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Research article

On an extended Hardy-Littlewood-Polya's inequality

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Abstract: By utilization of the weight coefficients, the idea of introducing parameters and Euler-Maclaurin summation formula, an extended Hardy-Littlewood-Polya's inequality and its equivalent form are established. The equivalent statements of the best possible constant factor involving several parameters, and some particular cases are provided. The operator expressions of the obtained results are also considered.

Keywords: weight coefficient; Hardy-Littlewood-Polya's inequality; Euler-Maclaurin summation formula; equivalent statement; parameter
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1. Introduction

Assuming that $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m, b_n \geq 0, 0<\sum_{m=1}^{\infty} a_m^p<\infty$ *and* $0<\sum_{n=1}^{\infty} b_n^q<\infty$, then we have the following Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}} \tag{1}$$

and have the following Hardy-Littlewood-Polya's inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq(\sum_{m=1}^{\infty}a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}b_n^q)^{\frac{1}{q}}, \tag{2}$$

where the constant factor *pq* is the best possible (cf [1], Theorem 315 and Theorem 341).

In 2006, by introducing parameters $\lambda_i \in (0,2]\,(i=1,2), \lambda_1+\lambda_2=\lambda\in(0,4]$, an extension of (1) was provided by Krnić and Pečarić [2], as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}}, \tag{3}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible ($B(u,v)=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt\ (u,v>0)$ is the beta function). For $\lambda=1, \lambda_1=\frac{1}{q}, \lambda_2=\frac{1}{p}$, inequality (3) reduces to inequality (1); for $p=q=2, \lambda_1=\lambda_2=\frac{\lambda}{2}$, inequality (3) reduces to Yang's work in [3]. Recently, by applying inequality (2), a new inequality with the kernel $\frac{1}{(m+n)^{\lambda}}$ involving partial sums was given in [4].

If $f(x), g(y)\ge 0, 0<\int_0^{\infty}f^p(x)dx<\infty$ and $0<\int_0^{\infty}g^q(y)dy<\infty$, then we have the following Hardy -Hilbert's integral inequality (cf. [1], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}(\int_0^{\infty}f^p(x)dx)^{1/p}(\int_0^{\infty}g^q(y)dy)^{1/q} \tag{4}$$

where the constant factor $\pi/\sin(\frac{\pi}{p})$ is the best possible. Inequalities (1), (2) and (3) with their extensions and reverses are important in mathematical analysis and its applications (cf. [5–15]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 351):

If $K(t)\ (t>0)$ is decreasing, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^{\infty}K(t)t^{s-1}dt<\infty$, then we have

$$\int_0^{\infty}x^{p-2}(\sum_{n=1}^{\infty}K(nx)a_n)^p dx<\phi^p(\tfrac{1}{q})\sum_{n=1}^{\infty}a_n^p. \tag{5}$$

Some new extensions of inequality (5) and their reverses were provided in [16–20].

In 2016, by means of the technique of real analysis, Hong and Wen [21] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. The other similar works concerned with inequalities (2), (4) and (5) were investigated in [22–27].

In this paper, following the way of [2,21], by making use of the weight coefficients, the idea of introducing parameters and Euler-Maclaurin summation formula, an extension of inequality (2) with parameters as well as the equivalent form are provided in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to several parameters and some particular cases are discussed in Theorem 2 and Remark 2. The operator expressions are considered in Theorem 3.

2. Some lemmas

In what follows, we assume that $p>1\ (q>1), \frac{1}{p}+\frac{1}{q}=1,\ \lambda\in(0,3],\ \lambda_i\in(0,\frac{11}{8}]\cap(0,\lambda)\quad(i=1,2)$. We also assume that $a_m, b_n\ge 0\ (m,n\in \mathrm{N}=\{1,2,\cdots\})$ such that

$$0<\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p<\infty \ and \ 0<\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} b_n^q<\infty.$$

Lemma 1. Define the weight coefficient:

$$\varpi_\lambda(\lambda_2,m):=m^{\lambda-\lambda_2}\sum_{n=1}^{\infty}\frac{n^{\lambda_2-1}}{(\max\{m,n\})^\lambda} \quad (\mathrm{m}\in \mathrm{N}) \tag{6}$$

Then, we have the following inequality:

$$k_\lambda(\lambda_2)(1-\frac{1}{\lambda_2 k_\lambda(\lambda_2)m^{\lambda_2}})<\varpi_\lambda(\lambda_2,m)<k_\lambda(\lambda_2):=\frac{\lambda}{\lambda_2(\lambda-\lambda_2)} \quad (\mathrm{m}\in \mathrm{N}) \tag{7}$$

Proof. For fixed $m\in \mathrm{N}$, we set function $g_m(t):=\frac{t^{\lambda_2-1}}{(\max\{m,t\})^\lambda}\ (t>0)$. Thereby we get

$$g_m(t)=\begin{cases}\frac{t^{\lambda_2-1}}{m^\lambda},0<t<m,\\ t^{\lambda_2-\lambda-1},t\geq m\end{cases}, g_m'(t)=\begin{cases}\frac{(\lambda_2-1)t^{\lambda_2-2}}{m^\lambda},0<t<m,\\ (\lambda_2-\lambda-1)t^{\lambda_2-\lambda-2},t>m\end{cases}.$$

(i) For $\lambda_2\in(0,1]$, by the property of monotone decreasing, we obtain

$$\varpi_\lambda(\lambda_2,m)<m^{\lambda-\lambda_2}\int_0^\infty\frac{t^{\lambda_2-1}dt}{(\max\{m,t\})^\lambda}=m^{\lambda-\lambda_2}[\int_0^m\frac{t^{\lambda_2-1}}{m^\lambda}dt+\int_m^\infty\frac{t^{\lambda_2-1}}{t^\lambda}dt]=k_\lambda(\lambda_2).$$

$$\varpi_\lambda(\lambda_2,m)>m^{\lambda-\lambda_2}\int_1^\infty\frac{t^{\lambda_2-1}dt}{(\max\{m,t\})^\lambda}=m^{\lambda-\lambda_2}[\int_0^\infty\frac{t^{\lambda_2-1}dt}{(\max\{m,t\})^\lambda}-\int_0^1\frac{t^{\lambda_2-1}dt}{(\max\{m,t\})^\lambda}]$$
$$=k_\lambda(\lambda_2)-m^{\lambda-\lambda_2}\int_0^1\frac{t^{\lambda_2-1}}{m^\lambda}dt=k_\lambda(\lambda_2)(1-\frac{1}{\lambda_2 k_\lambda(\lambda_2)m^{\lambda_2}}).$$

Thus, in this case the inequality (7) is proved.

(ii) For $\lambda_2\in(1,\frac{11}{8}]$, by using Euler-Maclaurin summation formula (cf. [20]), for $\rho(t):=t-[t]-\frac{1}{2}$, we have

$$\sum_{n=2}^{m}g_m(n)=\int_1^m g_m(t)dt+\frac{1}{2}g_m(t)|_1^m+\int_1^m\rho(t)g_m'(t)dt$$
$$=\int_1^m g_m(t)dt+\frac{1}{2}g_m(t)|_1^m+\frac{\lambda_2-1}{m^\lambda}\int_1^m\rho(t)t^{\lambda_2-2}dt$$
$$=\int_1^m g_m(t)dt+\frac{1}{2}g_m(t)|_1^m+\frac{\lambda_2-1}{m^\lambda}\frac{\varepsilon}{12}t^{\lambda_2-2}|_1^m$$
$$\leq\int_1^m g_m(t)dt+\frac{1}{2}g_m(t)|_1^m \ (\lambda_2-1>0,0<\varepsilon<1),$$

$$\sum_{n=m+1}^{\infty}g_m(n)=\int_m^\infty g_m(t)dt+\frac{1}{2}g_m(t)|_m^\infty+\int_m^\infty\rho_1(t)g_m'(t)dt$$
$$=\int_m^\infty g_m(t)dt+\frac{1}{2}g_m(t)|_m^\infty+\frac{\lambda_2-\lambda-1}{12}\varepsilon_1 t^{\lambda_2-\lambda-2}|_m^\infty$$
$$<\int_m^\infty g_m(t)dt+\frac{1}{2}g_m(t)|_m^\infty+\frac{\lambda-\lambda_2+1}{12}m^{\lambda_2-\lambda-2} \ (\lambda>\lambda_2,0<\varepsilon_1<1),$$

and then it follows that

$$\sum_{n=1}^{\infty} g_m(n) < \int_1^{\infty} g_m(t)dt + \tfrac{1}{2} g_m(1) + \tfrac{\lambda-\lambda_2+1}{12} m^{\lambda_2-\lambda-2}$$

$$= \int_0^{\infty} g_m(t)dt - h_m(\lambda, \lambda_2),$$

in which, for $h(\lambda_2) := 12 - 10\lambda_2 + \lambda_2^2$,

$$h_m(\lambda, \lambda_2) := \int_0^1 g_m(t)dt - \tfrac{1}{2} g_m(1) - \tfrac{\lambda-\lambda_2+1}{12 m^{\lambda-\lambda_2+2}}$$

$$= \tfrac{1}{\lambda_2 m^{\lambda}} - \tfrac{1}{2m^{\lambda}} - \tfrac{\lambda-\lambda_2+1}{12 m^{\lambda+2-\lambda_2}} > \left(\tfrac{1}{\lambda_2} - \tfrac{1}{2} - \tfrac{4-\lambda_2}{12}\right)\tfrac{1}{m^{\lambda}} = \tfrac{h(\lambda_2)}{12\lambda_2 m^{\lambda}}.$$

Since $h'(\lambda_2) := -10 + 2\lambda_2 < 0$ $(\lambda_2 \in (1, \frac{11}{8}])$, we have

$$h_m(\lambda, \lambda_2) > \tfrac{h(\lambda_2)}{12\lambda_2 m^{\lambda}} \geq \tfrac{12 - 10\times(\frac{11}{8}) + (\frac{11}{8})^2}{12\lambda_2 m^{\lambda}} = \tfrac{3}{256\lambda_2 m^{\lambda}} > 0.$$

We obtain

$$\varpi_{\lambda}(\lambda_2, m) = m^{\lambda-\lambda_2} \sum_{n=1}^{\infty} g_m(n) < m^{\lambda-\lambda_2} \int_0^{\infty} g_m(t)dt = k_{\lambda}(\lambda_2) = \tfrac{\lambda}{\lambda_2(\lambda-\lambda_2)}.$$

On the other hand, we have

$$\sum_{n=2}^{m} g_m(n) = \int_1^{m} g_m(t)dt + \tfrac{1}{2} g_m(t)\,\big|_1^m + \tfrac{\lambda_2-1}{m^{\lambda}} \tfrac{\varepsilon}{12} t^{\lambda_2-2}\,\big|_1^m$$

$$\geq \int_1^{m} g_m(t)dt + \tfrac{1}{2} g_m(t)\,\big|_1^m + \tfrac{\lambda_2-1}{12 m^{\lambda}} (m^{\lambda_2-2} - 1),$$

$$\sum_{n=m+1}^{\infty} g_m(n) = \int_m^{\infty} g_m(t)dt + \tfrac{1}{2} g_m(t)\,\big|_m^{\infty} + \tfrac{\lambda_2-\lambda-1}{12} \varepsilon_1 t^{\lambda_2-\lambda-2}\,\big|_m^{\infty}$$

$$> \int_m^{\infty} g_m(t)dt + \tfrac{1}{2} g_m(t)\,\big|_m^{\infty},$$

and then for $\frac{1}{2m^{\lambda}} - \frac{\lambda_2-1}{12m^{\lambda}} > \frac{1}{2m^{\lambda}} - \frac{1}{12m^{\lambda}} > 0$ $(\lambda_2 < 2)$, we obtain

$$\sum_{n=1}^{\infty} g_m(n) > \int_1^{\infty} g_m(t)dt + \tfrac{1}{2} g_m(1) + \tfrac{\lambda_2-1}{12 m^{\lambda}} (m^{\lambda_2-2} - 1)$$

$$> \int_1^{\infty} g_m(t)dt + \left(\tfrac{1}{2m^{\lambda}} - \tfrac{\lambda_2-1}{12 m^{\lambda}}\right) > \int_0^{\infty} g_m(t)dt - \int_0^1 g_m(t)dt.$$

Hence, in view of (i), we still have the inequality (7). This completes the proof of Lemma 1. □

Lemma 2. The following extended Hardy-Littlewood-Polya's inequality holds true:

$$I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tfrac{a_m b_n}{(\max\{m,n\})^{\lambda}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2) k_{\lambda}^{\frac{1}{q}}(\lambda_1)$$

$$\times \{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} b_n^q\}^{\frac{1}{q}}. \tag{8}$$

Proof. In the same way as the proof of inequality (7), under the assumption conditions $\lambda \in (0,3]$, $\lambda_1 \in (0, \frac{11}{8}] \cap (0, \lambda)$, we can deduce the following inequality for the weight coefficient:

$$k_\lambda(\lambda_1)(1 - \frac{1}{\lambda_1 k_\lambda(\lambda_1) n^{\lambda_1}}) < \omega(\lambda_1, n) := n^{\lambda - \lambda_1} \sum_{m=1}^{\infty} \frac{n^{\lambda_1 - 1}}{(\max\{m,n\})^\lambda} < k_\lambda(\lambda_1) \quad (n \in \mathrm{N}). \tag{9}$$

By Hölder's inequality (cf. [28]), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(\max\{m,n\})^\lambda} [\frac{n^{(\lambda_2-1)/p}}{m^{(\lambda_1-1)/q}} a_m][\frac{m^{(\lambda_1-1)/q}}{n^{(\lambda_2-1)/p}} b_n] \\ &\le \{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(\max\{m,n\})^\lambda} \frac{n^{\lambda_2-1}}{m^{(\lambda_1-1)(p-1)}} a_m^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(\max\{m,n\})^\lambda} \frac{m^{\lambda_1-1}}{n^{(\lambda_2-1)(q-1)}} b_n^q\}^{\frac{1}{q}} \\ &= \{\sum_{m=1}^{\infty} \varpi(\lambda_2, m) m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} \omega(\lambda_1, n) n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} b_n^q\}^{\frac{1}{q}}. \end{aligned}$$

Then by (7) and (9), we obtain inequality (8). The proof of Lemma 2 is complete. □

Remark 1. By inequality (8), for $\lambda_1 + \lambda_2 = \lambda \in (0, \frac{11}{4}](\subset (0,3]), 0 < \lambda_i \le \frac{11}{8}$ $(i = 1,2)$, we have

$$0 < \sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p < \infty \ and \ 0 < \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q < \infty.$$

and the following inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(\max\{m,n\})^\lambda} < \frac{\lambda}{\lambda_1\lambda_2} \{\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\}^{\frac{1}{q}}. \tag{10}$$

Lemma 3. For $\lambda_1 + \lambda_2 = \lambda \in (0, \frac{11}{4}]$, the constant factor $\frac{\lambda}{\lambda_1\lambda_2}$ in (10) is the best possible.

Proof. For any $0 < \varepsilon < p\lambda_1$, we set

$$\widetilde{a}_m := m^{\lambda_1 - \frac{\varepsilon}{p} - 1}, \widetilde{b}_n := n^{\lambda_2 - \frac{\varepsilon}{q} - 1} \ (m, n \in \mathrm{N}).$$

If there exists a constant $M \le \frac{\lambda}{\lambda_1\lambda_2}$ such that (10) is valid when replacing $\frac{\lambda}{\lambda_1\lambda_2}$ by M, then in particular, substitution of $a_m = \widetilde{a}_m$ *and* $b_n = \widetilde{b}_n$ in (10), we have

$$\widetilde{I} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\widetilde{a}_m \widetilde{b}_n}{(\max\{m,n\})^\lambda} < M \{\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} \widetilde{a}_m^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} \widetilde{b}_n^q\}^{\frac{1}{q}}. \tag{11}$$

By (11) and the decreasingness property, we obtain

$$\begin{aligned} \widetilde{I} &< M \{\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} m^{p\lambda_1 - \varepsilon - p}\}^{\frac{1}{p}} [\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} n^{q\lambda_2 - \varepsilon - q}]^{\frac{1}{q}} \\ &= M(1 + \sum_{m=2}^{\infty} m^{-\varepsilon-1})^{\frac{1}{p}} (1 + \sum_{n=2}^{\infty} n^{-\varepsilon-1})^{\frac{1}{q}} \\ &< M(1 + \int_1^{\infty} x^{-\varepsilon-1} dx)^{\frac{1}{p}} (1 + \int_1^{\infty} y^{-\varepsilon-1} dy)^{\frac{1}{q}} = \frac{M}{\varepsilon}(\varepsilon + 1). \end{aligned}$$

By (9), setting $\hat{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} \in (0, \frac{11}{8}) \cap (0, \lambda)$ $(0 < \hat{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} = \lambda - \hat{\lambda}_1 < \lambda)$, we get

$$\widetilde{I} = \sum_{n=1}^{\infty}[n^{(\lambda_2+\frac{\varepsilon}{p})}\sum_{m=1}^{\infty}\frac{1}{(\max\{m,n\})^{\lambda}} m^{(\lambda_1-\frac{\varepsilon}{p})-1}]n^{-\varepsilon-1}$$
$$= \sum_{n=1}^{\infty}\omega(\hat{\lambda}_1,n)n^{-\varepsilon-1} > \frac{\lambda}{\hat{\lambda}_1\hat{\lambda}_2}\sum_{n=1}^{\infty}(1-\frac{\hat{\lambda}_2}{\lambda n^{\hat{\lambda}_1}})n^{-\varepsilon-1}$$
$$= \frac{\lambda}{\hat{\lambda}_1\hat{\lambda}_2}(\sum_{n=1}^{\infty}n^{-\varepsilon-1} - \frac{\hat{\lambda}_2}{\lambda}\sum_{n=1}^{\infty}\frac{1}{n^{\lambda_1+\frac{\varepsilon}{q}+1}}) > \frac{\lambda}{\hat{\lambda}_1,\hat{\lambda}_2}(\int_1^{\infty}x^{-\varepsilon-1}dx - O(1))$$
$$= \frac{\lambda}{\varepsilon\hat{\lambda}_1\hat{\lambda}_2}(1-\varepsilon O(1)).$$

Then we have

$$\frac{\lambda}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2+\frac{\varepsilon}{p})}(1-\varepsilon O(1)) < \varepsilon\widetilde{I} < M(\varepsilon+1).$$

For $\varepsilon \to 0^+$, we find $\frac{\lambda}{\lambda_1\lambda_2} \le M$. Hence, $M = \frac{\lambda}{\lambda_1,\lambda_2}$ is the best possible constant factor of (10). This completes the proof of Lemma 3. □

Setting $\widetilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \widetilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we obtain

$$\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda.$$

Thus we can rewrite (8) as follows:

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\max\{m,n\})^{\lambda}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$$
$$\times[\sum_{m=1}^{\infty}m^{p(1-\widetilde{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1-\widetilde{\lambda}_2)-1}b_n^q]^{\frac{1}{q}}. \tag{12}$$

Lemma 4. If inequality (12) is valid and the constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (12) is the best possible, then we have $\lambda = \lambda_1 + \lambda_2$.

Proof. Note that

$$\widetilde{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} > 0, \widetilde{\lambda}_1 < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda,$$
$$0 < \widetilde{\lambda}_2 = \lambda - \widetilde{\lambda}_1 < \lambda.$$

Hence, we have $k_{\lambda}(\widetilde{\lambda}_1) = \frac{\lambda}{\widetilde{\lambda}_1(\lambda-\widetilde{\lambda}_1)} = \frac{\lambda}{\widetilde{\lambda}_1\widetilde{\lambda}_2} \in \mathrm{R}_+ = (0,\infty)$.

If the constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (12) is the best possible, then in view of (10), the unique best possible constant factor must be $\frac{\lambda}{\widetilde{\lambda}_1\widetilde{\lambda}_2} = k_{\lambda}(\widetilde{\lambda}_1)(\in \mathrm{R}_+)$, namely, $k_{\lambda}(\widetilde{\lambda}_1) = k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$.

By Hölder's inequality, we obtain

$$k_{\lambda}(\widetilde{\lambda}_1) = k_{\lambda}(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})$$
$$= \int_0^{\infty}\frac{1}{(\max\{1,u\})^{\lambda}}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du = \int_0^{\infty}\frac{1}{(\max\{1,u\})^{\lambda}}(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du$$
$$\le [\int_0^{\infty}\frac{1}{(\max\{1,u\})^{\lambda}}u^{\lambda-\lambda_2-1}du]^{\frac{1}{p}}[\int_0^{\infty}\frac{1}{(\max\{1,u\})^{\lambda}}u^{\lambda_1-1}du]^{\frac{1}{q}}$$
$$= [\int_0^{\infty}\frac{1}{(\max\{1,v\})^{\lambda}}v^{\lambda_2-1}dv]^{\frac{1}{p}}[\int_0^{\infty}\frac{1}{(\max\{1,u\})^{\lambda}}u^{\lambda_1-1}du]^{\frac{1}{q}}$$
$$= k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1) \tag{13}$$

We observe that (13) keeps the form of equality if and only if there exist constants A and B (not all zero) such that (cf. [28])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \ a.e. \text{ in } \mathrm{R}_+.$$

Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ *a.e.* in R_+, and then $\lambda - \lambda_2 - \lambda_1 = 0$, namely, $\lambda = \lambda_1 + \lambda_2$. The Lemma 4 is proved. □

3. Main results and some particular cases

Theorem 1. Inequality (8) is equivalent to the following inequality:

$$J := \{\sum_{n=1}^{\infty} n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}[\sum_{m=1}^{\infty}\frac{1}{(\max\{m,n\})^{\lambda}}a_m]^p\}^{\frac{1}{p}}$$

$$< k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}, \tag{14}$$

If the constant factor in (8) is the best possible, then so is the constant factor in (14).
Proof. Suppose that (14) is valid. By Hölder's inequality (cf. [28]), we have

$$I = \sum_{n=1}^{\infty}[n^{\frac{-1}{p}+(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}\sum_{m=1}^{\infty}\frac{1}{(\max\{m,n\})^{\lambda}}a_m][n^{\frac{1}{p}-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}b_n]$$

$$\leq J\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{\frac{1}{q}}. \tag{15}$$

Hence by (14), we obtain inequality (8).

On the other hand, assuming that (8) is valid, we set

$$b_n := n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}[\sum_{m=1}^{\infty}\frac{1}{(\max\{m,n\})^{\lambda}}a_m]^{p-1}, n \in \mathbf{N}.$$

If $J = \infty$, then (14) is naturally valid; if $J = 0$, then it is impossible to make (14) valid, namely, $J > 0$. Suppose that $0 < J < \infty$. By (8), we have

$$\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q = J^p = I < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$$

$$\times\{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{\frac{1}{q}},$$

$$J = \{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{\frac{1}{p}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}},$$

namely, inequality (14) follows. Hence, inequality (8) is equivalent to (14).

If the constant factor in (8) is the best possible, then so is the constant factor in (14). Otherwise, by (15), we would reach a contradiction that the constant factor in (8) is not the best possible. The proof of Theorem 1 is complete. □

Theorem 2. The following statements (i), (ii), (iii) and (iv) are equivalent:

(i) $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is independent of p, q;

(ii) $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral;

(iii) $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (8) is the best possible constant factor;

(iv) $\lambda = \lambda_1 + \lambda_2$.

If the statement (iv) follows, namely, $\lambda = \lambda_1 + \lambda_2$, then we have (10) and the following equivalent inequalities with the best possible constant factor $\frac{\lambda}{\lambda_1\lambda_2}$:

$$\{\sum_{n=1}^{\infty} n^{p\lambda_2-1}[\sum_{m=1}^{\infty}\frac{1}{(\max\{m,n\})^\lambda}a_m]^p\}^{\frac{1}{p}} < \frac{\lambda}{\lambda_1\lambda_2}[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}. \quad (16)$$

Proof. (i)$\Rightarrow$(ii). By (i) , we have

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to 1^+}\lim_{q\to\infty} k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_2).$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral

$$k_\lambda(\lambda_1) = \int_0^\infty \frac{1}{(\max\{1,u\})^\lambda}u^{\lambda_2-1}du.$$

(ii)$\Rightarrow$(iv). If $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a convergent single integral $k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (13) keeps the form of equality. In view of the proof of Lemma 4, it follows that $\lambda = \lambda_1 + \lambda_2$.

(iv)$\Rightarrow$(i). If $\lambda = \lambda_1 + \lambda_2$, then $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_1)$, which is independent of p, q. Hence, it follows that (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv).

(iii)$\Rightarrow$(iv). By Lemma 4, we have $\lambda = \lambda_1 + \lambda_2$.

(iv)$\Rightarrow$(iii). By Lemma 3, for $\lambda = \lambda_1 + \lambda_2$, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)(=\frac{\lambda}{\lambda_1\lambda_2})$ is the best possible constant factor of (8). Therefore, we have (iii)$\Leftrightarrow$(iv).

Hence, the statements (i), (ii), (iii) and (iv) are equivalent. This completes the proof of Theorem 2. □

Remark 2. (i) For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ in (11) and (17), we obtain the inequality (2) and the following equivalent inequality with the best possible constant factor pq:

$$[\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty}\frac{1}{\max\{m,n\}}a_m)^p]^{\frac{1}{p}} < pq(\sum_{m=1}^{\infty}a_m^p)^{\frac{1}{p}}. \quad (17)$$

It follows that (8) and (11) are extensions of (2).

(ii) For $\lambda = 1, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}$ in (11) and (17), we have the following equivalent inequalities with the best possible constant factor pq:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\max\{m,n\}}a_m b_n < pq(\sum_{m=1}^{\infty}m^{p-2}a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}n^{q-2}b_n^q)^{\frac{1}{q}}, \quad (18)$$

$$[\sum_{n=1}^{\infty} n^{p-2} (\sum_{m=1}^{\infty} \frac{1}{\max\{m,n\}} a_m)^p]^{\frac{1}{p}} < pq(\sum_{m=1}^{\infty} m^{p-2} a_m^p)^{\frac{1}{p}}. \tag{19}$$

(iii) For $p = q = 2$, Both inequality (2) and inequality (18) reduce to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\max\{m,n\}} a_m b_n < 4(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2)^{\frac{1}{2}}, \tag{20}$$

moreover, both inequality (17) and inequality (19) reduce to the equivalent form of (20), as follows:

$$[\sum_{n=1}^{\infty} (\sum_{m=1}^{\infty} \frac{1}{\max\{m,n\}} a_m)^2]^{\frac{1}{2}} < 4(\sum_{m=1}^{\infty} a_m^2)^{\frac{1}{2}}. \tag{21}$$

4. Operator expressions

We set functions

$$\varphi(m) := m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}, \psi(n) := n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1},$$

wherefrom, one has

$$\psi^{1-p}(n) = n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1} \ (m, n \in \mathrm{N}).$$

Define the following real normed spaces:

$$l_{p,\varphi} := \{a = \{a_m\}_{m=1}^{\infty}; \| a \|_{p,\varphi} := (\sum_{m=1}^{\infty} \varphi(m) | a_m |^p)^{\frac{1}{p}} < \infty\},$$

$$l_{q,\psi} := \{b = \{b_n\}_{n=1}^{\infty}; \| b \|_{q,\psi} := (\sum_{n=1}^{\infty} \psi(n) | b_n |^q)^{\frac{1}{q}} < \infty\},$$

$$l_{p,\psi^{1-p}} := \{c = \{c_n\}_{n=1}^{\infty}; \| c \|_{p,\psi^{1-p}} := (\sum_{n=1}^{\infty} \psi^{1-p}(n) | b_n |^p)^{\frac{1}{p}} < \infty\}.$$

Assuming that $a \in l_{p,\varphi}$, setting

$$c = \{c_n\}_{n=1}^{\infty}, c_n := \sum_{m=1}^{\infty} \frac{1}{(\max\{m,n\})^{\lambda}} a_m, n \in \mathrm{N},$$

we can rewrite (14) as follows:

$$\| c \|_{p,\psi^{1-p}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2) k_{\lambda}^{\frac{1}{q}}(\lambda_1) \| a \|_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 2. Define a Hardy-Littlewood-Polya's operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows:
For any $a \in l_{p,\varphi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta, b) := \sum_{n=1}^{\infty} [\sum_{m=1}^{\infty} \frac{1}{(\max\{m,n\})^{\lambda}} a_m]b_n,$$

$$\| T \| := \sup_{a(\neq\theta)\in l_{p,\varphi}} \frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\varphi}}.$$

By Theorem 1 and Theorem 2, we have

Theorem 3. If $a \in l_{p,\varphi}, b \in l_{q,\psi}, \| a \|_{p,\varphi}, \| b \|_{q,\psi} > 0$, then we have the following equivalent inequalities:

$$(Ta,b) < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1) \| a \|_{p,\varphi} \| b \|_{q,\psi}, \tag{22}$$

$$\| Ta \|_{p,\psi^{1-p}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1) \| a \|_{p,\varphi}. \tag{23}$$

Furthermore, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (22) and (23) is the best possible, namely,

$$\| T \| = k_{\lambda}(\lambda_1) = \frac{\lambda}{\lambda_1\lambda_2}. \tag{24}$$

5. Conclusion

Let us give a brief summary of this paper, by applying the weight coefficients, the idea of introducing parameters and Euler-Maclaurin summation formula, an extended Hardy-Littlewood-Polya's inequality and the equivalent form are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to several parameters, and some particular cases are considered in Theorem 2 and Remark 2. The operator expressions are given in Theorem 3. The lemmas and theorems depict some essential characters of this type of inequalities.

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Conflict of interest

The authors declare that they have no competing interests.

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