Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(2): 1476–1483.
DOI:10.3934/math.2020101
Received: 16 October 2019
Accepted: 14 January 2020
Published: 21 January 2020

Research article

Ostrowski type inequalities via some exponentially convex functions with applications

Naila Mehreen*and Matloob Anwar

School of Natural Sciences, National University of Sciences and Technology, H-12 Islamabad, Pakistan

* **Correspondence:** Email: nailamehreen@gmail.com; Tel: +923317136419.

Abstract: In this paper, we obtain ostrowski type inequalities for exponentially convex function and exponentially s-convex function in second sense. Applications to some special means are also obtain. Here we extend the results of some previous investigations.

Keywords: Osrowski's inequality; exponentially convex function; exponentially s-convex function
Mathematics Subject Classification: 26A15, 26A51, 26D10

1. Introduction

Let $\curlyvee : [h_1, h_2] \to \mathbb{R}$ be a differentiable mapping on (h_1, h_2) whose derivative $\curlyvee' : ((h_1, h_2) \to \mathbb{R}$ is bounded on (h_1, h_2), i.e., $|\curlyvee'(l)| \leq M$ for all $l \in (h_1, h_2)$. Then the following inequality holds:

$$\curlyvee(l) - \frac{1}{h_2 - h_1} \int_{h_1}^{h_2} \curlyvee(t)dt \leq \frac{M}{h_2 - h_1} \left[\frac{(l - h_1)^2 + (h_2 - l)^2}{2} \right], \tag{1.1}$$

for all $l \in [h_1, h_2]$. Many authors find the inequality (1.1) for other generalized convex functions. For more results and details see [1, 3–6, 8–10].

Awan et al. [2] introduced following new class of convex functions.

Definition 1.1. ([2]) A function $\curlyvee : \mathcal{H} \subseteq \mathbb{R} \to \mathbb{R}$ is called exponentially convex, if

$$\curlyvee(\tau h_1 + (1 - \tau)h_2) \leq \tau \frac{\curlyvee(h_1)}{e^{\alpha h_1}} + (1 - \tau) \frac{\curlyvee(h_2)}{e^{\alpha h_2}}, \tag{1.2}$$

for all $h_1, h_2 \in \mathcal{H}$, $\tau \in [0, 1]$ and $\alpha \in \mathbb{R}$. If the inequality (1.2) is in reversed order then $\curlyvee$ is called exponentially concave.

Mehreen and Anwar [7] introduced an other class of functions as:

Definition 1.2. ([7]) Let $s \in (0,1]$ and $\mathcal{H} \subset \mathbb{R}_0$ be an interval. A function $\curlyvee : \mathcal{H} \to \mathbb{R}$ is called exponentially s-convex in the second sense, if

$$\curlyvee(\tau h_1 + (1-\tau)h_2) \leq \tau^s \frac{\curlyvee(h_1)}{e^{\alpha h_1}} + (1-\tau)^s \frac{\curlyvee(h_2)}{e^{\alpha h_2}}, \tag{1.3}$$

for all $h_1, h_2 \in \mathcal{H}$, $\tau \in [0,1]$ and $\alpha \in \mathbb{R}$. If (1.3) is in reversed order then $\curlyvee$ is called exponentially s-concave.

Example 1.1. A function $\curlyvee : [0,\infty) \to \mathbb{R}$, defined by $\curlyvee(l) = \ln(l)$ for $s \in (0,1)$ is an exponentially s-convex in the second sense, for all $\alpha \leq -1$.

Mehreen and Anwar [7] proved following Hadamard's inequality for exponentially s-convex in second sense.

Theorem 1.2. *([7]) Let $\curlyvee : \mathcal{H} \subset \mathbb{R}_0 \to \mathbb{R}$ be an integrable exponentially s-convex function in the second sense on $\mathcal{H}^\circ$. Then for $h_1, h_2 \in \mathcal{H}$ with $h_1 < h_2$ and $\alpha \in \mathbb{R}$, we have*

$$2^{s-1} \curlyvee\left(\frac{h_1+h_2}{2}\right) \leq \frac{1}{h_2-h_1}\int_{h_1}^{h_2} \frac{\curlyvee(l)}{e^{\alpha l}} dl \leq A_1(\tau)\frac{\curlyvee(h_1)}{e^{\alpha h_1}} + A_2(\tau)\frac{\curlyvee(h_2)}{e^{\alpha h_2}}, \tag{1.4}$$

where

$$A_1(\tau) = \int_0^1 \frac{\tau^s d\tau}{e^{\alpha(\tau h_1 + (1-\tau)h_2)}} \quad \text{and} \quad A_2(\tau) = \int_0^1 \frac{(1-\tau)^s d\tau}{e^{\alpha(\tau h_1 + (1-\tau)h_2)}}.$$

2. Ostrowski type inequalities

First we state the following lemma given in [3].

Lemma 2.1. *([3]) Let $\curlyvee : \mathcal{H} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{H}^\circ$ for $h_1, h_2 \in \mathcal{H}$ with $h_1 < h_2$. If $\curlyvee' \in L_1[h_1, h_2]$, then the following equality holds:*

$$\begin{aligned} &\curlyvee(l) - \frac{1}{h_2-h_1}\int_{h_1}^{h_2} \curlyvee(t)dt \\ &\leq \frac{(l-h_1)^2}{h_2-h_1}\int_0^1 \tau \curlyvee'(\tau l + (1-\tau)h_1)d\tau - \frac{(h_2-l)^2}{h_2-h_1}\int_0^1 \tau \curlyvee'(\tau l + (1-\tau)h_2)d\tau, \end{aligned} \tag{2.1}$$

for each $l \in [h_1, h_2]$.

Now we prove the following theorem.

Theorem 2.1. *Let $\curlyvee : \mathcal{H} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{H}^\circ$ such that $\curlyvee' \in [h_1, h_2]$ for $h_1, h_2 \in \mathcal{H}$ with $h_1 < h_2$. If $\curlyvee'$ is exponentially s-convex in second sense on $[h_1, h_2]$ for some $s \in (0,1]$ and $|\curlyvee'(l)| \leq M$, $l \in [h_1, h_2]$, then the following inequality holds:*

$$\begin{aligned} &\left|\curlyvee(l) - \frac{1}{h_2-h_1}\int_{h_1}^{h_2} \curlyvee(t)dt\right| \\ &\leq \frac{M}{h_2-h_1}\left[\frac{(l-h_1)^2}{e^{\alpha l}(s+2)} + \frac{(l-h_1)^2}{e^{\alpha h_1}(s+1)(s+2)} + \frac{(h_2-l)^2}{e^{\alpha l}(s+2)} + \frac{(h_2-l)^2}{e^{\alpha h_2}(s+1)(s+2)}\right], \end{aligned} \tag{2.2}$$

for each $l \in [h_1, h_2]$.

Proof. Using Lemma 2.1 and since $\curlyvee'$ is exponentially s-convex, we have

$$\begin{aligned}
&\left|\curlyvee(l) - \frac{1}{h_2 - h_1}\int_{h_1}^{h_2} \curlyvee(t)dt\right| \\
&\leq \frac{(l-h_1)^2}{h_2-h_1}\int_0^1 \tau|\curlyvee'(\tau l + (1-\tau)h_1)|d\tau + \frac{(h_2-l)^2}{h_2-h_1}\int_0^1 \tau|\curlyvee'(\tau l + (1-\tau)h_2)|d\tau \\
&\leq \frac{(l-h_1)^2}{h_2-h_1}\int_0^1 \tau\left[\tau^s\frac{|\curlyvee'(l)|}{e^{\alpha l}} + (1-\tau)^s\frac{|\curlyvee'(h_1)|}{e^{\alpha h_1}}\right]d\tau \\
&\quad + \frac{(h_2-l)^2}{h_2-h_1}\int_0^1 \tau\left[\tau^s\frac{|\curlyvee'(l)|}{e^{\alpha l}} + (1-\tau)^s\frac{|\curlyvee'(h_2)|}{e^{\alpha h_2}}\right]d\tau \qquad (2.3)\\
&\leq \frac{M(l-h_1)^2}{h_2-h_1}\left[\frac{1}{e^{\alpha l}(s+2)} + \frac{1}{e^{\alpha h_1}(s+1)(s+2)}\right] \\
&\quad + \frac{M(h_2-l)^2}{h_2-h_1}\left[\frac{1}{e^{\alpha l}(s+2)} + \frac{1}{e^{\alpha h_2}(s+1)(s+2)}\right] \\
&= \frac{M}{h_2-h_1}\left[\frac{(l-h_1)^2}{e^{\alpha l}(s+2)} + \frac{(l-h_1)^2}{e^{\alpha h_1}(s+1)(s+2)} + \frac{(h_2-l)^2}{e^{\alpha l}(s+2)} + \frac{(h_2-l)^2}{e^{\alpha h_2}(s+1)(s+2)}\right].
\end{aligned}$$

Since

$$\int_0^1 \tau^{s+1}d\tau = \frac{1}{s+2} \quad \text{and} \quad \int_0^1 \tau(1-\tau)^s d\tau = \frac{1}{(s+1)(s+2)}.$$

This completes the proof. □

Remark 2.1. *In Theorem* 2.1*, by letting* $\alpha = 0$*, we get inequality* 2.1 *of Theorem* 2 *in* [1].

Corollary 2.1. *Under the similar considerations of Theorem* 2.1*, by taking* $s = 1$*, we get*

$$\begin{aligned}
&\curlyvee(l) - \frac{1}{h_2-h_1}\int_{h_1}^{h_2}\curlyvee(t)dt \\
&\leq \frac{M}{h_2-h_1}\left[\frac{(l-h_1)^2}{3e^{\alpha l}} + \frac{(l-h_1)^2}{6e^{\alpha h_1}} + \frac{(h_2-l)^2}{3e^{\alpha l}} + \frac{(h_2-l)^2}{6e^{\alpha h_2}}\right].
\end{aligned} \qquad (2.4)$$

Remark 2.2. *In Corollary* 2.1*, by letting* $\alpha = 0$*, we get inequality* (1.1)*.*

Theorem 2.2. *Let* $\curlyvee : \mathcal{H} \to \mathbb{R}$ *be a differentiable mapping on* $\mathcal{H}^\circ$ *such that* $\curlyvee' \in [h_1, h_2]$ *for* $h_1, h_2 \in \mathcal{H}$ *with* $h_1 < h_2$*. If* $|\curlyvee'|^q$ *is exponentially s-convex in the second sense on* $[h_1, h_2]$ *for some* $s \in (0, 1]$*,* p*,* $q > 1$*,* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $|\curlyvee'(l)| \leq M$*,* $l \in [h_1, h_2]$*, then the following inequality holds:*

$$\begin{aligned}
&\left|\curlyvee(l) - \frac{1}{h_2-h_1}\int_{h_1}^{h_2}\curlyvee(t)dt\right| \\
&\leq \frac{M}{(h_2-h_1)(1+p)^{\frac{1}{p}}}\left[(l-h_1)^2\left(\frac{1}{(s+1)e^{\alpha l}} + \frac{1}{(s+1)e^{\alpha h_1}}\right)^{\frac{1}{q}}\right. \qquad (2.5)\\
&\quad \left. + (h_2-l)^2\left(\frac{1}{(s+1)e^{\alpha l}} + \frac{1}{(s+1)e^{\alpha h_2}}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

for each $l \in [h_1, h_2]$*.*

Proof. Let $p > 1$. From Lemma 2.1 and using Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\curlyvee(l) - \frac{1}{h_2 - h_1}\int_{h_1}^{h_2} \curlyvee(t)dt\right| \\
&\leq \frac{(l-h_1)^2}{h_2-h_1}\int_0^1 \tau|\curlyvee'(\tau l + (1-\tau)h_1)|d\tau + \frac{(h_2-l)^2}{h_2-h_1}\int_0^1 \tau|\curlyvee'(\tau l + (1-\tau)h_2)|d\tau \\
&\leq \frac{(l-h_1)^2}{h_2-h_1}\left(\int_0^1 \tau^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |\curlyvee'(\tau l + (1-\tau)h_1)|^q d\tau\right)^{\frac{1}{q}} \\
&\quad + \frac{(h_2-l)^2}{h_2-h_1}\left(\int_0^1 \tau^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |\curlyvee'(\tau l + (1-\tau)h_2)|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.6}$$

Since $|\curlyvee'|^q$ is exponentially s-convex in the second sense and $|\curlyvee'(l)| \leq M$, then we have

$$\begin{aligned}
\int_0^1 |\curlyvee'(\tau l + (1-\tau)h_1)|^q d\tau &= \int_0^1 \left[\tau^s \frac{|\curlyvee'(l)|^q}{e^{\alpha l}} + (1-\tau)^s \frac{|\curlyvee'(h_1)|^q}{e^{\alpha h_1}}\right] d\tau \\
&\leq \frac{M^q}{(s+1)e^{\alpha l}} + \frac{M^q}{(s+1)e^{\alpha h_2}},
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\int_0^1 |\curlyvee'(\tau l + (1-\tau)h_2)|^q d\tau &= \int_0^1 \left[\tau^s \frac{|\curlyvee'(l)|^q}{e^{\alpha l}} + (1-\tau)^s \frac{|\curlyvee'(h_2)|^q}{e^{\alpha h_1}}\right] d\tau \\
&\leq \frac{M^q}{(s+1)e^{\alpha l}} + \frac{M^q}{(s+1)e^{\alpha h_2}}.
\end{aligned} \tag{2.8}$$

Hence using (2.7) and (2.8) in (2.6), we get (2.5) □

Remark 2.3. *In Theorem 2.2, by letting $\alpha = 0$, we get inequality 2.2 of Theorem 3 in [1].*

Corollary 2.2. *Under the assumptions of Theorem 2.2, by taking $s = 1$, we get*

$$\begin{aligned}
&\left|\curlyvee(l) - \frac{1}{h_2 - h_1}\int_{h_1}^{h_2} \curlyvee(t)dt\right| \\
&\leq \frac{M}{(h_2-h_1)(1+p)^{\frac{1}{p}}}\left[(l-h_1)^2\left(\frac{1}{2e^{\alpha l}} + \frac{1}{2e^{\alpha h_1}}\right)^{\frac{1}{q}} + (h_2-l)^2\left(\frac{1}{2e^{\alpha l}} + \frac{1}{2e^{\alpha h_2}}\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.9}$$

Remark 2.4. *In Corollary 2.2, by letting $\alpha = 0$, we get result for convex function.*

Theorem 2.3. *Let $\curlyvee : \mathcal{H} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{H}^\circ$ such that $\curlyvee' \in [h_1, h_2]$ for $h_1, h_2 \in \mathcal{H}$ with $h_1 < h_2$. If $|\curlyvee'|^q$ is exponentially s-convex in second sense on $[h_1, h_2]$ for some $s \in (0, 1]$, $q \geq 1$ and $|\curlyvee'(l)| \leq M$, $l \in [h_1, h_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\curlyvee(l) - \frac{1}{h_2 - h_1}\int_{h_1}^{h_2} \curlyvee(t)dt\right| \\
&\leq \frac{M}{(h_2-h_1)(2)^{1-\frac{1}{q}}}\left[(l-h_1)^2\left(\frac{1}{(s+2)e^{\alpha l}} + \frac{1}{(s+1)(s+2)e^{\alpha h_1}}\right)^{\frac{1}{q}}\right. \\
&\quad \left. + (h_2-l)^2\left(\frac{1}{(s+2)e^{\alpha l}} + \frac{1}{(s+1)(s+2)e^{\alpha h_2}}\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{2.10}$$

for each $l \in [h_1, h_2]$.

Proof. Let $p > 1$. From Lemma 2.1 and using power mean inequality, we obtain

$$\begin{aligned}
&\left|\curlyvee(l) - \frac{1}{h_2-h_1}\int_{h_1}^{h_2}\curlyvee(t)dt\right| \\
&\leq \frac{(l-h_1)^2}{h_2-h_1}\int_0^1 \tau|\curlyvee'(\tau l+(1-\tau)h_1)|d\tau + \frac{(h_2-l)^2}{h_2-h_1}\int_0^1 \tau|\curlyvee'(\tau l+(1-\tau)h_2)|d\tau \\
&\leq \frac{(l-h_1)^2}{h_2-h_1}\left(\int_0^1 \tau d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 \tau|\curlyvee'(\tau l+(1-\tau)h_1)|^q d\tau\right)^{\frac{1}{q}} \\
&\quad + \frac{(h_2-l)^2}{h_2-h_1}\left(\int_0^1 \tau d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 \tau|\curlyvee'(\tau l+(1-\tau)h_2)|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.11}$$

Since $|\curlyvee'|^q$ is exponentially s-convex in the second sense and $|\curlyvee'(l)| \leq M$, then we have

$$\begin{aligned}
\int_0^1 \tau|\curlyvee'(\tau l+(1-\tau)h_1)|^q d\tau &= \int_0^1 \tau\left[\tau^s\frac{|\curlyvee'(l)|^q}{e^{\alpha l}} + (1-\tau)^s\frac{|\curlyvee'(h_1)|^q}{e^{\alpha h_1}}\right]d\tau \\
&\leq \frac{M^q}{(s+2)e^{\alpha l}} + \frac{M^q}{(s+1)(s+2)e^{\alpha h_1}},
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
\int_0^1 \tau|\curlyvee'(\tau l+(1-\tau)h_2)|^q d\tau &= \int_0^1 \tau\left[\tau^s\frac{|\curlyvee'(l)|^q}{e^{\alpha l}} + (1-\tau)^s\frac{|\curlyvee'(h_2)|^q}{e^{\alpha h_2}}\right]d\tau \\
&\leq \frac{M^q}{(s+2)e^{\alpha l}} + \frac{M^q}{(s+1)(s+2)e^{\alpha h_2}}.
\end{aligned} \tag{2.13}$$

Hence using (2.12) and (2.13) in (2.11), we get (2.10) □

Remark 2.5. *In Theorem* 2.3*, by letting* $\alpha = 0$*, we get inequality* (2.3) *of Theorem* 4 *in* [1].

Corollary 2.3. *Under the assumptions of Theorem* 2.3*, by taking* $s = 1$*, we get*

$$\begin{aligned}
&\left|\curlyvee(l) - \frac{1}{h_2-h_1}\int_{h_1}^{h_2}\curlyvee(t)dt\right| \\
&\leq \frac{M}{(h_2-h_1)(2)^{1-\frac{1}{q}}}\left[(l-h_1)^2\left(\frac{1}{3e^{\alpha l}}+\frac{1}{6e^{\alpha h_1}}\right)^{\frac{1}{q}} + (h_2-l)^2\left(\frac{1}{3e^{\alpha l}}+\frac{1}{6e^{\alpha h_2}}\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.14}$$

Remark 2.6. *In Corollary 2.3, by letting* $\alpha = 0$*, we get result for convex function.*

Remark 2.7. *From previous inequalities, we can obtain several midpoint type inequalities by setting* $l = \frac{h_1+h_2}{2}$*. However, the details are omitted for interested reader.*

Theorem 2.4. *Let* $\curlyvee : \mathcal{H} \to \mathbb{R}$ *be a differentiable mapping on* $\mathcal{H}^\circ$ *such that* $\curlyvee' \in [h_1, h_2]$ *for* $h_1, h_2 \in \mathcal{H}$ *with* $h_1 < h_2$*. If* $|\curlyvee'|^q$ *is exponentially s-concave on* $[h_1, h_2]$*,* $p,\ q > 1$*,* $\frac{1}{p} + \frac{1}{q} = 1$*, then the following* *inequality holds:*

$$\begin{aligned}&\left|\curlyvee(l)-\frac{1}{h_2-h_1}\int_{h_1}^{h_2}\curlyvee(t)dt\right|\\&\leq\frac{2^{\frac{s-1}{q}}}{(h_2-h_1)(1+p)^{\frac{1}{p}}}\left[(l-h_1)^2\left|\curlyvee'\left(\frac{l+h_1}{2}\right)\right|+(h_2-l)^2\left|\curlyvee'\left(\frac{l+h_2}{2}\right)\right|\right],\end{aligned}\tag{2.15}$$

for each $l\in[h_1,h_2]$.

Proof. Let $p>1$. From Lemma 2.1 and using Hölder's inequality, we obtain

$$\begin{aligned}&\left|\curlyvee(l)-\frac{1}{h_2-h_1}\int_{h_1}^{h_2}\curlyvee(t)dt\right|\\&\leq\frac{(l-h_1)^2}{h_2-h_1}\int_0^1\tau|\curlyvee'(\tau l+(1-\tau)h_1)|d\tau+\frac{(h_2-l)^2}{h_2-h_1}\int_0^1\tau|\curlyvee'(\tau l+(1-\tau)h_2)|d\tau\\&\leq\frac{(l-h_1)^2}{h_2-h_1}\left(\int_0^1\tau^pd\tau\right)^{\frac{1}{p}}\left(\int_0^1|\curlyvee'(\tau l+(1-\tau)h_1)|^qd\tau\right)^{\frac{1}{q}}\\&\quad+\frac{(h_2-l)^2}{h_2-h_1}\left(\int_0^1\tau^pd\tau\right)^{\frac{1}{p}}\left(\int_0^1|\curlyvee'(\tau l+(1-\tau)h_2)|^qd\tau\right)^{\frac{1}{q}}.\end{aligned}\tag{2.16}$$

Since $|\curlyvee'|$ is exponentially s-concave and from inequality (1.4), we have

$$\int_0^1|\curlyvee'(\tau l+(1-\tau)h_1)|^qd\tau\geq2^{s-1}\left|\curlyvee'\left(\frac{l+h_1}{2}\right)\right|^q,\tag{2.17}$$

and

$$\int_0^1|\curlyvee'(\tau l+(1-\tau)h_2)|^qd\tau\geq2^{s-1}\left|\curlyvee'\left(\frac{l+h_2}{2}\right)\right|^q.\tag{2.18}$$

Using inequalities (2.18) and (2.17) in (2.16), we get (2.15). □

Remark 2.8. *If one takes* $l=\frac{h_1+h_2}{2}$ *in* (2.15) *then one gets inequality* (2.8) *in* [1].
Similarly, by letting $s=1$ *in* (2.15) *then one gets inequality* (2.9) *in* [1].

3. Applications to special means

Consider some special means of two positive numbers h_1,h_2, $h_1<h_2$:

(1). The arithmetic mean

$$A(h_1,h_2)=\frac{h_1+h_2}{2}.$$

(2). The Identric mean

$$I(h_1,h_2)=\begin{cases}\frac{1}{e}\left(\frac{h_2^{h_2}}{h_1^{h_1}}\right)^{\frac{1}{h_2-h_1}}, & h_1\neq h_2\\ h_1, & h_1=h_2\,.\end{cases}$$

where $h_1,h_2>0$.

(3). The p-logarithmic mean

$$L_p(h_1,h_2)=\left(\frac{h_2^{p+1}-h_1^{p+1}}{(p+1)(h_2-h_1)}\right)^{\frac{1}{p}},\ p\in\mathbb{R}\setminus\{-1,0\}.$$

Proposition 3.1. Let $0<h_1<h_2$, $q\geq 1$ and $0<s<1$. Then we have

$$\begin{aligned}&|A^s(h_1,h_2)-L_s^s(h_1,h_2)|\\ &\leq\frac{M}{(h_2-h_1)(2)^{1-\frac{1}{q}}}\left[(l-h_1)^2\left(\frac{1}{(s+2)e^{\alpha l}}+\frac{1}{(s+1)(s+2)e^{\alpha h_1}}\right)^{\frac{1}{q}}\right.\\ &\left.+(h_2-l)^2\left(\frac{1}{(s+2)e^{\alpha l}}+\frac{1}{(s+1)(s+2)e^{\alpha h_2}}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{3.1}$$

Proof. the result holds by letting $l=\frac{h_1+h_2}{2}$ in (2.10) with exponentially s-convex function in second sense $\curlyvee:(0,\infty)\to\mathbb{R}$, $\curlyvee(l)=l^s$ for all $\alpha\leq -1$. □

Proposition 3.2. Let $0<h_1<h_2$, $q\geq 1$ and $0<s<1$. Then we have

$$\begin{aligned}&|\ln A(h_1,h_2)-\ln I(h_1,h_2)|\\ &\leq\frac{M}{(h_2-h_1)(2)^{1-\frac{1}{q}}}\left[(l-h_1)^2\left(\frac{1}{(s+2)e^{\alpha l}}+\frac{1}{(s+1)(s+2)e^{\alpha h_1}}\right)^{\frac{1}{q}}\right.\\ &\left.+(h_2-l)^2\left(\frac{1}{(s+2)e^{\alpha l}}+\frac{1}{(s+1)(s+2)e^{\alpha h_2}}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{3.2}$$

Proof. the result holds by letting $l=\frac{h_1+h_2}{2}$ in (2.10) with exponentially s-convex function in second sense $\curlyvee:(0,\infty)\to\mathbb{R}$, $\curlyvee(l)=\ln(l)$ for all $\alpha\leq -1$. □

Remark 3.1. *From Corollary 2.3 one can gets more applications to some special means for exponentially convex functions.*

4. Conclusion

From Corollaries 2.1, 2.2 and 2.3, we obtained new ostrowski's type inequalities for exponentially convex function. From Theorems 2.1, 2.2, 2.3 and 2.4, we obtained new ostrowski's type inequalities for exponentially s-convex function. Some applications are obtained from Propositions 3.1 and 3.2.

Acknowledgement

This research article is supported by National University of Sciences and Technology(NUST), Islamabad, Pakistan.

Conflict of interest

The authors declare that there is no interest regarding the publication of this paper.

References

1. M. Alomari, M. Darus, S. S. Dragomir, et al. *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Appl. Math. Lett., **23** (2010), 1071–1076.
2. M. U. Awan, M. A. Noor, K. I. Noor, *Hermite-Hadamard inequalities for exponentially convex functions*, Appl. Math. Inf. Sci., **12** (2018), 405–409 .
3. P. Cerone, S. S. Dragomir, *Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions*, Demonstratio Math., **37** (2004), 299–308.
4. S. S. Dragomir, Th. M. Rassias, *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
5. S. S. Dragomir, *Some Companions of Ostrowski's inequality for absplutely continuous functions and applications, Facta Universitatis*, Ser. Math. Inform., **19** (2004), 1–16.
6. S. S. Dragomir and S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules*, Appl. Math. Lett., **11** (1998), 105–109.
7. N. Mehreen, M. Anwar, *Hermite–Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications*, J. Inequal. Appl., **2019** (2019), 92.
8. M. E. Ozdemir, H. Kavurmacı, E. Set, *Ostrowski's type inequalities for* (α, m)*-convex functions*, Kyungpook Math. J., **50** (2010), 371–378.
9. B. G. Pachpatte, *On an inequality of Ostrowski type in three independent variables*, J. Math. Anal. Appl., **249** (2000), 583–591.
10. E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Comput. Math. Appl., **63** (2012), 1147–1154.