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Research article

Certain generalized fractional integral inequalities

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Abstract: The principal aim of this article is to establish certain generalized fractional integral inequalities by utilizing the Marichev-Saigo-Maeda (MSM) fractional integral operator. Some new classes of generalized fractional integral inequalities for a class of n ($n \in \mathbb{N}$) positive continuous and decreasing functions on $[a, b]$ by using the MSM fractional integral operator also derived.

Keywords: Marichev-Saigo-Maeda fractional integral operator; fractional integral inequalities
Mathematics Subject Classification: 6D10, 26A33, 26D53

1. Introduction

Fractional integral inequalities (FII in short) have made a great impact on scientists and mathematicians because of its potential applications in various fields. This subject plays a vital role in the development of differential equations and related problems in applied mathematics. In recent few decades, a variety of various integral inequalities and their generalizations have been established by utilizing fractional integral, fractional derivative operators and their generalizations are found in [4–6, 10, 14–16, 19–21, 29, 35]. Also, the applications of (k, s)-Riemann-Liouville (R-L) fractional integral is found in [30]. In the past few years, various researchers have established the generalization of some classical inequalities by using different mathematical techniques. The generalized Hermite-Hadamard type inequalities with fractional integral operators and Hermite-Hadamard type inequalities by using the generalized k-fractional integrals are given in [34] and [2] respectively.

In [1], the authors established FII for a class of n decreasing positive functions where $n \in \mathbb{N}$ by using (k, s)-fractional integral operator. Recently, the researchers [17, 18, 22–26] have established certain inequalities by employing some recent type(proportional and conformable) of fractional integrals. Without any doubt one can state that fractional and k-fractional calculus have become a very powerful tool for the modern studies, see for example [36, 37].
To move towards our main results, we recall the following definitions [9, 27, 31].

Definition 1.1. *Let $f(\tau)$, $\tau \geq 0$, real valued function, is said to be in the space $C_\mu([a, b])$, $\mu \in \mathbb{R}$ if there exist $p \in \mathbb{R}$ such that $p > \mu$ and $f(\tau) = \tau^p f_1(\tau)$ where $f_1(\tau) \in C([a, b])$.*

Definition 1.2. *Let ν, $\acute{\nu}$, $\xi, \acute{\xi} \in \mathbb{C}$ such that $R(\vartheta) > 0$ and $x \in \mathbb{R}$. Then MSM fractional integral is defined by*

$$\left(\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta} f\right)(x) = \frac{x^{-\nu}}{\Gamma(\eta)} \int_a^x (x-t)^{\eta-1} t^{-\nu'} F_3\left(\nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t) dt \tag{1.1}$$

where $F_3(.)$ represents the Appell function (or Horn function) which is given in [8] as

$$F_3(\nu, \nu', \xi, \xi'; \vartheta; x; y) = \sum_{m,n=0}^{\infty} \frac{(\nu)_m (\nu')_n (\xi)_m (\xi')_n}{(\vartheta)_{m+n}} \frac{x^m y^n}{m! n!}, \max\{|x|, |y|\} < 1,$$

and $(\nu)_m = \nu(\nu+1)\cdots(\nu+m-1)$ is the Pochhammer symbol.

The operator (1.1) is introduced in [13] and extended in [31, 32]. The use of this function in connection with special functions is appeared in many recent papers [3, 11, 12].

2. Main results

In this section, we employ the MSM fractional integral operator to establish the generalization of some classical inequalities. Recalling the following Theorem which will be used to establish our main result.

Theorem 1. *(see [28], Theorem 1) If $\nu, \nu', \xi, \xi', \eta \in \mathbb{R}$ such that $\eta > \max\{\nu, \nu', \xi, \xi'\} > 0$, then the following inequality holds*

$$F_3\left(\nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) > 0, \tag{2.1}$$

provided $-1 < (1 - \frac{t}{x}) < 0$ and $0 < (1 - \frac{x}{t}) < \frac{1}{2}$. Also, if $f(x) > 0$, then

$$\left(\Im_{0,x}^{\nu,\nu',\xi,\xi',\eta} f\right)(x) > 0.$$

Theorem 2. *Let g be a positive continuous and decreasing function on the interval $[a, b]$. Let $\nu, \nu', \xi, \xi', \eta \in \mathbb{R}$ such that $\eta > \max\{\nu, \nu', \xi, \xi'\} > 0$, $a < x \leq b$, $\vartheta > 0$ and $\sigma \geq \gamma > 0$. Then for MSM fractional integral operator* (1.1)*, we have*

$$\frac{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^\sigma(x)\right]}{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^\gamma(x)\right]} \geq \frac{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^\vartheta g^\sigma(x)\right]}{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^\vartheta g^\gamma(x)\right]}, \tag{2.2}$$

provided $-1 < (1 - \frac{t}{x}) < 0$ and $0 < (1 - \frac{x}{t}) < \frac{1}{2}$.

Proof. Since g be a positive continuous and decreasing functions on the interval $[a,b]$. Therefore, we have

$$\left((\rho-a)^{\vartheta}-(t-a)^{\vartheta}\right)(g^{\sigma-\gamma}(t)-g^{\sigma-\gamma}(\rho))\geq 0, \tag{2.3}$$

where $a<t,\rho\leq b$, $\vartheta>0$, $\sigma\geq\gamma>0$.
By (2.3), we have

$$(\rho-a)^{\vartheta}g^{\sigma-\gamma}(t)+(t-a)^{\vartheta}g^{\sigma-\gamma}(\rho)-(\rho-a)^{\vartheta}g^{\sigma-\gamma}(\rho)-(t-a)^{\vartheta}g^{\sigma-\gamma}(t)\geq 0. \tag{2.4}$$

Define a function

$$\begin{aligned}\mathfrak{F}(x,t)&=(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\\&=(x-t)^{\eta-1}t^{-\nu'}\left[1+\frac{(\nu\prime)(\xi)}{(\eta)}(1-\frac{x}{t})+\frac{(\nu)(\xi)}{(\eta)}(1-\frac{t}{x})+\cdots\right].\end{aligned} \tag{2.5}$$

In view of Theorem 1, we observe that the function $\mathfrak{F}(x,t)$ remain positive for all $t\in(a,x)$, $x>a$, since each term of the above function is positive in view of conditions stated in Theorem 2. Therefore multiplying (2.4) by

$$\mathfrak{F}(x,t)g^{\gamma}(t)=(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)g^{\gamma}(t),\ t\in(a,x),a<x\leq b,$$

we get

$$\begin{aligned}&\mathfrak{F}(x,t)\left[(\rho-a)^{\vartheta}g^{\sigma-\gamma}(t)+(t-a)^{\vartheta}g^{\sigma-\gamma}(\rho)-(\rho-a)^{\vartheta}g^{\sigma-\gamma}(\rho)-(t-a)^{\vartheta}g^{\sigma-\gamma}(t)\right]g^{\gamma}(t)\\&=(\rho-a)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)g^{\gamma}(t)g^{\sigma-\gamma}(t)\\&+(t-a)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)g^{\gamma}(t)g^{\sigma-\gamma}(\rho)\\&-(\rho-a)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)g^{\gamma}(t)g^{\sigma-\gamma}(\rho)\\&-(t-a)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)g^{\gamma}(t)g^{\sigma-\gamma}(t)\geq 0.\end{aligned} \tag{2.6}$$

Integrating (2.6) with respect to t over (a,x), we have

$$\begin{aligned}&(\rho-a)^{\vartheta}\int_a^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)g^{\sigma}(t)dt\\&+g^{\sigma-\gamma}(\rho)\int_a^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)(t-a)^{\vartheta}g^{\gamma}(t)dt\\&-(\rho-a)^{\vartheta}g^{\sigma-\gamma}(\rho)\int_a^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)g^{\gamma}(t)dt\\&-\int_a^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)(t-a)^{\vartheta}g^{\sigma}(t)dt\geq 0.\end{aligned} \tag{2.7}$$

Multiplying (2.7) by $\frac{x^{-\nu}}{\Gamma(\eta)}$, we get

$$(\rho-a)^{\vartheta}\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]+g^{\sigma-\gamma}(\rho)\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\gamma}(x)\right]$$

$$-(\rho-a)^{\vartheta}g^{\sigma-\gamma}(\rho)\left[\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}g^{\gamma}(x)\right]-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]. \tag{2.8}$$

Multiplying (2.8) by

$$\frac{x^{-\nu}}{\Gamma(\eta)}\mathfrak{F}(x,\rho)g^{\gamma}(\rho)=\frac{x^{-\nu}}{\Gamma(\eta)}(x-\rho)^{\eta-1}\rho^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)g^{\gamma}(\rho)$$

where $\mathfrak{F}(x,\rho)$ is defined by (2.5) and integrating the resultant identity with respect to ρ over (a,x), we get

$$\begin{aligned}&\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\gamma}(x)\right]\\&-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right]\geq 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\gamma}(x)\right]\\&\geq\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right].\end{aligned}$$

Dividing the above equation by $\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\gamma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right]$, we get the desired inequality (2.2). □

Remark 2.1. *The inequality in Theorem 2 will reverse if g is an increasing function on the interval* $[a,b]$.

Theorem 3. *Let g be a positive continuous and decreasing function on the interval* $[a,b]$. *Let $a<x\leq b$, $\vartheta>0$, $\sigma\geq\gamma>0$. Then for the MSM fractional integral* (1.1)*, we have*

$$\frac{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}g^{\gamma}(x)\right]+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\gamma}(x)\right]}{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\gamma}(x)\right]+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right]}\geq 1, \tag{2.9}$$

where $\alpha,\beta,\zeta,\zeta',\lambda,\nu,\ \acute{\nu},\ \xi,\xi',\ \eta\in\mathbb{R}$ such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$ and $\lambda>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof. By multiplying both sides of (2.8) by

$$\frac{x^{-\alpha}}{\Gamma(\lambda)}\mathfrak{F}(x,\rho)g^{\gamma}(\rho)=\frac{x^{-\alpha}}{\Gamma(\lambda)}(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)g^{\gamma}(\rho)$$

where $\mathfrak{F}(x,\rho)$ is defined by (2.5) and integrating the resultant identity with respect to ρ over (a,x), we have

$$\begin{aligned}&\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}g^{\gamma}(x)\right]\\&+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\gamma}(x)\right]\\&-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\gamma}(x)\right]\\&-\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right]\geq 0.\end{aligned} \tag{2.10}$$

Hence, dividing (2.10) by

$$\begin{aligned}&\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\gamma}(x)\right]\\&+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right],\end{aligned}$$

we get the required results. □

Remark 2.2. *Applying Theorem 3 for $\alpha = \nu$, $\beta = \nu'$, $\zeta = \xi$, $\zeta' = \xi'$, $\lambda = \eta$, we get Theorem 2.*

Theorem 4. *Let g and h be positive continuous functions on the interval $[a, b]$ such that h is increasing and g be decreasing functions on the interval $[a, b]$. Let $a < x \leq b$, $\vartheta > 0$, $\sigma \geq \gamma > 0$. Then for the MSM fractional integral* (1.1)*, we have*

$$\frac{\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]}{\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right]} \geq 1, \tag{2.11}$$

where ν, $\acute{\nu}$, ξ, ξ', $\eta \in \mathbb{R}$ such that $\eta > \max\{\nu, \nu', \xi, \xi'\} > 0$.

Proof. Under the conditions stated in Theorem 4, we can write

$$\left(h^{\vartheta}(\rho) - h^{\vartheta}(t)\right)\left(g^{\sigma-\gamma}(t) - g^{\sigma-\gamma}(\rho)\right) \geq 0 \tag{2.12}$$

where $a < x \leq b$, $\vartheta > 0$, $\sigma \geq \gamma > 0$.
From (2.12), we have

$$h^{\vartheta}(\rho)g^{\sigma-\gamma}(t) + h^{\vartheta}(t)g^{\sigma-\gamma}(\rho) - h^{\vartheta}(\rho)g^{\sigma-\gamma}(\rho) - h^{\vartheta}(t)g^{\sigma-\gamma}(t) \geq 0. \tag{2.13}$$

Multiplying both sides of (2.13)

$$\mathfrak{F}(x,t)g^{\gamma}(t) = (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)g^{\gamma}(t),\ t \in (a, x), a < x \leq b,$$

where $\mathfrak{F}(x, t)$ is defined by (2.5), we get

$$\begin{aligned}
&\mathfrak{F}(x,t)g^{\gamma}(t)\left[h^{\vartheta}(\rho)g^{\sigma-\gamma}(t) + h^{\vartheta}(t)g^{\sigma-\gamma}(\rho) - h^{\vartheta}(\rho)g^{\sigma-\gamma}(\rho) - h^{\vartheta}(t)g^{\sigma-\gamma}(t)\right]\\
=&h^{\vartheta}(\rho)(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)g^{\sigma}(t)\\
&+h^{\vartheta}(t)(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)g^{\sigma-\gamma}(\rho)g^{\sigma}(t)\\
&-h^{\vartheta}(\rho)(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)g^{\sigma-\gamma}(\rho)g^{\sigma}(t)\\
&-h^{\vartheta}(t)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)g^{\sigma}(t) \geq 0.
\end{aligned} \tag{2.14}$$

Integrating (2.14) with respect to t over (a, x), we have

$$\begin{aligned}
&h^{\vartheta}(\rho)\int_a^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)g^{\sigma}(t)dt\\
&+g_p^{\sigma-\gamma_p}(\rho)\int_a^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)h^{\vartheta}(t)g^{\gamma}(t)dt\\
&-h^{\vartheta}(\rho)g^{\sigma-\gamma}(\rho)\int_a^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)g^{\gamma}(t)dt\\
&-\int_a^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)h^{\vartheta}(t)g^{\sigma}(t)dt \geq 0.
\end{aligned} \tag{2.15}$$

Multiplying (2.15) by $\frac{x^{-\nu}}{\Gamma(\eta)}$, we get

$$h^{\vartheta}(\rho)\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]+g^{\sigma-\gamma}(\rho)\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]$$
$$-h^{\vartheta}(\rho)g^{\sigma-\gamma}(\rho)\left[\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}g^{\gamma}(x)\right]-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]\geq 0. \tag{2.16}$$

Again, multiplying (2.16) by

$$\frac{x^{-\nu}}{\Gamma(\eta)}\mathfrak{F}(x,\rho)g^{\gamma}(\rho)=\frac{x^{-\nu}}{\Gamma(\eta)}(x-\rho)^{\eta-1}\rho^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)g^{\gamma}(\rho)$$

and integrating the resultant identity with respect to ρ over (a,x), we get

$$\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]$$
$$-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right]\geq 0$$

which completes the desired inequality (2.11) of Theorem 4. □

Theorem 5. *Let g and h be positive continuous functions on the interval* $[a,b]$ *such that h is increasing and g be decreasing functions on the interval* $[a,b]$*. Let* $a<x\leq b$*,* $\vartheta>0$*,* $\sigma\geq\gamma>0$*. Then for the MSM fractional integral* (1.1)*, we have*

$$\frac{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]}{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\gamma}(x)\right]+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right]}\geq 1, \tag{2.17}$$

where $\alpha,\beta,\zeta,\zeta',\lambda,\nu,\ \acute{\nu},\ \xi,\xi',\ \eta\in\mathbb{R}$ *such that* $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$ *and* $\lambda>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof. Multiplying (2.16) by

$$\frac{x^{-\alpha}}{\Gamma(\lambda)}\mathfrak{F}(x,\rho)g^{\gamma}(\rho)=\frac{x^{-\alpha}}{\Gamma(\lambda)}(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)g^{\gamma}(\rho)$$

(where $\mathfrak{F}(x,\rho)$ is defined by (2.5)) and integrating the resultant identity with respect to ρ over (a,x), we get

$$\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]$$
$$+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]$$
$$-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\gamma}(x)\right]$$
$$-\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right]\geq 0.$$

It follows that

$$\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]$$
$$+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\gamma}(x)\right]$$
$$\geq\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\gamma}(x)\right]$$

$$+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right].$$

Dividing both sides by

$$\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[g^{\gamma}(x)\right]$$
$$+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)g^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[g^{\gamma}(x)\right],$$

which gives the desired inequality (2.32). □

Remark 2.3. *Applying Theorem 5 for* $\alpha = \nu$, $\beta = \nu'$, $\zeta = \xi$, $\zeta' = \xi'$, $\lambda = \eta$, *we get Theorem 4.*

Now, we use the MSM fractional integral fractional integral operator to present some inequalities for a class of n-decreasing positive functions.

Theorem 6. *Let* $(g_i)_{i=1,2,3,\cdots,n}$ *be n positive continuous and decreasing functions on the interval* $[a,b]$. *Let* $a < x \leq b$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ *for any fixed* $p \in \{1,2,3,\cdots,n\}$. *Then for MSM fractional integral operator* (1.1)*, we have*

$$\frac{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]}{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]} \geq \frac{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]}{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]}, \tag{2.18}$$

where ν, $\acute{\nu}$, ξ, ξ', $\eta \in \mathbb{R}$ *such that* $\eta > \max\{\nu,\nu',\xi,\xi'\} > 0$.

Proof. Since $(g_i)_{i=1,2,3,\cdots,n}$ be n positive continuous and decreasing functions on the interval $[a,b]$. Therefore, we have

$$\left((\rho-a)^{\vartheta}-(t-a)^{\vartheta}\right)\left(g_p^{\sigma-\gamma_p}(t)-g_p^{\sigma-\gamma_p}(\rho)\right) \geq 0 \tag{2.19}$$

where $a < x \leq b$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ and for any fixed $p \in \{1,2,3,\cdots,n\}$.
By (2.19), we have

$$(\rho-a)^{\vartheta}g_p^{\sigma-\gamma_p}(t)+(t-a)^{\vartheta}g_p^{\sigma-\gamma_p}(\rho)-(\rho-a)^{\vartheta}g_p^{\sigma-\gamma_p}(\rho)-(t-a)^{\vartheta}g_p^{\sigma-\gamma_p}(t) \geq 0. \tag{2.20}$$

Therefore multiplying both sides of (2.20)

$$\mathfrak{F}(x,t)\prod_{i=1}^{n} g_i^{\gamma_i}(t) = (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t),\ t\in(a,x),\ a<x\leq b,$$

where $\mathfrak{F}(x,t)$ is defined by (2.5), we have

$$\mathfrak{F}(x,t)\left[(\rho-a)^{\vartheta}g^{\sigma-\gamma}(t)+(t-a)^{\vartheta}g^{\sigma-\gamma}(\rho)-(\rho-a)^{\vartheta}g^{\sigma-\gamma}(\rho)-(t-a)^{\vartheta}g^{\sigma-\gamma}(t)\right]\prod_{i=1}^{n} g_i^{\gamma_i}(t)$$
$$=(\rho-a)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(t)$$
$$+(t-a)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(\rho)$$

$$-(\rho-a)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\prod_{i=1}^{n}g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(\rho)$$
$$-(t-a)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\prod_{i=1}^{n}g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(t)\geq 0. \tag{2.21}$$

Integrating (2.21) with respect to t over (a,x), we have

$$(\rho-a)^{\vartheta}\int_a^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\prod_{i=1}^{n}g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(t)dt$$
$$+g_p^{\sigma-\gamma_p}(\rho)\int_a^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)(t-a)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(t)dt$$
$$-(\rho-a)^{\vartheta}g_p^{\sigma-\gamma_p}(\rho)\int_a^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)\prod_{i=1}^{n}g_i^{\gamma_i}(t)dt$$
$$-\int_a^x(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)(t-a)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(t)dt\geq 0. \tag{2.22}$$

Multiplying (2.22) by $\frac{x^{-\nu}}{\Gamma(\eta)}$, we get

$$(\rho-a)^{\vartheta}\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]+g_p^{\sigma-\gamma_p}(\rho)\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]$$
$$-(\rho-a)^{\vartheta}g_p^{\sigma-\gamma_p}(\rho)\left[\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\geq 0. \tag{2.23}$$

Multiplying (2.23) by

$$\frac{x^{-\nu}}{\Gamma(\eta)}\mathfrak{F}(x,\rho)\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)=\frac{x^{-\nu}}{\Gamma(\eta)}(x-\rho)^{\eta-1}\rho^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)$$

(where $\mathfrak{F}(x,\rho)$ is defined by (2.5)) and integrating the resultant identity with respect to ρ over (a,x), we get

$$\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]$$
$$-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\geq 0$$

which completes the desired inequality (2.18). □

Remark 2.4. *The inequality in Theorem 6 will reverse if* $(g_i)_{i=1,2,3,\cdots,n}$ *are increasing functions on the interval* $[a,b]$.

Theorem 7. *Let $(g_i)_{i=1,2,3,\cdots,n}$ be n positive continuous and decreasing functions on the interval $[a,b]$. Let $a < x \leq b$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\cdots,n\}$. Then for MSM fractional integral* (1.1)*, we have*

$$\frac{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]+}{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]+}$$
$$\frac{\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]}{\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]} \geq 1, \tag{2.24}$$

where $\alpha,\beta,\zeta,\zeta',\lambda,\nu,\ \acute{\nu},\ \xi,\xi',\ \eta \in \mathbb{R}$ such that $\eta > \max\{\nu,\nu',\xi,\xi'\} > 0$ and $\lambda > \max\{\nu,\nu',\xi,\xi'\} > 0$.

Proof. By multiplying both sides of (2.23) by

$$\frac{x^{-\alpha}}{\Gamma(\lambda)}\mathfrak{F}(x,\rho)\prod_{i=1}^{n} g_i^{\gamma_i}(\rho) = \frac{x^{-\alpha}}{\Gamma(\lambda)}(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)$$

(where $\mathfrak{F}(x,\rho)$ is defined by (2.5)) and integrating the resultant identity with respect to ρ over (a,x), we have

$$\begin{aligned}
&\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]\\
&+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]\\
&-\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]\\
&-\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] \geq 0.
\end{aligned} \tag{2.25}$$

Hence, dividing (2.25) by

$$\begin{aligned}
&\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]\\
&+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right],
\end{aligned}$$

which completes the desired proof. □

Remark 2.5. *Applying Theorem 7 for $\alpha = \nu$, $\beta = \nu'$, $\zeta = \xi$, $\zeta' = \xi'$, $\lambda = \eta$, we get Theorem 6.*

Theorem 8. *Let $(g_i)_{i=1,2,3,\cdots,n}$ and h be positive continuous functions on the interval $[a,b]$ such that h is increasing and $(g_i)_{i=1,2,3,\cdots,n}$ be decreasing functions on the interval $[a,b]$. Let $a < x \leq b$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\cdots,n\}$. Then for the MSM fractional integral* (1.1)*, we have*

$$\frac{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]}{\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]} \geq 1, \tag{2.26}$$

where ν, $\acute{\nu}$, ξ, ξ', $\eta \in \mathbb{R}$ *such that* $\eta > \max\{\nu, \nu', \xi, \xi'\} > 0$.

Proof. Under the conditions stated in Theorem 8, we can write

$$\left(h^{\vartheta}(\rho) - h^{\vartheta}(t)\right)\left(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\right) \geq 0 \tag{2.27}$$

where $a < x \leq b$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ and for any fixed $p \in \{1, 2, 3, \cdots, n\}$.
From (2.27), we have

$$h^{\vartheta}(\rho)g_p^{\sigma-\gamma_p}(t) + h^{\vartheta}(t)g_p^{\sigma-\gamma_p}(\rho) - h^{\vartheta}(\rho)g_p^{\sigma-\gamma_p}(\rho) - h^{\vartheta}(t)g_p^{\sigma-\gamma_p}(t) \geq 0. \tag{2.28}$$

Multiplying both sides of (2.28)

$$\mathfrak{F}(x,t)\prod_{i=1}^{n} g_i^{\gamma_i}(t) = (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)$$

(where $\mathfrak{F}(x,\rho)$ is defined by (2.5)), we get

$$\begin{aligned}
&h^{\vartheta}(\rho)(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(t)\\
&+h^{\vartheta}(t)(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(\rho)\\
&-h^{\vartheta}(\rho)(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(\rho)\\
&-h^{\vartheta}(t)^{\vartheta}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(t) \geq 0.
\end{aligned} \tag{2.29}$$

Integrating (2.29) with respect to t over (a, x), we have

$$\begin{aligned}
&h^{\vartheta}(\rho)\int_a^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(t)dt\\
&+g_p^{\sigma-\gamma_p}(\rho)\int_a^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)h^{\vartheta}(t)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(t)dt\\
&-h^{\vartheta}(\rho)g_p^{\sigma-\gamma_p}(\rho)\int_a^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)\prod_{i=1}^{n} g_i^{\gamma_i}(t)dt\\
&-\int_a^x (x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu, \nu', \xi, \xi'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right)h^{\vartheta}(t)\prod_{i=1}^{n} g_i^{\gamma_i}(t)g_p^{\sigma-\gamma_p}(t)dt \geq 0.
\end{aligned} \tag{2.30}$$

Multiplying (2.30) by $\frac{x^{-\nu}}{\Gamma(\eta)}$, we get

$$h^{\vartheta}(\rho)\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\sigma}(x)\right] + g_p^{\sigma-\gamma_p}(\rho)\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]$$

$$-h^{\vartheta}(\rho)g_p^{\sigma-\gamma_p}(\rho)\left[\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]-\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\geq 0. \tag{2.31}$$

Again, multiplying (2.31) by

$$\frac{x^{-\nu}}{\Gamma(\eta)}\mathfrak{F}(x,\rho)\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)=\frac{x^{-\nu}}{\Gamma(\eta)}(x-\rho)^{\eta-1}\rho^{-\nu'}F_3\left(\nu,\nu',\xi,\xi';\eta;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)$$

(where $\mathfrak{F}(x,\rho)$ is defined by (2.5)) and integrating the resultant identity with respect to ρ over (a,x), we get

$$\begin{aligned}&\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\\&-\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\geq 0\end{aligned}$$

which completes the desired inequality (2.26) of Theorem 8. □

Theorem 9. *Let $(g_i)_{i=1,2,3,\cdots,n}$ and h be positive continuous functions on the interval $[a,b]$ such that h is increasing and $(g_i)_{i=1,2,3,\cdots,n}$ be decreasing functions on the interval $[a,b]$. Let $a<x\leq b$, $\vartheta>0$, $\sigma\geq\gamma_p>0$ for any fixed $p\in\{1,2,3,\cdots,n\}$. Then for MSM fractional integral* (1.1)*, we have*

$$\frac{\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]+}{\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]+}$$
$$\frac{\mathfrak{I}_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]}{\mathfrak{I}_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]}\geq 1, \tag{2.32}$$

where $\alpha,\beta,\zeta,\zeta',\lambda,\nu,\ \acute{\nu},\ \xi,\xi',\ \eta\in\mathbb{R}$ such that $\eta>\max\{\nu,\nu',\xi,\xi'\}>0$ and $\lambda>\max\{\nu,\nu',\xi,\xi'\}>0$.

Proof. Multiplying (2.31) by

$$\frac{x^{-\alpha}}{\Gamma(\lambda)}\mathfrak{F}(x,\rho)\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)=\frac{x^{-\alpha}}{\Gamma(\lambda)}(x-\rho)^{\lambda-1}\rho^{-\beta}F_3\left(\alpha,\beta,\zeta,\zeta';\lambda;1-\frac{\rho}{x},1-\frac{x}{\rho}\right)\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)$$

(where $\mathfrak{F}(x,\rho)$ is defined by (2.5)) and integrating the resultant identity with respect to ρ over (a,x), we get

$$\begin{aligned}&\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\\&+\mathfrak{I}_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\\&-\mathfrak{I}_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\mathfrak{I}_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\end{aligned}$$

$$-\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\geq 0.$$

It follows that

$$\begin{aligned}
&\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\\
&+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\\
&\geq\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\\
&+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right].
\end{aligned}$$

Dividing both sides by

$$\begin{aligned}
&\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\\
&+\Im_{a,x}^{\alpha,\beta,\zeta,\zeta',\lambda}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\Im_{a,x}^{\nu,\nu',\xi,\xi',\eta}\left[\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right],
\end{aligned}$$

which gives the desired inequality (2.32). □

Remark 2.6. *Applying Theorem 9 for* $\alpha = \nu$, $\beta = \nu'$, $\zeta = \xi$, $\zeta' = \xi'$, $\lambda = \eta$, *we get Theorem 8.*

Remark 2.7. *The results presented in this paper generalize some previous works cited therein.*

3. Concluding remarks

In this present paper, the we introduced certain inequalities by employing the MSM fractional integral operator. Also, they presented some inequalities for a class of n positive continuous and decreasing functions on the interval $[a, b]$. The inequalities obtained in this present paper are more general than the classical inequalities available in the literature. The MSM operator defined by (1.1) was introduced by [13] as Mellin type convolution operator with a special function $F_3(.)$ in the kernel. This MSM operator was re-discovered by Saigo [31] which is the generalized form of Saigo fractional integral operator [11]. The MSM operator (1.1) will led to the Saigo fractional integral operator [31] due to the following relation $\Im_{a,x}^{\nu,0,\xi,\xi',\eta}(x) = \Im_{a,x}^{\eta,\nu-\eta,-\xi}(x), (\gamma \in \mathbb{C})$. Thus, the inequalities obtained in this paper will reduce to the inequalities integral inequalities involving Saigo fractional integral operators recently defined by Houas [7].

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Conflict of interest

All authors declare no conflicts of interest.

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