^2})(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u)\Big)dx\\
=&\int_{\mathbb{R}^n}(\Delta u)\Big((\frac{2}{k^2}\rho'(\frac{|x|^2}{k^2})+\frac{4x^2}{k^4}\rho''(\frac{|x|^2}{k^2}))(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u)\\
&+2\cdot\frac{2|x|}{k^2}\rho'(\frac{|x|^2}{k^2})\nabla(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u)+\rho(\frac{|x|^2}{k^2})\Delta(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u)\Big)dx\\
\geq&-\int_{k<x<\sqrt{2}k}(\frac{2\mu_1}{k^2}+\frac{4\mu_2x^2}{k^4})|(\Delta u)v|dx-\int_{k<x<\sqrt{2}k}\frac{4\mu_1x}{k^2}|(\Delta u)(\nabla v)|dx\\
&+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx\\
\geq&-\int_{\mathbb{R}^n}(\frac{2\mu_1+8\mu_2}{k^2})|(\Delta u)v|dx-\int_{\mathbb{R}^n}\frac{4\sqrt{2}\mu_1}{k}|(\Delta u)(\nabla v)|dx+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx\\
&+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx
\end{aligned}
$$

$$\geq -\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)-\frac{4\sqrt{2}\mu_1}{k}\|\Delta u\|\|\nabla v\|+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx-\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$\geq -\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)-\frac{2\sqrt{2}\mu_1}{k}(\|\Delta u\|^2+\|\nabla v\|^2)+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx$$
$$-(|\varepsilon||z(\theta_t\omega)|-\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx. \quad (2.3.56)$$

For the fifth term on the left-hand side of (2.3.54), we have

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)vdx=\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)u)dx$$
$$=\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx$$
$$-\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx. \quad (2.3.57)$$

By (2.2.5), we see that

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx\geq c_2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\phi_2(x)dx, \quad (2.3.58)$$

On the other hand, by (2.2.4) and (2.2.6),

$$\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)udx$$
$$\leq c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx+c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx$$
$$+c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\phi_1|^2+|\phi_3|)dx. \quad (2.3.59)$$

Similar to (2.3.9) and (2.3.10) in Lemma 2.3.1, we get

$$\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx+\beta_2|\varepsilon||z(\theta_t\omega)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u||v|dx$$
$$\leq(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|u|^2+|v|^2)dx. \quad (2.3.60)$$
$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g(x,t)vdx\leq\frac{1}{2(\beta_1-\delta)}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,t)|^2dx+\frac{\beta_1-\delta}{2}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx. \quad (2.3.61)$$

Assemble together (2.3.54)–(2.3.61) to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v|^2+(\lambda+\delta^2-\beta_2\delta)|u|^2+|\Delta u|^2+2F(x,u))dx$$

$$+ \delta \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2)dx + \delta c_2 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})F(x,u)dx$$
$$\leq \frac{\mu_1 + 4\mu_2}{k^2}(|\Delta u|^2 + |v|^2) + \frac{2\sqrt{2}\mu_1}{k}(|\Delta u|^2 + |\nabla v|^2) + \Big(\frac{1}{2}(3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)|$$
$$+ \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\Big)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|u|^2 + |v|^2)dx + c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|g(x,t)|^2 dx$$
$$+ c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})F(x,u)dx + c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|u|^2 dx$$
$$+ c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|\phi_1|^2 + |\phi_3|)dx + c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\phi_2(x)dx$$
$$+ |\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v|^2 + (\lambda + \delta^2 + \beta_2\delta)|u|^2 + |\Delta u|^2)dx. \tag{2.3.62}$$

Since that $\phi_1 \in L^2(\mathbb{R}^n)$, $\phi_2,\ \phi_3 \in L^1(\mathbb{R}^n)$, for given $\eta > 0$, there exists $K_0 = K_0(\eta) \geq 1$ such that for all $k \geq K_0$,

$$\begin{aligned} & c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|\phi_1|^2 + |\phi_2| + |\phi_3|)dx \\ =& c\int_{|x|\geq k} \rho(\frac{|x|^2}{k^2})(|\phi_1|^2 + |\phi_2| + |\phi_3|)dx \\ \leq& c\int_{|x|\geq k} (|\phi_1|^2 + |\phi_2| + |\phi_3|)dx \\ \leq& \eta. \end{aligned} \tag{2.3.63}$$

Using the expression (2.3.15), we conclude from (2.3.62) that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2 + 2F(x,u))dx \\ \leq& -\varrho(t,\omega)\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2 + 2F(x,u))dx \\ &+ \frac{\mu_1 + 4\mu_2}{k^2}(|\Delta u|^2 + |v|^2) + \frac{2\sqrt{2}\mu_1}{k}(|\Delta u|^2 + |\nabla v|^2) \\ &+ c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|g(x,t)|^2 dx + \eta(1 + |\varepsilon||z(\theta_t\omega)|). \end{aligned} \tag{2.3.64}$$

Integrating (2.3.64) over $(\tau - t, \tau)$ for $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, we get

$$\begin{aligned} & \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\omega,v_0)|^2 + (\lambda + \delta^2 - \beta_2\delta)|u(\tau,\tau-t,\omega,u_0)|^2)dx \\ &+ \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\Big(|\Delta u(\tau,\tau-t,\omega,u_0)|^2 + 2F(x,u(\tau,\tau-t,\omega,u_0))\Big)dx \\ \leq& e^{2\int_\tau^{\tau-t}\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v_0(x)|^2 + (\lambda + \delta^2 - \beta_2\delta)|u_0(x)|^2)dx \end{aligned}$$

$$
\begin{aligned}
&+ e^{2\int_\tau^{\tau-t}\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Big(|\Delta u_0(x)|^2+2F(x,u_0(x))\Big)dx\\
&+ c\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s)|^2dsdx+\eta\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu,\omega)d\mu}(1+|\varepsilon||z(\theta_s\omega)|)ds\\
&+\frac{\mu_1+4\mu_2}{k^2}\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu,\omega)d\mu}(|\Delta u(s,\tau-t,\omega,u_0)|^2+|v(s,\tau-t,\omega,v_0)|^2)ds\\
&+\frac{2\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu,\omega)d\mu}(|\Delta u(s,\tau-t,\omega,u_0)|^2+|\nabla v(s,\tau-t,\omega,v_0)|^2)ds. \qquad (2.3.65)
\end{aligned}
$$

Replacing ω by $\theta_{-\tau}\omega$ in (2.3.65) and by (2.3.51) we obtain, for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R}$, and $\omega\in\Omega$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx\\
&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Big(|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))\Big)dx\\
\le& e^{2\int_\tau^{\tau-t}\varrho(\mu-\tau,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v_0(x)|^2+(\lambda+\delta^2-\beta_2\delta)|u_0(x)|^2)dx\\
&+ e^{2\int_\tau^{\tau-t}\varrho(\mu-\tau,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Big(|\Delta u_0(x)|^2+2F(x,u_0(x))\Big)dx\\
&+ c\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s)|^2dsdx+\eta\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(1+|\varepsilon||z(\theta_{s-\tau}\omega)|)ds\\
&+\frac{\mu_1+4\mu_2}{k^2}\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds\\
&+\frac{2\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds\\
\le& e^{2\int_0^{-t}\varrho(\mu,\omega)d\mu}\Big(\|v_0(x)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0(x)\|^2+\|\Delta u_0(x)\|^2+2\widetilde{F}(x,u_0(x))\Big)dx\\
&+ c\int_{-t}^{0} e^{2\int_0^s\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s+\tau)|^2dsdx+\eta\int_{-t}^{0} e^{2\int_0^s\varrho(\mu,\omega)d\mu}(1+|\varepsilon||z(\theta_s\omega)|)ds\\
&+\frac{\mu_1+4\mu_2}{k^2}\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds\\
&+\frac{2\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds. \qquad (2.3.66)
\end{aligned}
$$

It is similar to (2.3.26), for an arbitrarily given $\eta>0$, there exists $T=T(\tau,\omega,D,\eta)$ such that for all $t\ge T$,

$$
e^{2\int_0^{-t}\varrho(\mu,\omega)d\mu}\Big(\|v_0(x)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0(x)\|^2+\|\Delta u_0(x)\|^2+2\widetilde{F}(x,u_0(x))\Big)dx\le\eta. \qquad (2.3.67)
$$

For the second and third terms on the right-hand of (2.3.66), by Lemma 2.3.1 and Lemma 2.3.3, for all $t\ge\max\{T_2,\ T_4\}$,

$$
\frac{\mu_1+4\mu_2}{k^2}\int_{\tau-t}^{\tau} e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds
$$

$$+\frac{2\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds$$
$$\le\eta(R_1^2(\tau,\omega)+R_5^2(\tau,\omega)). \tag{2.3.68}$$

For the fourth term on the right-hand side of (2.3.66), there exists $K_1=K_1(\tau,\omega)\ge 1$ such that for all $k\ge K_1$, by (2.3.22), we get

$$\int_{-\infty}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s+\tau)|^2dsdx$$
$$\le\int_{-\infty}^{-T_1}e^{2\int_0^s\varrho(\mu,\omega)d\mu}\int_{|x|\ge k}|g(x,s+\tau)|^2dsdx$$
$$+\int_{-T_1}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}\int_{|x|\ge k}|g(x,s+\tau)|^2dsdx$$
$$\le\int_{-\infty}^{-T_1}e^{\sigma s}\int_{|x|\ge k}|g(x,s+\tau)|^2dsdx+e^{c^*}\int_{-T_1}^{0}e^{\sigma s}\int_{|x|\ge k}|g(x,s+\tau)|^2dsdx, \tag{2.3.69}$$

where $c^*>0$ is a random variable independent of $\tau\in\mathbb{R}$ and $D\in\mathcal{D}$, i.e.

$$c^*=\Big(\frac{\sigma}{2}+|\varepsilon|\max_{-T_1\le\mu\le0}|z(\theta_\mu\omega)|+\gamma_2(\frac{1}{2}\varepsilon^2\max_{-T_1\le\mu\le0}z^2(\theta_\mu\omega)+\gamma_3|\varepsilon|\max_{-T_1\le\mu\le0}|z(\theta_\mu\omega)|)\Big)T_1.$$

Therefore, by (2.2.21) there exists $K_2(\tau,\omega)\ge K_1$ such that for all $k\ge K_2$, we obtain

$$c\int_{-\infty}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|g(x,s+\tau)|^2dsdx$$
$$\le e^c\int_{-\infty}^{0}e^{\sigma s}\int_{|x|\ge k}|g(x,s+\tau)|^2dsdx\le\eta. \tag{2.3.70}$$

Let

$$R_7(\tau,\omega)=\int_{-\infty}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}(1+|\varepsilon||z(\theta_s\omega)|)ds, \tag{2.3.71}$$

by (2.3.22), we know that the integral of (2.3.71) is convergent.
Together with (2.3.66)–(2.3.70), we have

$$\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx$$
$$+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))dx \tag{2.3.72}$$
$$\le2\eta(1+R_1^2(\tau,\omega)+R_5^2(\tau,\omega)+R_7(\tau,\omega)).$$

It follows from (2.3.25) and (2.3.72) that there exists $K_3=K_3(\tau,\omega)\ge K_2$, such for all $k\ge K_3$, $t\ge\max\{T_2,\ T_4\}$,

$$\int_{|x|\ge\sqrt{2}k}\rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx$$
$$+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx$$
$$\le3\eta(1+R_1^2(\tau,\omega)+R_5^2(\tau,\omega)+R_7(\tau,\omega)),$$

which implies (2.3.50). □

We now derive uniform estimates of solutions in bounded domains. These estimates will be used to establish pullback asymptotic compactness. Let $\widehat{\rho} = 1 - \rho$ with ρ given by (2.3.51). Fix $k \geq 1$, and set

$$\begin{cases} \widehat{u}(t,\tau,\omega,\widehat{u_0}) = \widehat{\rho}(\frac{|x|^2}{k^2})u(t,\tau,\omega,u_0), \\ \widehat{v}(t,\tau,\omega,\widehat{v_0}) = \widehat{\rho}(\frac{|x|^2}{k^2})v(t,\tau,\omega,v_0). \end{cases} \tag{2.3.73}$$

By (2.2.11)–(2.2.13) we find that $\widehat{u}$ and $\widehat{v}$ satisfy the following system in $\mathbb{B}_{2k} = \{x \in \mathbb{R}^n : |x| < 2k\}$:

$$\frac{d\widehat{u}}{dt} = \widehat{v} + \varepsilon\widehat{u}z(\theta_t\omega) - \delta\widehat{u}, \tag{2.3.74}$$

$$\begin{aligned} &\frac{d\widehat{v}}{dt} - \delta\widehat{v} + (\delta^2 + \lambda + A)\widehat{u} + \widehat{\rho}(\frac{|x|^2}{k^2})f(u) \\ =&\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t) - \widehat{\rho}(\frac{|x|^2}{k^2})h(v + \varepsilon uz(\theta_t\omega) - \delta u) - \varepsilon(\widehat{v} - 3\delta\widehat{u} + \varepsilon\widehat{u}z(\theta_t\omega))z(\theta_t\omega) \\ &+ 4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\nabla u + 6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\Delta u + 4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\Delta\nabla u + u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}), \end{aligned} \tag{2.3.75}$$

with boundary conditions

$$\widehat{u} = \widehat{v} = 0 \quad \text{for} \quad |x| = 2k. \tag{2.3.76}$$

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(\mathbb{B}_{2k})$ such that $Ae_n = \lambda_n e_n$ with zero boundary condition in $\mathbb{B}_{2k}$. Given n, let $X_n = \text{span}\{e_1, \cdots, e_n\}$ and $P_n : L^2(\mathbb{B}_{2k}) \to X_n$ be the projection operator.

Lemma 2.3.5 *Under Assumptions* I *and* II*, for every $\eta > 0, \tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\eta) > 0, K = K(\tau,\omega,\eta) \geq 1$ and $N = N(\tau,\omega,\eta) \geq 1$ such that for all $t \geq T,\ k \geq K$ and $n \geq N$, the solution of problem* (2.3.74)–(2.3.76) *satisfies*

$$\|(I - P_n)\widehat{Y}(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-\tau}\omega))\|^2_{E(\mathbb{B}_{2k})} \leq \eta.$$

Proof. Let $\widehat{u}_{n,1} = P_n\widehat{u},\ \widehat{u}_{n,2} = (I - P_n)\widehat{u},\ \widehat{v}_{n,1} = P_n\widehat{v},\ \widehat{v}_{n,2} = (I - P_n)\widehat{v}$. Applying $I - P_n$ to (2.3.74), we obtain

$$\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \varepsilon z(\theta_t\omega)\widehat{u}_{n,2}. \tag{2.3.77}$$

Then applying $I - P_n$ to (2.3.75) and taking the inner product with $\widehat{v}_{n,2}$ in $L^2(\mathbb{B}_{2k})$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\widehat{v}_{n,2}\|^2 - (\delta - \varepsilon z(\theta_t\omega))\|\widehat{v}_{n,2}\|^2 + (\lambda + \delta^2 + A)(\widehat{u}_{n,2}, \widehat{v}_{n,2}) \\ &+ ((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), \widehat{v}_{n,2}) \\ =&((I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t), \widehat{v}_{n,2}) + \varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(\widehat{u}_{n,2}, \widehat{v}_{n,2}) \\ &- (I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v + \varepsilon z(\theta_t\omega) - \delta u), \widehat{v}_{n,2}) \\ &+ (4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\nabla u + 6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\Delta u + 4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\Delta\nabla u + u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}), \widehat{v}_{n,2}). \end{aligned} \tag{2.3.78}$$

Substituting $\widehat{v}_{n,2}$ in (2.3.77) into the third term on the left-hand side of (2.3.78), we have

$$
\begin{aligned}
(\widehat{u}_{n,2},\widehat{v}_{n,2}) &= (\widehat{u}_{n,2},\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-\varepsilon z(\theta_t\omega)\widehat{u}_{n,2})\\
&\geq \frac{1}{2}\frac{d}{dt}\|\widehat{u}_{n,2}\|^2+\delta\|\widehat{u}_{n,2}\|^2-|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|^2, \qquad (2.3.79)
\end{aligned}
$$

and then

$$
\begin{aligned}
(A\widehat{u}_{n,2},\widehat{v}_{n,2}) &= (\Delta\widehat{u}_{n,2},\Delta(\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-\varepsilon z(\theta_t\omega)\widehat{u}_{n,2}))\\
&\geq \frac{1}{2}\frac{d}{dt}\|\Delta\widehat{u}_{n,2}\|^2+\delta\|\Delta\widehat{u}_{n,2}\|^2-|\varepsilon||z(\theta_t\omega)|\|\Delta\widehat{u}_{n,2}\|^2. \qquad (2.3.80)
\end{aligned}
$$

For the fourth term on the left-hand side of (2.3.78), we have

$$
\begin{aligned}
&((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{v}_{n,2})\\
=&((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-\varepsilon z(\theta_t\omega)\widehat{u}_{n,2})\\
=&\frac{d}{dt}((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})-((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t,\widehat{u}_{n,2})\\
&+(\delta-\varepsilon z(\theta_t\omega))((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}). \qquad (2.3.81)
\end{aligned}
$$

For the third term on the right-hand side of (2.3.78), we have

$$
\begin{aligned}
&-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v+\varepsilon z(\theta_t\omega)-\delta u),\widehat{v}_{n,2})\\
=&-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v+\varepsilon z(\theta_t\omega)-\delta u)-h(0),\widehat{v}_{n,2})\\
\leq&-\beta_1\|\widehat{v}_{n,2}\|^2+h'(\vartheta)\delta(\widehat{u}_{n,2},\widehat{v}_{n,2})+\beta_2|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|. \qquad (2.3.82)
\end{aligned}
$$

Using the Cauchy-Schwarz inequality and Young's inequality, we get

$$
\begin{aligned}
&\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))(\widehat{u}_{n,2},\widehat{v}_{n,2})+\beta_2|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|\\
=&(3\delta\varepsilon z(\theta_t\omega)-\varepsilon^2z^2(\theta_t\omega))(\widehat{u}_{n,2},\widehat{v}_{n,2})+\beta_2|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|\\
\leq&(3\delta|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|+\beta_2|\varepsilon||z(\theta_t\omega)|\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|\\
=&((3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\|\widehat{u}_{n,2}\|\|\widehat{v}_{n,2}\|\\
\leq&(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)(\|\widehat{u}_{n,2}\|^2+\|\widehat{v}_{n,2}\|^2), \qquad (2.3.83)
\end{aligned}
$$

and

$$
\begin{aligned}
&((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t),\widehat{v}_{n,2})\\
\leq&\frac{\beta_1-\delta}{4}\|\widehat{v}_{n,2}\|^2+\frac{1}{\beta_1-\delta}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2. \qquad (2.3.84)
\end{aligned}
$$

Now, we estimate the last term in (2.3.78)

$$
\begin{aligned}
&(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\nabla u+6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta u+4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta\nabla u+u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2})\\
=&(4\nabla u\cdot(\frac{12|x|}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{8|x|^3}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2}))+6\Delta u\cdot(\frac{2}{k^2}\widehat{\rho}\,'(\frac{|x|^2}{r^2})\\
&+\frac{4x^2}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2}))+\frac{8|x|}{k^2}\Delta\nabla u\cdot\widehat{\rho}\,'(\frac{|x|^2}{k^2})+u(\frac{12}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{48x^2}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2})\\
&+\frac{16x^4}{k^8}\widehat{\rho}\,''''(\frac{|x|^2}{k^2})),\widehat{v}_{n,2})\\
\le&\frac{16\sqrt{2}(3\mu_2+4\mu_3)}{k^3}\|\nabla u\|\cdot\|\widehat{v}_{n,2}\|+\frac{12(\mu_1+4\mu_2)}{k^2}\|\Delta u\|\cdot\|\widehat{v}_{n,2}\|\\
&+\frac{8\sqrt{2}\mu_1}{k}\|A^{\frac{3}{4}}u\|\cdot\|\widehat{v}_{n,2}\|+\frac{4(3\mu_2+24\mu_3+16\mu_4)}{k^4}\|u\|\cdot\|\widehat{v}_{n,2}\|\\
\le&\frac{8(48\mu_2+64\mu_3)^2}{(\beta_1-\delta)k^6}\|\nabla u\|^2+\frac{4(12\mu_1+48\mu_2)^2}{(\beta_1-\delta)k^4}\|\Delta u\|^2+\frac{512\mu_1^2}{(\beta_1-\delta)k^2}\|A^{\frac{3}{4}}u\|^2\\
&+\frac{4(12\mu_2+96\mu_3+64\mu_4)^2}{(\beta_1-\delta)k^8}\|u\|^2+\frac{\beta_1-\delta}{4}\|\widehat{v}_{n,2}\|^2.
\end{aligned}
\tag{2.3.85}
$$

Assemble together (2.3.78)–(2.3.85) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})]\\
&+(\delta-|\varepsilon||z(\theta_t\omega)|)[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2\\
&+((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(u),\widehat{u}_{n,2})]\\
\le&(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+\|\widehat{u}_{n,2}\|^2)\\
&+\frac{2}{\beta_1-\delta}(\frac{4(48\mu_2+64\mu_3)^2}{k^6}\|\nabla u\|^2+\frac{2(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2+\frac{256\mu_1^2}{k^2}\|A^{\frac{3}{4}}u\|^2\\
&+\frac{2(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2+\frac{1}{2}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2)\\
&+\frac{3\delta-\beta_1}{2}\|\widehat{v}_{n,2}\|^2+((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t,\widehat{u}_{n,2}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{d}{dt}\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})\Big]\\
\le&2\Big(-\delta+|\varepsilon||z(\theta_t\omega)|+\gamma_2(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)\Big)\\
&\cdot\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})\Big]\\
&+\frac{4}{\beta_1-\delta}\Big(\frac{4(48\mu_2+64\mu_3)^2}{k^6}\|\nabla u\|^2+\frac{2(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{256\mu_1^2}{k^2}\|A^{\frac{3}{4}}u\|^2+\frac{2(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2\\
&+\frac{1}{2}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2\Big)+2\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f_u'(x,u)u_t,\widehat{u}_{n,2}\Big)\\
&+2\Big(\delta-|\varepsilon||z(\theta_t\omega)|-\gamma_2(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)\Big)\\
&\cdot((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})\\
\leq&-2\varrho(\tau,\omega)\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2\\
&+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})\Big]+2\Big[-\delta+\sigma-(\gamma_1+\gamma_2-1)|\varepsilon||z(\theta_t\omega)|\Big]\\
&\cdot\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2\Big]+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f_u'(x,u)u_t,\widehat{u}_{n,2})\\
&+\frac{4}{\beta_1-\delta}\Big(\frac{4(48\mu_2+64\mu_3)^2}{k^6}\|\nabla u\|^2+\frac{2(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2+\frac{256\mu_1^2}{k^2}\|A^{\frac{3}{4}}u\|^2\\
&+\frac{2(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2+\frac{1}{2}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2\Big)\\
&+4(\sigma-\frac{1}{2}\delta)\cdot\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\Big)+4\Big[-\frac{\gamma_1}{2}|\varepsilon||z(\theta_t\omega)|\\
&-\frac{\gamma_2}{2}\Big(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta+4)|\varepsilon||z(\theta_t\omega)|)\Big)\Big]\\
&\cdot\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\Big).
\end{aligned}
\tag{2.3.86}
$$

Let $\theta=\frac{n(\gamma-1)}{4(\gamma+1)}$. Since $1\leq\gamma\leq\frac{n+4}{n-4}$, we find that $0\leq\theta\leq1$. Then by (2.2.4) and interpolation inequalities, the last term on the right hand of (2.3.86) is bounded by

$$
\begin{aligned}
&4\Big[-\frac{\gamma_1}{2}|\varepsilon||z(\theta_t\omega)|-\frac{\gamma_2}{2}\Big(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2+(\frac{1}{2}(3\delta+\beta_2+4\beta_2\delta+4)|\varepsilon||z(\theta_t\omega)|)\Big)\Big]\\
&\cdot\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\Big)\\
\leq& c(1+|z(\theta_t\omega)|^2)\Big[c_1\int_{\mathbb{R}^n}\widehat{\rho}(\frac{|x|^2}{k^2})|u|^\gamma|\widehat{u}_{n,2}|dx+\int_{\mathbb{R}^n}\widehat{\rho}(\frac{|x|^2}{k^2})|\eta_1(x)||\widehat{u}_{n,2}|dx\Big]\\
\leq& c(1+|z(\theta_t\omega)|^2)(c_1\|u\|_{\gamma+1}^\gamma\|\widehat{u}_{n,2}\|_{\gamma+1}+\|\eta_1\|\|\widehat{u}_{n,2}\|)\\
\leq& c(1+|z(\theta_t\omega)|^2)(c_1\|u\|_{\gamma+1}^\gamma\|\Delta\widehat{u}_{n,2}\|^\theta\|\widehat{u}_{n,2}\|^{1-\theta}+\lambda_{n+1}^{-\frac{1}{2}}\|\eta_1\|\|\Delta\widehat{u}_{n,2}\|)\\
\leq& c(1+|z(\theta_t\omega)|^2)\Big[\lambda_{n+1}^{-\frac{1}{2}}\|\Delta\widehat{u}_{n,2}\|(c_1\lambda_{n+1}^{\frac{\theta}{2}}\|u\|_{H^2}^\gamma+\|\eta_1\|))\Big]\\
\leq&\frac{1}{6}(\delta-\sigma)\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}(1+|z(\theta_t\omega)|^{18}+\lambda_{n+1}^\theta\|u\|_{H^2(\mathbb{R}^n)}^{18}).
\end{aligned}
\tag{2.3.87}
$$

Similarly we have

$$
4(\sigma-\frac{1}{2}\delta)\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}\Big)
$$

$$\leq\frac{1}{6}(\delta-\sigma)\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}(1+\lambda_{n+1}^{\theta}\|u\|_{H^2(\mathbb{R}^n)}^{18}). \tag{2.3.88}$$

On the other hand, by (2.2.7), using H$\ddot{o}$lder inequality and Young's inequality, we obtain

$$\begin{aligned}
&2\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f_u'(x,u)u_t,\widehat{u}_{n,2}\Big)\\
\leq&\frac{1}{6}(\delta-\sigma)\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}\|u_t\|^2\\
\leq&\frac{1}{6}(\delta-\sigma)\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}(\|u\|^2+\|v\|^2+\|u\|^4+|z(\theta_t\omega)|^4).
\end{aligned} \tag{2.3.89}$$

Then by (2.3.86)–(2.3.89), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})\Big]\\
\leq&-2\varrho(\tau,\omega)[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})]\\
&+c\lambda_{n+1}^{-1}[1+\|v\|^{18}+(1+\lambda_{n+1}^{\theta})\|u\|_{H^2(\mathbb{R}^n)}^{18}+|z(\theta_t\omega)|^{18}]+\frac{c}{k^6}\|\nabla u\|^2+\frac{c}{k^4}\|\Delta u\|^2\\
&+\frac{c}{k^2}\|A^{\frac{3}{4}}u\|^2+\frac{c}{k^8}\|u\|^2+c\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2.
\end{aligned} \tag{2.3.90}$$

Note that $\lambda_n\to\infty$ when $n\to\infty$. Therefore, given $\eta>0$, by Lemma 2.3.1 and 2.3.3, we know there exist $N_1=N_1(\eta)\geq1$ and $K_4=K_4(\eta)\geq1$ such for all $n\geq N_1$ and $k\geq K_4$,

$$\begin{aligned}
&\frac{d}{dt}\Big[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})\Big]\\
\leq&-2\varrho(\tau,\omega)[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})]\\
&+\eta(1+\|v\|^{18}+\|u\|_{H^2(\mathbb{R}^n)}^{18}+|z(\theta_t\omega)|^{18})+c\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2.
\end{aligned} \tag{2.3.91}$$

Integrating (2.3.91) over $(\tau-t,\tau)$ with $t\geq0$, we get for all $n\geq N_1$ and $k\geq K_4$,

$$\begin{aligned}
&\|\widehat{v}_{n,2}(\tau,\tau-t,\omega)\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\omega)\|^2+\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\omega)\|^2\\
&+2\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\omega)\Big)\\
\leq&ce^{2\int_{\tau}^{\tau-t}\varrho(\mu,\omega)d\mu}(1+\|v_0\|^2+\|u_0\|_{H^2(\mathbb{R}^n)}^2)\\
&+\eta\int_{\tau-t}^{\tau}e^{2\int_{\tau}^{s}\varrho(\mu,\omega)d\mu}(\|u(s,\tau-t,\omega,u_0)\|_{H^2(\mathbb{R}^n)}^{18}+\|v(s,\tau-t,\omega,v_0)\|^{18})ds\\
&+\eta\int_{\tau-t}^{\tau}e^{2\int_{\tau}^{s}\varrho(\mu,\omega)d\mu}(1+|z(\theta_s\omega)|^{18})ds\\
&+c\int_{\tau-t}^{\tau}e^{2\int_{\tau}^{s}\varrho(\mu,\omega)d\mu}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s))\|^2ds.
\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above we obtain, for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R},\ \omega\in\Omega,\ n\geq N_1$ and $k\geq K_4$,

$$\|\widehat{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2$$

$$
\begin{aligned}
&+\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+2\Big((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\Big)\\
\leq& ce^{2\int_\tau^{\tau-t}\varrho(\mu-\tau,\omega)d\mu}(1+\|v_0\|^2+\|u_0\|^{\gamma+1}_{H^2(\mathbb{R}^n)})\\
&+\eta\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)}+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^{18})ds\\
&+\eta\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}(1+|z(\theta_{s-\tau}\omega)|^{18})ds\\
&+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho(\mu-\tau,\omega)d\mu}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s))\|^2ds\\
\leq& ce^{2\int_0^{-t}\varrho(\mu,\omega)d\mu}(1+\|v_0\|^2+\|u_0\|^{\gamma+1}_{H^2(\mathbb{R}^n)})\\
&+\eta\int_{-t}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}(\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)}\\
&+\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^{18})ds+\eta\int_{-t}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}(1+|z(\theta_s\omega)|^{18})ds\\
&+c\int_{-t}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s+\tau))\|^2ds.
\end{aligned}
\tag{2.3.92}
$$

We now estimate every term on the right-hand side of (2.3.92). For the first term, as in (2.3.26), we find that there exists $\widetilde{T}=\widetilde{T}(\tau,\omega,D,\eta)>0$ such for all $t\geq\widetilde{T}$,

$$
ce^{2\int_0^{-t}\varrho(\mu,\omega)d\mu}(1+\|v_0\|^2+\|u_0\|^{\gamma+1}_{H^2(\mathbb{R}^n)})\leq\eta.
\tag{2.3.93}
$$

For the second term on the right-hand side of (2.3.92), by Lemma 2.3.2 we have

$$
\begin{aligned}
&\eta\int_{-t}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}(\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)}+\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^{18})ds\\
\leq&\eta c\int_{-t}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}ds+\eta R_3^9(\tau,\omega)\int_{-t}^{0}e^{-16\int_0^s\varrho(\mu,\omega)d\mu}ds\\
\leq&\eta c\int_{-\infty}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}ds+\frac{\eta}{8\sigma}R_3^9(\tau,\omega),
\end{aligned}
\tag{2.3.94}
$$

where $R_3(\tau,\omega)$ is the random variable given in Lemma 2.3.2. Note that by (2.3.22) the above integral is well defined, and so is the following one

$$
\int_{-\infty}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}(1+|z(\theta_s\omega)|^{18})ds<\infty.
\tag{2.3.95}
$$

For the last term on the right-hand side of (2.3.92), by (2.2.20) and (2.3.22), since $g\in L^2(\mathbb{R}^n)$, there exists $N_2=N_2(\tau,\omega,\eta)\geq N_1$, such that for all $n\geq N_2$,

$$
\int_{-\infty}^{0}e^{2\int_0^s\varrho(\mu,\omega)d\mu}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s+\tau))\|^2ds<\eta.
\tag{2.3.96}
$$

According to (2.3.92)–(2.3.96) we find that, for every $\tau\in\mathbb{R},\ \omega\in\Omega,\ t\geq\widetilde{T},\ n\geq N_2$ and $k\geq K_4$,

$$\begin{aligned}&\|\widehat{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2\\&+\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+2((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega))\\\leq&\eta R_8(\tau,\omega),\end{aligned}\tag{2.3.97}$$

where $R_8(\tau,\omega)$ is a positive random variable. The proof is completed by (2.2.4) and (2.3.97). □

3. Results

In this section, we prove existence and uniqueness of $\mathcal{D}$- pullback attractors for the stochastic system (2.2.11)–(2.2.13). First we apply the Lemmas shown in Section 4 to prove the asymptotic compactness of solutions of (2.2.11)–(2.2.13) in E.

Lemma 3.1 *Under Assumptions* I *and* II*, for every $\tau\in\mathbb{R},\ \omega\in\Omega$, the sequence of weak solutions of* (2.2.11)–(2.2.13)*, $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\}_{m=1}^{\infty}$ has a convergent subsequence in E whenever $t_m\to\infty$ and $Y_0(\theta_{-t_m}\omega)\in D(\tau-t_m,\theta_{-t_m}\omega)$ with $D\in\mathcal{D}$.*

Proof. Let $t_m\to\infty$ and $Y_0(\theta_{-t_m}\omega)\in D(\tau-t_m,\theta_{-t_m}\omega)$ with $D\in\mathcal{D}$. By Lemma 2.3.1, there exists $m_1=m_1(\tau,\omega,D)>0$ such for all $m\geq m_1$, we have

$$\|Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\|_E^2\leq R_1(\tau,\omega).\tag{3.1}$$

By Lemma 2.3.4, for every $\eta>0$, there exist $k_0=k_0(\tau,\omega,\eta)\geq 1$ and $m_2=m_2(\tau,\omega,D,\eta)\geq m_1$ such for all $m\geq m_2$,

$$\|Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\|^2_{E(\mathbb{R}^n\setminus\mathbb{B}_{k_0})}\leq\eta.\tag{3.2}$$

By Lemma 2.3.5, there exist $k_1=k_1(\tau,\omega,\eta)\geq k_0$ and $m_3=m_3(\tau,\omega,D,\eta)\geq m_2$ and $n_1=n_1(\tau,\omega,\eta)\geq 0$ such for all $m\geq m_3$,

$$\|(I-P_n)\widehat{Y}(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-\tau}\omega))\|^2_{E(\mathbb{B}_{2k_1})}\leq\eta.\tag{3.3}$$

Using (2.3.73) and (3.1), we get

$$\|P_n\widehat{Y}(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-\tau}\omega))\|^2_{P_nE(\mathbb{B}_{2k_1})}\leq c_{19}R_1(\tau,\omega),\tag{3.4}$$

which together with (3.3) implies that $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\}$ is precompact in $E(\mathbb{B}_{2k_1})$. Note that $\widehat{\rho}(\frac{|x|^2}{k_1^2})=1$ for $|x|\leq k_1$. Therefore, $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\}$ is precompact in $E(\mathbb{B}_{k_1})$, which along with (5.2) shows the precompactness of this sequence in E. □

Theorem 3.1 *Under Assumptions* I *and* II*, the random dynamical system* Φ *generated by the stochastic plate Eq.* (2.2.11)–(2.2.13) *has a unique pullback $\mathcal{D}$-attractor $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\ \omega\in\Omega\}\in\mathcal{D}$ in the space E.*

Proof. Note that the cocycle Φ is pullback $\mathcal{D}$-asymptotically compact in E by Lemma 3.1. On the other hand, the cocycle Φ has a pullback $\mathcal{D}$-absorbing set by Lemma 2.3.1. Then the existence and uniqueness of a pullback $\mathcal{D}$-attractor of Φ follow from Proposition 2.1.8 immediately. □

4. Conclusion

Using the uniform estimates on the tails of solutions and the splitting technique as well as the compactness methods, we obtained the existence of pullback attractor for the problem (1.1)–(1.2). It is well-known that the pullback random attractors are employed to describe long-time behavior for an non-autonomous dynamical system with random term, while the $\mathcal{D}$-pullback attractor that we obtained can characterize the asymptotic behavior of the equation like (1.1)–(1.2), which is featured with both stochastic term and non-autonomous term.

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Conflict of interest

The author declares that there is no conflict of interest.

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