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Research article

Non-autonomous 3D Brinkman-Forchheimer equation with singularly oscillating external force and its uniform attractor

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Abstract: For $\rho \in [0, 1)$ and $\varepsilon > 0$, the non-autonomous 3D Brinkman-Forchheimer equation with singularly oscillating external force

$$\begin{aligned} &u_t - \nu\Delta u + au + b|u|u + c|u|^\beta u + \nabla p = f_0(x,t) + \varepsilon^{-\rho} f_1(x, \frac{t}{\varepsilon}),\\ &\mathrm{div} u = 0 \end{aligned}$$

are considered, together with the averaged equation

$$\begin{aligned} &u_t - \nu\Delta u + au + b|u|u + c|u|^\beta u + \nabla p = f_0(x,t),\\ &\mathrm{div} u = 0 \end{aligned}$$

formally corresponding to the limiting case $\varepsilon = 0$. First, within the restriction $\rho < 1$ and under suitable translation-compactness assumptions on the external forces, the uniform (w.r.t.ε) boundedness of the related uniform attractors $\mathcal{A}^\varepsilon$ is established when $1 < \beta \leq 4/3$. This fact is not at all intuitive, since in principle the blow up of the oscillation amplitude might overcome the averaging effect due to the term $\frac{t}{\varepsilon}$ in f_1. Next, the convergence of the attractor $\mathcal{A}^\varepsilon$ of the first equation to the attractor $\mathcal{A}^0$ of the second one as $\varepsilon \to 0^+$ is established.

Keywords: 3D Brinkman-Forchheimer equation; singularly oscillating external force; uniform attractor; translation bounded function
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1. Introduction

For $\rho \in [0,1)$ and $\varepsilon > 0$, we consider the following non-autonomous Brinkman-Forchheimer equation with a singularly oscillating external force on $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} u_t - \nu\Delta u + au + b|u|u + c|u|^\beta u + \nabla p = f_0(x,t) + \varepsilon^{-\rho} f_1(x, \frac{t}{\varepsilon}), & in\ \Omega \times (\tau, T), \\ \text{div} u = 0, & in\ \Omega \times (\tau, T), \\ u|_{t=\tau} = u_\tau, & in\ \Omega, \\ u = 0, & on\ \partial\Omega \times (\tau, T), \end{cases} \tag{1.1}$$

where Ω is an open, bounded domain of $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. $u = (u_1, u_2, u_3)$ is the fluid velocity vector, ν is the Brinkman coefficient, $a > 0$ is the Darcy coefficient, $b > 0, c > 0$ are the Forchheimer coefficients, p is the pressure, $\beta \in (1, \frac{4}{3}]$ is a constant.

Along with (1.1), we consider the averaged Brinkman-Forchheimer equation

$$\begin{cases} u_t - \nu\Delta u + au + b|u|u + c|u|^\beta u + \nabla p = f_0(x,t), & in\ \Omega \times (\tau, T), \\ \text{div} u = 0, & in\ \Omega \times (\tau, T), \\ u|_{t=\tau} = u_\tau, & in\ \Omega, \\ u = 0, & on\ \partial\Omega \times (\tau, T), \end{cases} \tag{1.2}$$

without by rapid and singular oscillations, which formally corresponds to $\varepsilon = 0$.

The Brinkman-Forchheimer equation describes the motion of fluid flow in a saturated porous medium and has been studied by many researchers. We should note that most of the previous studies have been focused on physical viewpoint or numerical simulation viewpoint(see [1–8]), or continuous dependence of solutions on the coefficients ν, b and c (see [9–15]). The asymptotic behavior of solutions was examined in [16–22], where [16–21] were mainly for the case of the parameter $\beta = 2$. In [16], using condition (C) method, Uğurlu showed the existence of global attractor in $H_0^1(\Omega)$. In [17], Wang and Lin showed that Brinkman-Forchheimer equation has a global attractor in $H^2(\Omega)$ by a very clever way. In [18], using the method of regularization of solutions and the compact embedding to deduce the uniformly ω-limit compactness of the associated evolutionary process, You, Zhao and Zhou proved the existence of uniform attractor in $H_0^1(\Omega)$. In [19], the existence of $\mathcal{D}$-pullback attractors was deduced by establishing the $\mathcal{D}$-pullback asymptotical compactness of θ-cocycle. In [20], Song and Qiao proved the existence and structure of the uniform attractor in $H_0^1(\Omega)$ for the processes associated to the fluid when the external force $f_0(x,t)$ is translation compact in $L^2_{\text{loc}}(\mathbb{R}, (L^2(\Omega))^3)$ and investigated the averaging problems of the equations with oscillating external forces. In [21], Zhang, Su and Wen investigated the existence of global attractor and uniform attractor for the 3D autonomous and nonautonomous Brinkman-Forchheimer equations. For $\beta \neq 2$, in [22], using condition (C) method, Ouyang and Yang proved the existence of global attractor in $H_0^1(\Omega)$ when $1 < \beta \leq \frac{4}{3}$.

Stability of attractors for a dynamical system with some oscillating (or perturbed) external forces is very important in natural phenomenon. Indeed, this issue has been considered by some mathematicians and engineers. Chepyzhov et al. [23] studied the non-autonomous sine-Gordon type equations with rapidly oscillating external force. Efendiev and Zelik [24, 25] considered the reaction-diffusion systems with rapidly oscillating coefficients and nonlinear rapidly oscillating in time. Chepyzhov and Vishik [26–28] investigated the Navier-Stokes equations with terms that rapidly oscillate with respect to spatial and time variables. Qin et al. [29] investigated the uniform attractors for a 3D non-autonomous Navier-Stokes-Voight equation with singularly oscillating forces. Anh and Toan [30] considered the nonclassical diffusion equation on $\mathbb{R}^N (N \geq 3)$ with a singularly oscillating external force. Medjo [31] and [32] used different methods to discuss non-autonomous planetary 3D geostrophic equation with oscillating external force and its global attractor. Medjo [33] investigated a non-autonomous two-phase flow model with oscillating external force and its global attractor. As far as we know, there is almost no paper dealing with 3D Brinkman-Forchheimer equations with rapidly oscillating terms have been published.

Motivated by [22] and [23–33], we consider the properties of (1.1), depending on the small parameter ε, which reflects the rate of fast time oscillation in the term $\varepsilon^{-\rho} f_1(\frac{t}{\varepsilon}, x)$, having the growing amplitude of order $\varepsilon^{-\rho}$. By using the method in [27, 29, 30, 32, 33], under suitable assumptions on the external force, we prove the stability of the uniform attractor $\mathcal{A}^{\varepsilon}(0 < \varepsilon \leq 1)$ associated to problem (1.1)-(1.2) as $\varepsilon \to 0^+$ in space H. The uncertainty of parameter β brings a lot of trouble to our proof, because when β is too large, some Sobolev embedding inequlaities can not be used. In the proof process of this paper, we finally determine that $1 < \beta \leq \frac{4}{3}$, just like in [22].

The main purpose of this paper is to show:
(1) the uniform (w.r.t.ε) boundedness of the family $\mathcal{A}^{\varepsilon}$ in H which is defined in Section 3:

$$\sup_{\varepsilon \in [0,1]} \| \mathcal{A}^{\varepsilon} \|_H < +\infty;$$

(2)the convergence of $\mathcal{A}^{\varepsilon}$ to $\mathcal{A}^0$ as $\varepsilon \to 0^+$ in the standard Hausdorff semidistance in H, i.e.,:

$$\lim_{\varepsilon \to 0^+} \operatorname{dist}_H(\mathcal{A}^{\varepsilon}, \mathcal{A}^0) = 0.$$

This paper is organized as follows. In Section 2, we present the notations and preliminaries that are required for this study. In Section 3, we show the existence of uniform attractor $\mathcal{A}^{\varepsilon}$ and we demonstrate the structure of the uniform attractor. In Section 4, we verify the uniform boundedness of the uniform attractor $\mathcal{A}^{\varepsilon}$. In Section 5, we prove the convergence $\mathcal{A}^{\varepsilon} \to \mathcal{A}^0$ as $\varepsilon \to 0^+$.

Nomenclature			
u	fluid velocity vector(m/s)	p	pressure
$\mathbb{R}^3$	three-dimensional whole space	Ω	open, bounded domain of $\mathbb{R}^3$
u_1, u_2, u_3	velocity components	t	time
$\partial\Omega$	boundary of a domain	ν	Brinkman coefficient
a	Darcy coefficient	b, c	Forchheimer coefficient
β	power of nonlinear term	f	external force

2. Notations and preliminaries

Given a space X, we usually denote the norm in X by $\| \cdot \|_X$, and we indicate by

$$\operatorname{dist}_X(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \| b_1 - b_2 \|_X,$$

the Hausdorff semidistance in X from a set B_1 to a set B_2. Throughout this paper, we set $\mathbb{R}_\tau = [\tau, +\infty), \tau \in \mathbb{R}$. C will stand for a generic positive constant, which is different from line to line or even in the same line.

The mathematical setting of our problem is similar to that of the Navier-Stokes equations. Let us introduce the following spaces

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \mathrm{div} u = 0\}, H = \mathrm{cl}_{(L^2(\Omega))^3}\mathcal{V}, V = \mathrm{cl}_{(H_0^1(\Omega))^3}\mathcal{V},$$

where cl_X denotes the closure in the space X. Operator P is the Helmholtz-Leray orthogonal projection from $(L^2(\Omega))^3$ onto H. $A := -P\Delta$ is the Stokes operator subject to the nonslip homogeneous Dirichlet boundary with the domain $(H^2(\Omega))^3 \cap V$, and A is a self-adjoint positively defined operator on H. We define, for $\sigma \in \mathbb{R}$, the scale of Hilbert spaces

$$H^\sigma := D(A^{\frac{\sigma}{2}})$$

with inner products and norms

$$\langle u, v\rangle_\sigma := \langle A^{\frac{\sigma}{2}} u, A^{\frac{\sigma}{2}} v\rangle_{(L^2(\Omega))^3}, \| u \|_{H^\sigma} := \| A^{\frac{\sigma}{2}} u \|_{(L^2(\Omega))^3} .$$

In particular,

$$H^0 = H, H^1 = V, H^2 = D(A),$$

and we have the generalized Poincaré inequality

$$\| u \|_{H^{\sigma+1}} \geq \lambda_1^{\frac{1}{2}} \| u \|_{H^\sigma}, \forall u \in H^{\sigma+1}, \tag{2.1}$$

where λ_1 is the first eigenvalue of the Stokes operator A.

In this paper, we use $(\cdot, \cdot)$ and $\| \cdot \|$ denote the product and the norm of H, i.e.,

$$(u, v) = \int_\Omega u \cdot v \mathrm{d}x, \forall u, v \in H, \ \| u \| = (u, u)^{\frac{1}{2}},$$

$((\cdot, \cdot))$ and $\| \cdot \|_V$ denote the product and norm of V, i.e.,

$$((u, v)) = \sum_{i=1}^{3} \int_\Omega \nabla u_i \cdot \nabla v_i \mathrm{d}x, \forall u, v \in V, \| u \|_V = ((u, u))^{\frac{1}{2}}.$$

In this paper, $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$, and we use $\| \cdot \|_p$ to denote the norm in $\mathbf{L}^p(\Omega)$ (for $p \neq 2$).

Assumptions on the external forces The function $f_0(x, t)$ and $f_1(x, t)$ are taken from the space $L_b^2(\mathbb{R}; H)$ of translation bounded functions in $L_{\mathrm{loc}}^2(\mathbb{R}; H)$, with

$$\| f_0 \|_{L_b^2}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \| f_0(s) \|^2 \mathrm{d}s = M_0^2, \tag{2.2}$$

$$\| f_1 \|_{L_b^2}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \| f_1(s) \|^2 \mathrm{d}s = M_1^2, \tag{2.3}$$

for some constants $M_0, M_1 \geq 0$.

Putting

$$f^{\varepsilon}(x,t) := \begin{cases} f_0(x,t) + \varepsilon^{-\rho} f_1(x, \frac{t}{\varepsilon}), & \varepsilon > 0, \\ f_0(x,t), & \varepsilon = 0. \end{cases} \tag{2.4}$$

It is easy to check that $f^{\varepsilon} \in L^2_b(\mathbb{R}; H)$, and

$$\| f^{\varepsilon} \|_{L^2_b} \leq Q_{\varepsilon} := \begin{cases} M_0 + \sqrt{2} M_1 \varepsilon^{-\rho}, & \varepsilon > 0, \\ M_0, & \varepsilon = 0. \end{cases}$$

Now we recall some inequalities and a Gronwall-type lemma that will be needed in the sequel.

Lemma 2.1. *([34]) Let $p \in [2, \infty)$. Then for every $a, b \in \mathbb{R}$,*

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq 2^{-2-p} 3^{-p/2} |a-b|^p.$$

Lemma 2.2. *([27]) For every $\tau \in \mathbb{R}$, every nonnegative locally summable function φ on $\mathbb{R}_\tau$ and every $\beta > 0$, we have*

$$\int_\tau^t \varphi(s) e^{-\beta(t-s)} \mathrm{d}s \leq \frac{1}{1-e^{-\beta}} \sup_{\theta \geq \tau} \int_\theta^{\theta+1} \varphi(s) \mathrm{d}s, \tag{2.5}$$

for all $t \geq \tau$.

Lemma 2.3. *([23]) Let a real function $z(t), t \geq 0$ be uniformly continuous and satisfy the inequality*

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \gamma z(t) \leq f(t), \forall t \geq 0,$$

where $\gamma > 0, f(t) \geq 0$ for all $t \geq 0$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$. Suppose also that

$$\int_t^{t+1} f(s) \mathrm{d}s \leq M, \forall t \geq 0.$$

Then

$$z(t) \leq z(0) e^{-\gamma t} + M(1+\gamma^{-1}), \forall t \geq 0.$$

3. On the existence and structure of the uniform attractor $\mathcal{A}^{\varepsilon}$

We rewrite (1.1) and (1.2) in the abstract form

$$\begin{cases} u_t + \nu A u + a u + G(u) = f^{\varepsilon}(x,t), \\ u|_{t=\tau} = u_\tau, \end{cases} \tag{3.1}$$

where the pressure p has disappeared by force of the application of the Helmholtz-Leray projection P, and $G(u) = P(b|u|u + c|u|^{\beta}u)$, $f^{\varepsilon}(x,t) = f_0(x,t) + \varepsilon^{-\rho} f_1(x, \frac{t}{\varepsilon})$ and $\varepsilon > 0$ is fixed.

Proposition 3.1. *Given $f^{\varepsilon} \in L^2_b(\mathbb{R}; H)$ and $u_\tau \in H$. Then the system (3.1) has a unique weak solution $u(t)$ satisfying*

$$u \in C(\mathbb{R}_\tau; H) \cap L^{\infty}(\mathbb{R}_\tau; H) \cap L^2_{\mathrm{loc}}(\mathbb{R}_\tau; V).$$

Proof. We can prove the global existence and uniqueness result by using the Faedo-Galerkin method (see [35, 36]). □

If the functions $f_0(t)$ and $f_1(t)$ are translation bounded, i.e. conditions (2.2) and (2.3) hold, equation (3.1) generates the dynamical process

$$\{U_{f^\varepsilon}(t,\tau), t \geq \tau, \tau \in \mathbb{R}\}$$

acting on H by the formula

$$U_{f^\varepsilon}(t,\tau)u_\tau = u(t), t \geq \tau,$$

where $u(t)$ is the solution to (3.1).

Proposition 3.2. *For any $\varepsilon > 0$, the process $\{U_{f^\varepsilon}(t,\tau)\}$ associated to (3.1) is uniformly compact in H and it has a uniform (with respect to $\tau \in \mathbb{R}$) absorbing set B_ε in H,*

$$B_\varepsilon = \{u \in H | \parallel u \parallel \leq C_0 Q_\varepsilon\}, \tag{3.2}$$

where the constant C_0 depends on ν and λ_1. Moreover, there exists a uniform attractor $\mathcal{A}^\varepsilon$ in H.

Proof. The proof process is similar to the proof in [21], so we omit it here. □

We consider the hull $\mathcal{H}(f^\varepsilon)$ of $f^\varepsilon(x,t)$ in the space $L^2_{\text{loc}}(\mathbb{R};H)$:

$$\mathcal{H}(f^\varepsilon) = [\{f^\varepsilon(\cdot, t+h) | h \in \mathbb{R}\}]_{L^2_{\text{loc}}(\mathbb{R};H)}. \tag{3.3}$$

Recall that $\mathcal{H}(f^\varepsilon)$ is compact in $L^2_{\text{loc}}(\mathbb{R};H)$ and each element $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)$ can be written as

$$\hat{f}^\varepsilon(x,t) = \hat{f}_0(x,t) + \varepsilon^{-\rho}\hat{f}_1(x, \frac{t}{\varepsilon}), \tag{3.4}$$

with $\hat{f}_0 \in \mathcal{H}(f_0)$ and $\hat{f}_1 \in \mathcal{H}(f_1)$, where $\mathcal{H}(f_0)$ and $\mathcal{H}(f_1)$ are the hulls of f_0 and f_1 in $L^2_{\text{loc}}(\mathbb{R};H)$ respectively.

We also note that (see [23])

$$\parallel \hat{f}_0 \parallel_{L^2_b(\mathbb{R};H)} \leq \parallel f_0 \parallel_{L^2_b(\mathbb{R};H)}, \forall \hat{f}_0 \in \mathcal{H}(f_0), \tag{3.5}$$

$$\parallel \hat{f}_1 \parallel_{L^2_b(\mathbb{R};H)} \leq \parallel f_1 \parallel_{L^2_b(\mathbb{R};H)}, \forall \hat{f}_1 \in \mathcal{H}(f_1). \tag{3.6}$$

It follows that

$$\parallel \hat{f}^\varepsilon \parallel_{L^2_b(\mathbb{R};H)} \leq \parallel f_0 \parallel_{L^2_b(\mathbb{R};H)} + \frac{C}{\varepsilon^\rho} \parallel f_1 \parallel_{L^2_b(\mathbb{R};H)}, \forall \hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon), \tag{3.7}$$

where C is independent of f_0, f_1, ρ and ε.

To describe the structure of the attractor $\mathcal{A}_{\hat{f}^\varepsilon}$, we consider the family of equations

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} + \nu A\hat{u} + a\hat{u} + G(\hat{u}) = \hat{f}^\varepsilon(x,t), \tag{3.8}$$

with the external force $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)$.

For $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)$, the Eq.(3.8) generates a process $\{U_{\hat{f}^\varepsilon}(t,\tau)\}$ that satisfies the same properties as $\{U_{f^\varepsilon}(t,\tau)\}$. Moreover, the process $\{U_{\hat{f}^\varepsilon}(t,\tau)\}$ has a uniform attractor $\mathcal{A}_{\hat{f}^\varepsilon}$ that satisfies $\mathcal{A}_{\hat{f}^\varepsilon} \subset \mathcal{A}_{f^\varepsilon}$.

Proposition 3.3. *Let $f_0(x,t), f_1(x,t)$ be translation compact in the space $L^2_{\rm loc}(\mathbb{R};H)$. Then for any fixed $\varepsilon, 0<\varepsilon\le 1$, the family of processes $\{U_{\hat f^\varepsilon}(t,\tau)\}, \hat f^\varepsilon \in \mathcal{H}(f^\varepsilon)$ corresponding to (3.8) has an absorbing set B_ε, which is bounded in H and satisfies*

$$|B_\varepsilon|_H \le C + C\varepsilon^{-\rho}. \tag{3.9}$$

The family $\{U_{\hat f^\varepsilon}(t,\tau)\}, \hat f^\varepsilon \in \mathcal{H}(f^\varepsilon)$ is $(H\times\mathcal{H}(f^\varepsilon);H)$-continuous. That is, if

$$\hat f^\varepsilon_n \to \hat f^\varepsilon \ in\ L^2_{\rm loc}(\mathbb{R};H), u_{\tau n}\to u_\tau \ in\ H, \tag{3.10}$$

then

$$U_{\hat f^\varepsilon_n}(t,\tau)u_{\tau n} \to U_{\hat f^\varepsilon}(t,\tau)u_\tau \ in\ H. \tag{3.11}$$

Proof. The first part of the proposition is proved in Proposition 3.2. Now, let us prove the second part. Let $w_n = \hat u_n - \hat u = U_{\hat f^\varepsilon_n}(t,\tau)u_{\tau n} - U_{\hat f^\varepsilon}(t,\tau)u_\tau$. Then w_n satisfies

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} + \nu A w_n + a w_n + G(\hat u_n) - G(\hat u) = \hat f^\varepsilon_n - \hat f^\varepsilon. \tag{3.12}$$

Multiplying (3.12) by w_n we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\parallel w_n\parallel^2 + \nu\parallel\nabla w_n\parallel^2 + a\parallel w_n\parallel^2 + (G(\hat u_n)-G(\hat u), w_n) = (\hat f^\varepsilon_n - \hat f^\varepsilon, w_n). \tag{3.13}$$

Noticing $b|u|u + c|u|^\beta u$ is monotonic, i.e.,

$$(G(\hat u_n)-G(\hat u), w_n) = (b|\hat u_n|\hat u_n + c|\hat u_n|^\beta \hat u_n - b|\hat u|\hat u - c|\hat u|^\beta \hat u, \hat u_n - \hat u) \ge 0, \tag{3.14}$$

then (3.13) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\parallel w_n\parallel^2 + a\parallel w_n\parallel^2 \le \frac{1}{a}\parallel \hat f^\varepsilon_n - \hat f^\varepsilon\parallel^2. \tag{3.15}$$

Applying Gronwall Lemma to (3.15) we have

$$\begin{aligned}\parallel w_n(t)\parallel^2 &\le \parallel w_n(\tau)\parallel^2 e^{-a(t-\tau)} + \frac{1}{a}\int_\tau^t \parallel \hat f^\varepsilon_n - \hat f^\varepsilon\parallel^2 e^{-a(t-s)}\mathrm{d}s \\ &\le \parallel w_n(\tau)\parallel^2 + \frac{1}{a}\int_\tau^t \parallel \hat f^\varepsilon_n - \hat f^\varepsilon\parallel^2 \mathrm{d}s, \forall t\ge\tau. \end{aligned}\tag{3.16}$$

Note that

$$\hat f^\varepsilon_n \to \hat f^\varepsilon \ in\ L^2_{\rm loc}(\mathbb{R};H)\ and\ u_{\tau n}\to u_\tau\ in\ H\ as\ n\to\infty, \tag{3.17}$$

therefore, it follows from (3.16) that

$$\parallel w_n(t)\parallel = \parallel \hat u_n(t) - \hat u(t)\parallel \to 0\ as\ n\to\infty,$$

and (3.11) is proved, i.e., the family of processes $\{U_{\hat f^\varepsilon}(t,\tau)\}, \hat f^\varepsilon\in\mathcal{H}(f^\varepsilon)$ is $(H\times\mathcal{H}(f^\varepsilon);H)$-continuous. □

We denote by $\mathcal{K}_{\hat{f}^\varepsilon}$ the kernel of (3.8) with the external force $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)$. Let us recall that $\mathcal{K}_{\hat{f}^\varepsilon}$ is the family of all complete solutions $\{\hat{u}(t), t \in \mathbb{R}\}$ of (3.8), which are uniformly bounded in H. The set

$$\mathcal{K}_{\hat{f}^\varepsilon}(s) = \{\hat{u}(s) | \hat{u} \in \mathcal{K}_{\hat{f}^\varepsilon}\} \subset H$$

is called the kernel section of $\mathcal{K}_{\hat{f}^\varepsilon}$ at time $t = s$.

For every $\varepsilon \in [0, 1]$, the following representation of the uniform attractor $\mathcal{A}^\varepsilon$ of equation (3.1) holds:

$$\mathcal{A}^\varepsilon = \bigcup_{\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)} \mathcal{K}_{\hat{f}^\varepsilon}(0). \tag{3.18}$$

Actually, $\mathcal{K}_{\hat{f}^\varepsilon}(0)$ can be replaced by $\mathcal{K}_{\hat{f}^\varepsilon}(\tau)$, for an arbitrary $\tau \in \mathbb{R}$.

4. Uniform boundedness of $\mathcal{A}^\varepsilon$ in H

First, we consider the auxiliary linear equation with nonautonomous external force and give some useful estimates and then prove the uniform boundedness of $\mathcal{A}^\varepsilon$ in H.

Considering the linear equation

$$\mathcal{V}_t + \nu A\mathcal{V} + a\mathcal{V} = K(t), \quad \mathcal{V}|_{t=\tau} = 0, \tag{4.1}$$

we get the following lemma.

Lemma 4.1. *If $K \in L^2_{\text{loc}}(\mathbb{R}; V)$, then the above problem has a unique solution*

$$\mathcal{V} \in C(\mathbb{R}_\tau; H^2) \cap L^2_{\text{loc}}(\mathbb{R}_\tau; H^3).$$

Moreover, the inequalities

$$\| \mathcal{V}(t) \|^2 \leq C \int_\tau^t e^{-a(t-s)} \| K(s) \|^2 \, \mathrm{d}s, \tag{4.2}$$

$$\| A\mathcal{V}(t) \|^2 \leq C \int_\tau^t e^{-2a(t-s)} \| K(s) \|_V^2 \, \mathrm{d}s, \tag{4.3}$$

$$\int_t^{t+1} \| \nabla \mathcal{V}(s) \|^2 \, \mathrm{d}s \leq C(\| \mathcal{V}(t) \|^2 + \int_t^{t+1} \| K(s) \|^2 \, \mathrm{d}s) \tag{4.4}$$

$$\int_t^{t+1} \| A^{\frac{3}{2}}\mathcal{V}(s) \|^2 \, \mathrm{d}s \leq C(\| A\mathcal{V}(t) \|^2 + \int_t^{t+1} \| K(s) \|_V^2 \, \mathrm{d}s) \tag{4.5}$$

hold for every $t \geq \tau$ and some constant $C > 0$, independent of the initial time $\tau \in \mathbb{R}$.

Proof. Multiplying the equation (4.1) by $\mathcal{V}$ and $A^2\mathcal{V}$ respectively, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| \mathcal{V} \|^2 + \nu \| \nabla \mathcal{V} \|^2 + a \| \mathcal{V} \|^2 &= (K(t), \mathcal{V}) \\ &\leq \frac{a}{2} \| \mathcal{V} \|^2 + \frac{1}{2a} \| K(t) \|^2, \end{aligned} \tag{4.6}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| A\mathcal{V} \|^2 + \nu \| A^{\frac{3}{2}}\mathcal{V} \|^2 + a \| A\mathcal{V} \|^2 = (K(t), A^2\mathcal{V})$$

$$\leq \frac{1}{2\nu} \| K(t) \|_V^2 + \frac{\nu}{2} \| A^{\frac{3}{2}} \mathcal{V} \|^2 . \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \| \mathcal{V} \|^2 + a \| \mathcal{V} \|^2 \leq \frac{1}{a} \| K(t) \|^2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \| A\mathcal{V} \|^2 + 2a \| A\mathcal{V} \|^2 \leq \frac{1}{\nu} \| K(t) \|_V^2 .$$

Applying Gronwall lemma it yields

$$\| \mathcal{V}(t) \|^2 \leq \frac{1}{a} \int_\tau^t e^{-a(t-s)} \| K(s) \|^2 \, \mathrm{d}s,$$

$$\| A\mathcal{V}(t) \|^2 \leq \frac{1}{\nu} \int_\tau^t e^{-2a(t-s)} \| K(s) \|_V^2 \, \mathrm{d}s.$$

From (4.6) and (4.7) we also can get

$$\frac{\mathrm{d}}{\mathrm{d}t} \| \mathcal{V} \|^2 + 2\nu \| \nabla\mathcal{V} \|^2 \leq \frac{1}{a} \| K(t) \|^2, \tag{4.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \| A\mathcal{V} \|^2 + \nu \| A^{\frac{3}{2}}\mathcal{V} \|^2 \leq \frac{1}{\nu} \| K(t) \|_V^2 . \tag{4.9}$$

Integrating (4.8) and (4.9) on $[t, t+1]$ respectively, we obtain

$$2\nu \int_t^{t+1} \| \nabla\mathcal{V}(s) \|^2 \, \mathrm{d}s \leq \| \mathcal{V}(t) \|^2 + \frac{1}{a} \int_t^{t+1} \| K(s) \|^2 \, \mathrm{d}s, \tag{4.10}$$

$$\nu \int_t^{t+1} \| A^{\frac{3}{2}}\mathcal{V}(s) \|^2 \, \mathrm{d}s \leq \| A\mathcal{V}(t) \|^2 + \frac{1}{\nu} \int_t^{t+1} \| K(s) \|_V^2 \, \mathrm{d}s.$$

The proof is finished. □

Setting $F(t,\tau) = \int_\tau^t f_1(s)\mathrm{d}s, t \geq \tau$, we assume that

$$\sup_{t\geq\tau, \tau\in\mathbb{R}} \Big(\| F(t,\tau) \|^2 + \int_t^{t+1} \| F(s,\tau) \|_V^2 \, \mathrm{d}s \Big) \leq l^2. \tag{4.11}$$

Lemma 4.2. *Assume that $f_1 \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ and satisfies (4.11). Then the solution $v(t)$ to the Cauchy problem*

$$v_t + \nu A v + a v = f_1\Big(\frac{t}{\varepsilon}\Big), \ v|_{t=\tau} = 0 \tag{4.12}$$

with $\varepsilon \in (0, 1]$, satisfies the inequality

$$\| v(t) \|^2 + \int_t^{t+1} \| \nabla v(s) \|^2 \, \mathrm{d}s \leq C l^2 \varepsilon^2, \forall t \geq \tau, \tag{4.13}$$

where $C > 0$ is a constant independent of f_1.

Proof. Without loss of generality, we may assume $\tau = 0$. Denoting $\mathcal{V}(t) = \int_0^t v(s)\mathrm{d}s$, we have, for any $t \geq 0$,

$$\partial_t \mathcal{V}(t) = v(t) = \int_0^t \partial_t v(s)\mathrm{d}s,$$

as $v(0) = 0$. Integrating (4.12) in time, we see that the function $\mathcal{V}(t)$ solves the problem

$$\partial_t \mathcal{V} + \nu A\mathcal{V} + a\mathcal{V} = F_\varepsilon(t), \mathcal{V}|_{t=0} = 0, \tag{4.14}$$

with external force

$$F_\varepsilon(t) = \int_0^t f_1(\frac{s}{\varepsilon})\mathrm{d}s = \varepsilon \int_0^{\frac{t}{\varepsilon}} f_1(s)\mathrm{d}s = \varepsilon F(\frac{t}{\varepsilon}, 0).$$

It follows from (4.11) that

$$\sup_{t\geq 0} \| F_\varepsilon(t) \| \leq l\varepsilon$$

and

$$\int_t^{t+1} \| F_\varepsilon(s) \|_V^2 \mathrm{d}s = \varepsilon^3 \int_{\frac{t}{\varepsilon}}^{\frac{t+1}{\varepsilon}} \| F(s,0) \|_V^2 \mathrm{d}s \leq 2\varepsilon^2 \sup_{t\geq 0} \Big\{ \int_t^{t+1} \| F(s,0) \|_V^2 \mathrm{d}s \Big\} \leq 2l^2\varepsilon^2.$$

By (2.5) we have

$$\int_0^t e^{-a(t-s)} \| F_\varepsilon(s) \|^2 \mathrm{d}s \leq Cl^2\varepsilon^2, \int_0^t e^{-2a(t-s)} \| F_\varepsilon(s) \|_V^2 \mathrm{d}s \leq Cl^2\varepsilon^2.$$

So applying Lemma 5.1, we obtain

$$\| \mathcal{V}(t) \|^2 + \| A\mathcal{V}(t) \|^2 + \int_t^{t+1} \| \nabla\mathcal{V}(s) \|^2 \mathrm{d}s + \int_t^{t+1} \| A^{\frac{3}{2}}\mathcal{V}(s) \|^2 \mathrm{d}s \leq Cl^2\varepsilon^2.$$

Hence, on account of (4.14) we have

$$\| v(t) \| = \| \partial_t\mathcal{V}(t) \| \leq \| F_\varepsilon(t) \| + \nu \| A\mathcal{V}(t) \| + a \| \mathcal{V}(t) \| \leq Cl\varepsilon$$

and

$$\| \nabla v(s) \|^2 = \| \nabla(\partial_t\mathcal{V}(s)) \|^2 \leq 3 \| F_\varepsilon(s) \|_V^2 + 3\nu^2 \| A^{\frac{3}{2}}\mathcal{V}(s) \|^2 + 3a^2 \| \nabla\mathcal{V}(s) \|^2,$$

from which we derive the integral estimate

$$\int_t^{t+1} \| \nabla v(s) \|^2 \mathrm{d}s \leq Cl^2\varepsilon^2.$$

This finishes the proof. □

Theorem 4.1. *Let (4.11) holds true. Then the uniform attractors $\mathcal{A}^\varepsilon$ are uniformly (w.r.t. ε) bounded in H, that is,*

$$\sup_{\varepsilon\in[0,1]} \| \mathcal{A}^\varepsilon \| < \infty.$$

Proof. Let u be the solution to (3.1) with initial data $u_\tau \in H$. For $\varepsilon > 0$, we consider the problem

$$v_t + \nu Av + av = \varepsilon^{-\rho} f_1(\frac{t}{\varepsilon}), \quad v|_{t=\tau} = 0. \tag{4.15}$$

Lemma 4.2 provides the estimate

$$\| v(t) \|^2 + \int_t^{t+1} \| \nabla v(s) \|^2 \, \mathrm{d}s \le cl^2 \varepsilon^{2(1-\rho)}, \forall t \ge \tau. \tag{4.16}$$

Then, the function $w(t) = u(t) - v(t)$ clearly satisfies the equation

$$w_t - \nu \Delta w + aw + b|w + v|(w + v) + c|w + v|^\beta (w + v) + \nabla p = f_0 \tag{4.17}$$

with initial condition $w|_{t=\tau} = u_\tau$. Taking the inner product of (4.17) with w in H, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| w \|^2 + \nu \| \nabla w \|^2 + a \| w \|^2 + b(|w + v|(w + v) - |v|v, w) + c(|w + v|^\beta (w + v) - |v|^\beta v, w) \\
&\qquad = -b(|v|v, w) - c(|v|^\beta v, w) + (f_0, w).
\end{aligned} \tag{4.18}$$

By Lemma 2.1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| w \|^2 + \nu \| \nabla w \|^2 + a \| w \|^2 + b \cdot 2^{-5} 3^{-\frac{3}{2}} \| w \|_3^3 + c \cdot 2^{-4-\beta} 3^{-\frac{\beta+2}{2}} \| w \|_{\beta+2}^{\beta+2} \\
&\qquad \le -b(|v|v, w) - c(|v|^\beta v, w) + (f_0, w).
\end{aligned} \tag{4.19}$$

Noticing

$$(f_0, w) \le \frac{a}{2} \| w \|^2 + \frac{1}{2a} \| f_0 \|^2, \tag{4.20}$$

$$\begin{aligned}
b|(|v|v, w)| &= b\Big| \int_\Omega |v|vw \mathrm{d}x \Big| \\
&\le b\Big(\int_\Omega |w|^6 \mathrm{d}x \Big)^{1/6} \Big(\int_\Omega (|v|v)^{6/5} \mathrm{d}x \Big)^{5/6} \\
&= b \| w \|_6 \| v \|_{\frac{12}{5}}^2 \\
&\le \frac{\nu}{2d_0^2} \| w \|_6^2 + C \| v \|_{\frac{12}{5}}^4,
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
c|(|v|^\beta v, w)| &= c\Big| \int_\Omega |v|^\beta vw \mathrm{d}x \Big| \\
&\le c\Big(\int_\Omega |w|^6 \mathrm{d}x \Big)^{\frac{1}{6}} \cdot \Big(\int_\Omega (|v|^\beta v)^{6/5} \mathrm{d}x \Big)^{5/6} \\
&= c \| w \|_6 \| v \|_{\frac{6}{5}(\beta+1)}^{\beta+1} \\
&\le \frac{\nu}{2d_0^2} \| w \|_6^2 + C \| v \|_{\frac{6}{5}(\beta+1)}^{2(\beta+1)},
\end{aligned} \tag{4.22}$$

according to Sobolev inequality

$$\| v \|_p \leq d_0 \| \nabla v \|, 1 \leq p \leq 6, \tag{4.23}$$

combining (4.20)-(4.23) with (4.19) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \| w \|^2 + a \| w \|^2 \leq C \| v \|_{\frac{12}{5}}^4 + C \| v \|_{\frac{6}{5}(\beta+1)}^{2(\beta+1)} + \frac{1}{a} \| f_0 \|^2 . \tag{4.24}$$

Case I. $1 < \beta < \frac{4}{3}$.

Now we use Gagliardo-Nirenberg inequality to obtain

$$\| v \|_{\frac{12}{5}} \leq C \| \nabla v \|^{1/4} \| v \|^{3/4}, \tag{4.25}$$

$$\| v \|_{\frac{6}{5}(\beta+1)} \leq C \| \nabla v \|^{\frac{3\beta-2}{2(\beta+1)}} \| v \|^{\frac{4-\beta}{2(\beta+1)}} . \tag{4.26}$$

Considering $1 < \beta < \frac{4}{3}$, so $3\beta - 2 < 2$. Combining (4.25), (4.26) with (4.24), and according to (4.16), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \| w \|^2 + a \| w \|^2 &\leq C \| \nabla v \| \| v \|^3 + C \| \nabla v \|^{3\beta-2} \| v \|^{4-\beta} + \frac{1}{a} \| f_0 \|^2 \\ &\leq \| \nabla v \|^2 + C \| v \|^6 + C \| v \|^{\frac{2(4-\beta)}{4-3\beta}} + \frac{1}{a} \| f_0 \|^2 \\ &\leq \| \nabla v \|^2 + Cl^6 \varepsilon^{6(1-\rho)} + C(l^2 \varepsilon^{2(1-\rho)})^{\frac{4-\beta}{4-3\beta}} + \frac{1}{a} \| f_0 \|^2 .\end{aligned} \tag{4.27}$$

Let $g(s) = \| \nabla v(s) \|^2 + Cl^6 \varepsilon^{6(1-\rho)} + C(l^2 \varepsilon^{2(1-\rho)})^{\frac{4-\beta}{4-3\beta}} + \frac{1}{a} \| f_0 \|^2$. Noticing (4.16), we have

$$\begin{aligned}\int_t^{t+1} g(s) \mathrm{d}s &= \int_t^{t+1} [\| \nabla v(s) \|^2 + Cl^6 \varepsilon^{6(1-\rho)} + C(l^2 \varepsilon^{2(1-\rho)})^{\frac{4-\beta}{4-3\beta}} + \frac{1}{a} \| f_0 \|^2] \mathrm{d}s \\ &\leq C(l^2 \varepsilon^{2(1-\rho)} + l^6 \varepsilon^{6(1-\rho)} + (l^2 \varepsilon^{2(1-\rho)})^{\frac{4-\beta}{4-3\beta}} + M_0^2), \forall t \geq \tau.\end{aligned} \tag{4.28}$$

Applying Lemma 2.3 to (4.27) we have

$$\begin{aligned}\| w(t) \|^2 &\leq \| u_\tau \|^2 e^{-a(t-\tau)} + C(1 + \frac{1}{a})(l^2 \varepsilon^{2(1-\rho)} + l^6 \varepsilon^{6(1-\rho)} + (l^2 \varepsilon^{2(1-\rho)})^{\frac{4-\beta}{4-3\beta}} + M_0^2) \\ &\leq \| u_\tau \|^2 e^{-a(t-\tau)} + C(l^2 + l^6 + l^{\frac{2(4-\beta)}{4-3\beta}} + M_0^2), \forall t \geq \tau.\end{aligned} \tag{4.29}$$

Case II. $\beta = \frac{4}{3}$.

From (4.24)-(4.26) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \| w \|^2 + a \| w \|^2 &\leq C \| \nabla v \| \| v \|^3 + C \| \nabla v \|^2 \| v \|^{\frac{8}{3}} + \frac{1}{a} \| f_0 \|^2 \\ &\leq \| \nabla v \|^2 + C \| v \|^6 + C \| \nabla v \|^2 \| v \|^{\frac{8}{3}} + \frac{1}{a} \| f_0 \|^2 \\ &\leq [1 + C \| v \|^{\frac{8}{3}}] \| \nabla v \|^2 + C \| v \|^6 + \frac{1}{a} \| f_0 \|^2\end{aligned}$$

$$\leq (1 + Cl^{\frac{8}{3}}\varepsilon^{\frac{8}{3}(1-\rho)}) \parallel \nabla v \parallel^2 + Cl^6\varepsilon^{6(1-\rho)} + \frac{1}{a} \parallel f_0 \parallel^2 . \tag{4.30}$$

Similar to the derivation of (4.29), we get

$$\begin{aligned}\parallel w(t) \parallel^2 &\leq \parallel u_\tau \parallel^2 e^{-a(t-\tau)} + C(1 + \frac{1}{a})[(1 + l^{\frac{8}{3}}\varepsilon^{\frac{8}{3}(1-\rho)})l^2\varepsilon^{2(1-\rho)} + l^6\varepsilon^{6(1-\rho)} + M_0^2],\\ &\leq \parallel u_\tau \parallel^2 e^{-a(t-\tau)} + C(l^2 + l^{\frac{14}{3}} + l^6 + M_0^2), \forall t \geq \tau.\end{aligned} \tag{4.31}$$

Recalling that $u = w + v$, using (4.16), (4.29) and (4.31), we end up with

$$\parallel u(t) \parallel^2 \leq \parallel u_\tau \parallel^2 e^{-a(t-\tau)} + C(l^2 + l^6 + l^{\frac{2(4-\beta)}{4-3\beta}} + l^{\frac{14}{3}} + M_0^2), \forall t \geq \tau. \tag{4.32}$$

Thus, for every $\varepsilon \leq \varepsilon_0$, the process $\{U_{f^\varepsilon}(t, \tau)\}$ has the absorbing set

$$B_0 := \{u \in H | \parallel u \parallel^2 \leq C(l^2 + l^6 + l^{\frac{2(4-\beta)}{4-3\beta}} + l^{\frac{14}{3}} + M_0^2)\}.$$

On the other hand, if $\varepsilon_0 < \varepsilon \leq 1$, the process $\{U_{f^\varepsilon}(t, \tau)\}$ possesses also the absorbing set (cf (3.2))

$$B^{\varepsilon_0} = \{u \in H | \parallel u \parallel \leq C_0 Q_{\varepsilon_0}\}.$$

In conclusion, for every $\varepsilon \in [0, 1]$, the bounded set

$$B^* = B_0 \cup B^{\varepsilon_0}$$

is an absorbing set for $\{U_{f^\varepsilon}(t, \tau)\}$ which is independent of ε. Since $\mathcal{A}^\varepsilon \subset B^*$, the proof is completed. □

5. Convergence of the uniform attractors

The main result of this section is the following.

Theorem 5.1. *Let (4.11) hold. Then, the uniform attractor $\mathcal{A}^\varepsilon$ converges to $\mathcal{A}^0$ as $\varepsilon \to 0^+$ in the following sense:*

$$\lim_{\varepsilon \to 0^+} \text{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) = 0.$$

The proof of this theorem requires some steps. Now, we shall study the difference of two solutions to (3.1) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, sharing the same initial data. We denote

$$u^\varepsilon(t) = U_{f^\varepsilon}(t, \tau)u_\tau,$$

with u_τ belonging to the absorbing set B^* found in the previous section. Owing to (4.32), we have the uniform bound:

$$\parallel u^\varepsilon(t) \parallel^2 \leq R_1^2, \tag{5.1}$$

for some $R_1 = R_1(l, M_0)$ because the size of B^* depends on l and M_0. In particular, for $\varepsilon = 0$, since $u_\tau \in B^*$, we have the bound

$$\parallel u^0(t) \parallel^2 \leq R_0^2, \tag{5.2}$$

for some $R_0 = R_0(l, M_0)$.

Lemma 5.1. *For every $\varepsilon \in (0,1]$, every $\tau \in \mathbb{R}$ and every $u_\tau \in B^*$, the deviation $\tilde{w}(t) = u^\varepsilon(t) - u^0(t)$ with $u^\varepsilon(0) = u^0(0) = u_\tau$, fulfills the estimate*

$$\| \tilde{w}(t) \|^2 \leq Cl^2 \varepsilon^{2(1-\rho)}, \forall t \geq \tau, \tag{5.3}$$

for some positive constant C independent of ε.

Proof. Since the deviation $\tilde{w}(t)$ solves

$$\tilde{w}_t - \nu\Delta\tilde{w} + a\tilde{w} + b|u^\varepsilon|u^\varepsilon - b|u^0|u^0 + c|u^\varepsilon|^\beta u^\varepsilon - c|u^0|^\beta u^0 = \varepsilon^{-\rho} f_1(x, \frac{t}{\varepsilon}), \tilde{w}|_{t=\tau} = 0, \tag{5.4}$$

the difference $q(t) = \tilde{w}(t) - v(t)$, where $v(t)$ is the solution to (4.15), fulfills the Cauchy problem

$$q_t - \nu\Delta q + aq + b|u^\varepsilon|u^\varepsilon - b|u^0|u^0 + c|u^\varepsilon|^\beta u^\varepsilon - c|u^0|^\beta u^0 = 0, q|_{t=\tau} = 0. \tag{5.5}$$

At this point, we take the scalar product in H of (5.5) with q, so getting

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| q \|^2 + \nu \| \nabla q \|^2 + a \| q \|^2 + b(|u^\varepsilon|u^\varepsilon - |u^0|u^0, \tilde{w}) + c(|u^\varepsilon|^\beta u^\varepsilon - |u^0|^\beta u^0, \tilde{w}) \\ &\quad = b(|u^\varepsilon|u^\varepsilon - |u^0|u^0, v) + c(|u^\varepsilon|^\beta u^\varepsilon - |u^0|^\beta u^0, v). \end{aligned} \tag{5.6}$$

Noting the first term on the right-hand side of (5.6) is given by

$$b(|u^\varepsilon|u^\varepsilon - |u^0|u^0, v) = b(|u^\varepsilon|\tilde{w}, v) + b((|u^\varepsilon| - |u^0|)u^0, v), \tag{5.7}$$

we now proceed to estimate the first term on the right-hand side of (5.7). Since

$$\begin{aligned} b(|u^\varepsilon|\tilde{w}, v) &\leq b\int_\Omega |u^\varepsilon||\tilde{w}||v|\mathrm{d}x \\ &\leq b\int_\Omega |u^\varepsilon|(|v| + |q|)|v|\mathrm{d}x \\ &\leq b\int_\Omega |u^\varepsilon||v|^2\mathrm{d}x + b\int_\Omega |u^\varepsilon||q||v|\mathrm{d}x, \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} b\int_\Omega |u^\varepsilon||q||v|\mathrm{d}x &\leq b\Big(\int_\Omega |q|^6\mathrm{d}x\Big)^{\frac{1}{6}}\Big(\int_\Omega |u^\varepsilon|^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_\Omega |v|^3\mathrm{d}x\Big)^{\frac{1}{3}} \\ &= b \| q \|_6 \| u^\varepsilon \| \| v \|_3 \\ &\leq \frac{\nu}{4d_0^2} \| q \|_6^2 + C \| u^\varepsilon \|^2 \| v \|_3^2 \\ &\leq \frac{\nu}{4} \| \nabla q \|^2 + C \| u^\varepsilon \|^2 \| \nabla v \|^2, \end{aligned} \tag{5.9}$$

$$\begin{aligned} b\int_\Omega |u^\varepsilon||v|^2\mathrm{d}x &\leq b\Big(\int_\Omega |u^\varepsilon|^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_\Omega |v|^6\mathrm{d}x\Big)^{\frac{1}{6}}\Big(\int_\Omega |v|^3\mathrm{d}x\Big)^{\frac{1}{3}} \\ &= b \| u^\varepsilon \| \| v \|_6 \| v \|_3 \end{aligned}$$

$$\leq \frac{1}{2d_0^2} \parallel v \parallel_6^2 + C \parallel u^\varepsilon \parallel^2 \parallel v \parallel_3^2$$
$$\leq \frac{1}{2} \parallel \nabla v \parallel^2 + C \parallel u^\varepsilon \parallel^2 \parallel \nabla v \parallel^2, \tag{5.10}$$

it follows from (5.8)-(5.10) that

$$b(|u^\varepsilon|\tilde{w}, v) \leq \frac{\nu}{4} \parallel \nabla q \parallel^2 + \frac{1}{2} \parallel \nabla v \parallel^2 + C \parallel u^\varepsilon \parallel^2 \parallel \nabla v \parallel^2 . \tag{5.11}$$

Now, let us estimate the second term on the right-hand side of (5.7). Noting

$$\begin{aligned} b((|u^\varepsilon| - |u^0|)u^0, v) &\leq b \int_\Omega |u^\varepsilon - u^0||u^0||v| \mathrm{d}x \\ &= b \int_\Omega |\tilde{w}||u^0||v| \mathrm{d}x \\ &\leq b \int_\Omega (|q| + |v|)|u^0||v| \mathrm{d}x \\ &= b \int_\Omega |u^0||q||v| \mathrm{d}x + b \int_\Omega |v|^2 |u^0| \mathrm{d}x, \end{aligned} \tag{5.12}$$

similar arguments as (5.9) and (5.10), we have

$$b \int_\Omega |u^0||q||v| \mathrm{d}x \leq \frac{\nu}{4} \parallel \nabla q \parallel^2 + C \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2, \tag{5.13}$$

and

$$b \int_\Omega |v|^2 |u^0| \mathrm{d}x \leq \frac{1}{2} \parallel \nabla v \parallel^2 + C \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2 . \tag{5.14}$$

Hence, from (5.12)-(5.14) we get

$$b((|u^\varepsilon| - |u^0|)u^0, v) \leq \frac{\nu}{4} \parallel \nabla q \parallel^2 + \frac{1}{2} \parallel \nabla v \parallel^2 + C \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2 . \tag{5.15}$$

Combining (5.11), (5.15) with (5.7), we have

$$b(|u^\varepsilon|u^\varepsilon - |u^0|u^0, v) \leq \frac{\nu}{2} \parallel \nabla q \parallel^2 + \parallel \nabla v \parallel^2 + C(\parallel u^\varepsilon \parallel^2 + \parallel u^0 \parallel^2) \parallel \nabla v \parallel^2 . \tag{5.16}$$

Noting the second term on the right-hand side of (5.6) is given by

$$c(|u^\varepsilon|^\beta u^\varepsilon - |u^0|^\beta u^0, v) = c(|u^\varepsilon|^\beta \tilde{w}, v) + c((|u^\varepsilon|^\beta - |u^0|^\beta)u^0, v), \tag{5.17}$$

we now proceed to estimate the first term on the right-hand side of (5.17). Since

$$\begin{aligned} c(|u^\varepsilon|^\beta \tilde{w}, v) &\leq c \int_\Omega |u^\varepsilon|^\beta (|q| + |v|)|v| \mathrm{d}x \\ &\leq c \int_\Omega |u^\varepsilon|^\beta |q||v| \mathrm{d}x + c \int_\Omega |u^\varepsilon|^\beta |v|^2 \mathrm{d}x, \end{aligned} \tag{5.18}$$

so we should estimate the right-hand side of the last inequality in (5.18) term by term. Because

$$\begin{aligned} c\int_\Omega |u^\varepsilon|^\beta |q||v|\mathrm{d}x &\le c\Big(\int_\Omega |q|^6\mathrm{d}x\Big)^{\frac{1}{6}}\Big(\int_\Omega |u^\varepsilon|^2\mathrm{d}x\Big)^{\frac{\beta}{2}}\Big(\int_\Omega |v|^{\frac{6}{5-3\beta}}\mathrm{d}x\Big)^{\frac{5-3\beta}{6}} \\ &= c \parallel q \parallel_6 \parallel u^\varepsilon \parallel^\beta \parallel v \parallel_{\frac{6}{5-3\beta}} \\ &\le \frac{\nu}{4d_0^2} \parallel q \parallel_6^2 + C \parallel u^\varepsilon \parallel^{2\beta} \parallel v \parallel^2_{\frac{6}{5-3\beta}} \\ &\le \frac{\nu}{4} \parallel \nabla q \parallel^2 + C \parallel u^\varepsilon \parallel^{2\beta} \parallel \nabla v \parallel^2 \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} c\int_\Omega |u^\varepsilon|^\beta |v|^2\mathrm{d}x &\le c\Big(\int_\Omega |v|^6\mathrm{d}x\Big)^{\frac{1}{6}}\Big(\int_\Omega |u^\varepsilon|^2\mathrm{d}x\Big)^{\frac{\beta}{2}}\Big(\int_\Omega |v|^{\frac{6}{5-3\beta}}\mathrm{d}x\Big)^{\frac{5-3\beta}{6}} \\ &= c \parallel v \parallel_6 \cdot \parallel u^\varepsilon \parallel^\beta \cdot \parallel v \parallel_{\frac{6}{5-3\beta}} \\ &\le \frac{1}{2} \parallel \nabla v \parallel^2 + C \parallel u^\varepsilon \parallel^{2\beta} \parallel \nabla v \parallel^2, \end{aligned} \tag{5.20}$$

where the last inequalities in (5.19) and (5.20) are valid only if $\frac{6}{5-3\beta} \le 6$, i.e. $\beta \le \frac{4}{3}$, so combining (5.19), (5.20) with (5.18), we have

$$c(|u^\varepsilon|^\beta \tilde{w}, v) \le \frac{\nu}{4} \parallel \nabla q \parallel^2 + \frac{1}{2} \parallel \nabla v \parallel^2 + C \parallel u^\varepsilon \parallel^{2\beta} \parallel \nabla v \parallel^2 . \tag{5.21}$$

Now let us estimate the second term on the right-hand side of (5.17). Since

$$\begin{aligned} c((|u^\varepsilon|^\beta - |u^0|^\beta)u^0, v) &\le c\int_\Omega \Big||u^\varepsilon|^\beta - |u^0|^\beta\Big| |u^0||v|\mathrm{d}x \\ &\le C\int_\Omega \Big||u^\varepsilon|^{\beta-1} + |u^0|^{\beta-1}\Big| |\tilde{w}||u^0||v|\mathrm{d}x \\ &\le C\int_\Omega |u^\varepsilon|^{\beta-1}|\tilde{w}||u^0||v|\mathrm{d}x + C\int_\Omega |u^0|^{\beta-1}|\tilde{w}||u^0||v|\mathrm{d}x, \end{aligned} \tag{5.22}$$

in the second inequality of (5.22) we used the fact that

$$|x^p - y^p| \le Cp(x^{p-1} + y^{p-1})|x - y|$$

for any $x, y \ge 0$, where C is an absolute constant. So let us estimate the right-hand side of the last inequality in (5.22) term by term. For the first term, we have

$$C\int_\Omega |u^\varepsilon|^{\beta-1}|\tilde{w}||u^0||v|\mathrm{d}x \le C\int_\Omega |u^\varepsilon|^{\beta-1}|q||u^0||v|\mathrm{d}x + C\int_\Omega |u^\varepsilon|^{\beta-1}|v|^2|u^0|\mathrm{d}x, \tag{5.23}$$

and

$$\begin{aligned} C\int_\Omega |u^\varepsilon|^{\beta-1}|q||u^0||v|\mathrm{d}x &\le C\Big(\int_\Omega |q|^6\mathrm{d}x\Big)^{\frac{1}{6}}\Big(\int_\Omega |u^\varepsilon|^2\mathrm{d}x\Big)^{\frac{\beta-1}{2}}\Big(\int_\Omega |u^0|^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_\Omega |v|^{\frac{6}{5-3\beta}}\Big)^{\frac{5-3\beta}{6}} \\ &= C \parallel q \parallel_6 \parallel u^\varepsilon \parallel^{\beta-1} \parallel u^0 \parallel\parallel v \parallel_{\frac{6}{5-3\beta}} \end{aligned}$$

$$\le \frac{\nu}{8} \parallel \nabla q \parallel^2 + C \parallel u^\varepsilon \parallel^{2(\beta-1)} \parallel u^0 \parallel^2 \parallel v \parallel^2_{\frac{6}{5-3\beta}}$$
$$\le \frac{\nu}{8} \parallel \nabla q \parallel^2 + C \parallel u^\varepsilon \parallel^{2(\beta-1)} \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2, \tag{5.24}$$

similarly,

$$C \int_\Omega |u^\varepsilon|^{\beta-1} |v|^2 |u^0| \mathrm{d}x \le \frac{1}{4} \parallel \nabla v \parallel^2 + C \parallel u^\varepsilon \parallel^{2(\beta-1)} \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2 . \tag{5.25}$$

It follows from (5.23)-(5.25) that

$$C \int_\Omega |u^\varepsilon|^{\beta-1} |\tilde{w}| |u^0| |v| \mathrm{d}x \le \frac{\nu}{8} \parallel \nabla q \parallel^2 + \frac{1}{4} \parallel \nabla v \parallel^2 + C \parallel u^\varepsilon \parallel^{2(\beta-1)} \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2 . \tag{5.26}$$

Similar arguments as (5.23)-(5.26), for the second term on the right-hand side of inequality (5.22), we have

$$C \int_\Omega |u^0|^{\beta-1} |\tilde{w}| |u^0| |v| \mathrm{d}x \le \frac{\nu}{8} \parallel \nabla q \parallel^2 + \frac{1}{4} \parallel \nabla v \parallel^2 + C \parallel u^0 \parallel^{2(\beta-1)} \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2 . \tag{5.27}$$

Combining (5.26), (5.27) with (5.22), we have

$$c((|u^\varepsilon|^\beta - |u^0|^\beta)u^0, v) \le \frac{\nu}{4} \parallel \nabla q \parallel^2 + \frac{1}{2} \parallel \nabla v \parallel^2 + C(\parallel u^\varepsilon \parallel^{2(\beta-1)} + \parallel u^0 \parallel^{2(\beta-1)}) \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2 . \tag{5.28}$$

So it follows from (5.17), (5.21) and (5.28) that

$$c(|u^\varepsilon|^\beta u^\varepsilon - |u^0|^\beta u^0, v) \le \frac{\nu}{2} \parallel \nabla q \parallel^2 + \parallel \nabla v \parallel^2 + C \parallel u^\varepsilon \parallel^{2\beta} \parallel \nabla v \parallel^2$$
$$+ C(\parallel u^\varepsilon \parallel^{2(\beta-1)} + \parallel u^0 \parallel^{2(\beta-1)}) \parallel u^0 \parallel^2 \parallel \nabla v \parallel^2 . \tag{5.29}$$

Now, considering (5.6), (5.16) and (5.29), and according to (5.1) and (5.2), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel q \parallel^2 + 2a \parallel q \parallel^2 \le 4 \parallel \nabla v \parallel^2 + C(R_1^2 + R_0^2) \parallel \nabla v \parallel^2 + CR_1^{2\beta} \parallel \nabla v \parallel^2$$
$$+ C(R_1^{2(\beta-1)} + R_0^{2(\beta-1)})R_0^2 \parallel \nabla v \parallel^2$$
$$\le C(1 + R_0^2 + R_1^2 + R_1^{2\beta} + (R_0^{2(\beta-1)} + R_1^{2(\beta-1)})R_0^2) \parallel \nabla v \parallel^2 . \tag{5.30}$$

Recalling that $\parallel q(\tau) \parallel = 0$ and (4.16), Lemma 2.3 entails

$$\parallel q(t) \parallel^2 \le C(1 + \frac{1}{2a})(1 + R_0^2 + R_1^2 + R_1^{2\beta} + (R_0^{2(\beta-1)} + R_1^{2(\beta-1)})R_0^2) l^2 \varepsilon^{2(1-\rho)}$$
$$\le C l^2 \varepsilon^{2(1-\rho)}.$$

Finally, as $\tilde{w} = q + v$, using (4.16) to control $\parallel v \parallel$, we obtain the desired conclusion (5.3). □

In order to study the convergence of the uniform attractors, we actually need a generalization of Lemma 5.1, which applies to the whole family of equations

$$\hat{u}_t + \nu A \hat{u} + a \hat{u} + G(\hat{u}) = \hat{f}^\varepsilon, \hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon), \tag{5.31}$$

with the external force $\hat{f} = \hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)$. To this end, we observe that every function $\hat{f}_1 \in \mathcal{H}(f_1)$ fulfills the inequality (4.11). Defining

$$\hat{F}_1(t, \tau) = \int_\tau^t \hat{f}_1(s) \mathrm{d}s, t \ge \tau,$$

we have

$$\sup_{t\geq\tau,\tau\in\mathbb{R}}\Big\{\parallel \hat{F}_1(t,\tau)\parallel^2+\int_t^{t+1}\parallel \hat{F}_1(s,\tau)\parallel_V^2 \,\mathrm{d}s\Big\}\leq l^2. \tag{5.32}$$

For any $\varepsilon\in[0,1]$, let $\hat{u}^\varepsilon(t)=U_{\hat{f}^\varepsilon}(t,\tau)u_\tau$ be the solution to (5.31) with external force $\hat{f}^\varepsilon=\hat{f}_0+\varepsilon^{-\rho}\hat{f}_1(\cdot/\varepsilon)\in\mathcal{H}(f^\varepsilon)$ and $u_\tau\in B^*$. For $\varepsilon>0$, we consider the deviation $\hat{w}(t)=\hat{u}^\varepsilon(t)-\hat{u}^0(t)$.

Lemma 5.2. *The inequality*

$$\parallel \hat{w}(t)\parallel^2\leq Cl^2\varepsilon^{2(1-\rho)},\forall t\geq\tau, \tag{5.33}$$

holds, where C is independent of ε.

Proof. As the similar argument to the proof of Lemma 5.1, with $\hat{u}^\varepsilon,\hat{f}_0$ and $\hat{f}_1$ in place of u^ε, f_0 and f_1, respectively. Noting that (5.2) still holds for $\hat{u}^0$, and the family $\{U_{\hat{f}^\varepsilon}(t,\tau)\}(\hat{f}^\varepsilon\in\mathcal{H}(f^\varepsilon))$ is $(H\times\mathcal{H}(f^\varepsilon),H)$-continuous, and using (5.32) in place of (4.11), finally complete the proof of the lemma. □

We can now complete the proof of Theorem 5.1, using the following argument from [27], which we report in some detail for the reader's convenience.

Proof of Theorem 5.1 Let $\varepsilon>0$ and $u^\varepsilon\in\mathcal{A}^\varepsilon$. Thus, in view of (3.18), there exists a complete bounded trajectory $\hat{u}^\varepsilon(t)$ of (5.31), with the external force

$$\hat{f}^\varepsilon=\hat{f}_0+\varepsilon^{-\rho}\hat{f}_1(\cdot/\varepsilon)\in\mathcal{H}(f^\varepsilon),\hat{f}_0\in\mathcal{H}(f_0),\hat{f}_1\in\mathcal{H}(f_1)$$

such that $\hat{u}^\varepsilon(0)=u^\varepsilon$. For every $L\geq0$ to be specified later, consider the vector

$$\hat{u}^\varepsilon(-L)\in\mathcal{A}^\varepsilon\subset B^*.$$

From the straightforward equality

$$u^\varepsilon=U_{\hat{f}^\varepsilon}(0,-L)\hat{u}^\varepsilon(-L),$$

by applying Lemma 5.2, we have that

$$\parallel u^\varepsilon-U_{\hat{f}_0}(0,-L)\hat{u}^\varepsilon(-L)\parallel\leq Cl\varepsilon^{1-\rho}. \tag{5.34}$$

On the other hand, the set $\mathcal{A}^0$ attracts $U_{\hat{f}_0}(t,-L)B^*$, uniformly as $\hat{f}_0\in\mathcal{H}(f_0)$. Then, for every $\delta>0$, there is $T=T(\delta)\geq0$, independent of L, such that

$$\mathrm{dist}_H(U_{\hat{f}_0}(T-L,-L)\hat{u}^\varepsilon(-L),\mathcal{A}^0)\leq\delta. \tag{5.35}$$

Setting $L=T$, and collecting the two above inequalities, we readily get

$$\mathrm{dist}_H(u^\varepsilon,\mathcal{A}^0)\leq Cl\varepsilon^{1-\rho}+\delta.$$

Since $u^\varepsilon\in\mathcal{A}^\varepsilon$ and $\delta>0$ are arbitrary, taking the limit $\varepsilon\to0^+$, the conclusion follows. □

6. Conclusion

In this paper, we investigated a class of three-dimensional Brinkman-Forchheimer equation with oscillating external forces $f^\varepsilon(x,t)=f_0(x,t)+\varepsilon^{-\rho}f_1(x,\frac{t}{\varepsilon})$. Based on some translation-compactness assumptions on the external forces, we obtained the uniform boundedness of the uniform attractor $\mathcal{A}^\varepsilon$ of the system (1.1) in $(L^2(\Omega))^3$, and the convergence of $\mathcal{A}^\varepsilon$ to the attractor $\mathcal{A}^0$ of the system (1.2) as $\varepsilon\to0^+$. To prove the uniform boundedness and the convergence of the uniform attractors, the Gagliardo-Nirenberg inequality is needed. In the proof process, we concluded that the parameter $\beta\in(1,\frac{4}{3}]$.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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