AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(2): 1462–1475.
DOI:10.3934/math.2020100
Received: 25 September 2019
Accepted: 08 January 2020
Published: 21 January 2020

Research article

Fractional calculus of a product of multivariable Srivastava polynomial and multi-index Bessel function in the kernel F_3

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Abstract: In this article our main object to compute image formulas of generalized fractional hypergeometric operators, involving the product of multivariable Srivastava polynomial and multi-index Bessel function. The results obtained provide unification and an extension of known results given earlier by Agarwal and Nieto [1], Agarwal *et al.* [2] Mishra *et al.* [18], Saxena and Saigo [26], Suthar *et al.* [32]. We also consider certain special cases of derived results by specializing suitable value of the parameters.

Keywords: fractional calculus operators; multivariable Srivastava polynomial; multi–index Bessel function; Fox-Wright function; classical orthogonal polynomials
Mathematics Subject Classification: 33C20, 33B15

1. Introduction

Fractional calculus (FC) is a useful mathematical tool which deals with the study of fractional order integrals and derivatives. Excellent research on the topic of FC have given a useful concept of the theory and applications of FC operators of different areas of mathematical analysis having in mind for instance their meaningful role for aspect in the wave and diffusion equation and in the temperature field, among others. Since last few decades, many authors like Kilbas and Saigo [13], Kiryakova [16], Saigo [21], Saigo and Maeda [22], Saxena and Pogány [24], Srivastava and Saxena [30], and others have extensively studied the properties, applications and extensions of various fractional integral and differential operators of FC. It is notable that Baleanu *et al.* [4] obtained the images of the product of generalized Bessel functions in Saigo hypergeometric operators by providing extension of previous results due to Kilbas and Sebastian [14], Purohit *et al.* [20], Saxena *et al.* [25]. Lately, Mishra *et al.* [18] evaluated generalized FC formulas for the product of Srivastava polynomials and generalized Mittag-Leffler function via Marichev-Saigo-Maeda operators in terms of the Fox-Wright ${}_r\Psi_s$ function. Further, Suthar [31], has given an extension of a results by Mishra *et al.* [18]. Thus, many authors have explored new approach of applications by making use of FC operators to investigate image formulae involving special functions of one and more variables which are useful in the problem of applied science such as fractional diffusion, fractional reaction, fractional stochastic theory, dynamical systems theory and anomalous diffusion in complex systems etc., see for example [7–12, 30].

In this study we establish certain integral and derivation formulae of the product of multivariable Srivastava polynomial and multi–index Bessel function using generalized fractional hypergeometric operators. Moreover, we also find out some important special cases of our main results.

Throughout the usual notations have used $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the sets of complex, real, positive real numbers, positive and non–negative integers, respectively.

Firstly, we recall the generalized hypergeometric fractional integrals and derivatives, introduced by Marichev [17] and later extended by Saigo and Maeda [22]. These operators known as the Marichev–Saigo–Maeda operators. The generalized FC operators involving the third Appell function (or the Horn $F_3(\cdot)$ function in other words) in the kernel are defined in the following way.

Let $\mu, \mu', \varepsilon, \varepsilon' \in \mathbb{C}$, $\Re(\gamma) > 0$ and $u > 0$. Then the *left and right generalized Marichev–Saigo–Maeda–type fractional integral operators* $I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}$ and $I_{-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}$, respectively, are defined as [22, p. 393, Eqs. (4.12), (4.13)]

$$\left(I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} f\right)(u) = \frac{u^{-\mu}}{\Gamma(\gamma)} \int_0^u t^{-\mu'} (u-t)^{\gamma-1} F_3(\mu, \mu', \varepsilon, \varepsilon'; \gamma; 1 - t/u, 1 - u/t)\, f(t)\, \mathrm{d}t, \tag{1.1}$$

$$\left(I_{-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} f\right)(u) = \frac{u^{-\mu'}}{\Gamma(\gamma)} \int_u^\infty t^{-\mu} (t-u)^{\gamma-1} F_3(\mu, \mu', \varepsilon, \varepsilon'; \gamma; 1 - u/t, 1 - t/u)\, f(t)\, \mathrm{d}t,$$

where F_3 is the third Appell function defined by [29]

$$F_3\,(\mu, \mu', \varepsilon, \varepsilon'; \gamma; u, v) = \sum_{m,n \geq 0} \frac{(\mu)_m (\mu')_n (\varepsilon)_m (\varepsilon')_n}{(\gamma)_{m+n}} \frac{u^m}{m!} \frac{v^n}{n!}, \qquad (\max\{|u|, |v|\} < 1).$$

In turn, the argument transformation $t \mapsto t/u$ in the integrand results in

$$\left(I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} f\right)(u) = \frac{u^{\gamma-\mu-\mu'}}{\Gamma(\gamma)} \int_0^1 t^{-\mu'} (1-t)^{\gamma-1} F_3(\mu, \mu', \varepsilon, \varepsilon'; \gamma; 1-t, 1-1/t)\, f(ut)\, \mathrm{d}t, \tag{1.2}$$

$$\left(I_{-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} f\right)(u) = \frac{u^{\gamma-\mu-\mu'}}{\Gamma(\gamma)} \int_1^\infty t^{-\mu} (t-1)^{\gamma-1} F_3(\mu, \mu', \varepsilon, \varepsilon'; \gamma; 1-1/t, 1-t)\, f(ut)\, \mathrm{d}t,$$

which show that the considered integral operators are in fact weighted Beta–transforms of a suitable input function f at the 'wrapped' argument ut, with respect to the Appell F_3–kernel function.

These operators become for the Saigo fractional integral operators [21]:

$$\left(I_{0+}^{\mu+\varepsilon,0,-\tau,0,\mu} f\right)(u) = \left(I_{0+}^{\mu,\varepsilon,\tau} f\right)(u), \tag{1.3}$$

$$\left(I_{-}^{\mu+\varepsilon,0,-\tau,0,\mu} f\right)(u) = \left(I_{-}^{\mu,\varepsilon,\tau} f\right)(u). \tag{1.4}$$

Next, consider the same parameter space as above, that is assume $\mu, \mu', \varepsilon, \varepsilon' \in \mathbb{C}$, $\Re(\gamma) > 0$ and $u > 0$. Then the *left and right generalized Marichev–Saigo–Maeda-type fractional differential operators* $D_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}$ and $D_{-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}$, respectively, are defined by the integrals [22]

$$\begin{aligned}\left(D_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} f\right)(u) &= \left(I_{0+}^{-\mu',-\mu,-\varepsilon',-\varepsilon,-\gamma} f\right)(u) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n \left(I_{0+}^{-\mu',-\mu,-\varepsilon'+n,-\varepsilon,-\gamma+n} f\right)(u), \qquad (n = [\Re(\gamma] + 1) \\ &= \frac{1}{\Gamma(n-\gamma)} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n u^{\mu'} \int_0^u t^{\mu} (u-t)^{n-\gamma-1} \\ &\quad \times F_3\left(-\mu', -\mu, n-\varepsilon', -\varepsilon, n-\gamma; 1-t/u, 1-u/t\right) f(t)\,\mathrm{d}t, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\left(D_{-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} f\right)(u) &= \left(I_{-}^{-\mu',-\mu,-\varepsilon',-\varepsilon,-\gamma} f\right)(u) \\ &= (-1)^n \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n \left(I_{-}^{-\mu',-\mu,-\varepsilon',n-\varepsilon,n-\gamma} f\right)(u), \qquad (\Re(\gamma) > 0; n = [\Re(\gamma] + 1) \\ &= \frac{(-1)^n}{\Gamma(n-\gamma)} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n u^{\mu} \int_u^\infty t^{\mu'} (t-u)^{n-\gamma-1} \\ &\quad \times F_3(-\mu', -\mu, -\varepsilon', n-\varepsilon, n-\gamma; 1-u/t, 1-t/u) f(t)\,\mathrm{d}t; \end{aligned} \tag{1.6}$$

here, *and in what follows* $[x]$ denotes the integer part of some $x \in \mathbb{R}$. The previous fractional differentiation formulae possess equivalent forms:

$$\begin{aligned}\left(D_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} f\right)(u) &= \frac{1}{\Gamma(n-\gamma)} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n u^{n+\mu+\mu'-\gamma} \int_0^1 t^{\mu} (1-t)^{n-\gamma-1} \\ &\quad \times F_3(-\mu', -\mu, n-\varepsilon', -\varepsilon, n-\gamma; 1-t, 1-1/t) f(ut)\,\mathrm{d}t, \\ \left(D_{-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} f\right)(u) &= \frac{(-1)^n}{\Gamma(n-\gamma)} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n u^{n+\mu+\mu'-\gamma} \int_1^\infty t^{\mu'} (t-1)^{n-\gamma-1} \\ &\quad \times F_3(-\mu', -\mu, -\varepsilon', n-\varepsilon, n-\gamma; 1-1/t, 1-t) f(ut)\,\mathrm{d}t \end{aligned}$$

within the same parameter range. Moreover, these operators reduce to the Saigo fractional differential operators [21]

$$\left(D_{0+}^{\mu+\varepsilon,0,-\tau,0,\mu} f\right)(u) = \left(D_{0+}^{\mu,\varepsilon,\tau} f\right)(u), \tag{1.7}$$

$$\left(D_{-}^{\mu+\varepsilon,0,-\tau,0,\mu} f\right)(u) = \left(D_{-}^{\mu,\varepsilon,\tau} f\right)(u). \tag{1.8}$$

We recall two relations needful in obtaining our main results. Firstly [22, p. 394, Eq. (4.18)]

$$\left(I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} t^{\rho-1}\right)(u) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\mu-\mu'-\varepsilon)\Gamma(\rho+\varepsilon'-\mu')}{\Gamma(\rho+\gamma-\mu-\mu')\Gamma(\rho+\gamma-\mu'-\varepsilon)\Gamma(\rho+\varepsilon')}\, u^{\rho-\mu-\mu'+\gamma-1}, \tag{1.9}$$

valid under the constraint $\Re(\rho) > \max\{0, \Re(\mu+\mu'-\gamma), \Re(\mu'-\varepsilon')\}$ and secondly, for all $\Re(\rho) < 1 + \min\{\Re(-\varepsilon), \Re(\mu+\mu'-\gamma), \Re(\mu+\varepsilon'-\gamma)\}$ there holds [22, p. 394, Eq. (4.19)]

$$\left(I_{-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} t^{\rho-1}\right)(u) = \frac{\Gamma(1+\mu+\mu'-\gamma-\rho)\Gamma(1+\mu+\varepsilon'-\gamma-\rho)\Gamma(1-\varepsilon-\rho)}{\Gamma(1-\rho)\Gamma(1+\mu+\mu'+\varepsilon'-\gamma-\rho)\Gamma(1+\mu-\varepsilon-\rho)}\, u^{\rho-\mu-\mu'+\gamma-1}. \tag{1.10}$$

The series form of the Fox–Wright generalized hypergeometric function ${}_r\Psi_s$ [6, 33] is

$$ {}_r\Psi_s[z] = {}_r\Psi_s\Big[\begin{array}{c} (\gamma_1,\gamma_1'),\cdots,(\gamma_r,\gamma_r') \\ (l_1,l_1'),\cdots,(l_s,l_s') \end{array}\Big| z\Big] = \sum_{k\geq 0} \frac{\Gamma(\gamma_1+\gamma_1'k)\cdots\Gamma(\gamma_r+\gamma_r'k)}{\Gamma(l_1+l_1'k)\cdots\Gamma(l_r+l_r'k)}\frac{z^k}{k!}\,. \tag{1.11} $$

Here the coefficients $\gamma_1',\cdots,\gamma_r'$, $l_1',\cdots,l_s'$ are positive and the series absolutely converges for all $z\in\mathbb{C}$ when $\Delta = 1+\sum_{j=1}^s l_j' - \sum_{m=1}^r \gamma_m' > 0$, while in the case $\Delta = 0$ the convergence of the series (1.11) occur inside the circle [15, p. 56, Theorem 1.5]

$$ |z| < \prod_{j=1}^{r} {\gamma_j'}^{-\gamma_j'} \prod_{m=1}^{s} {l_m'}^{l_m'}\,. $$

We also point out the Fox's H function representation formula of the Fox–Wright generalized hypergeometric function [15, p. 67, Eq. (1.12.68)]

$$ {}_r\Psi_s[z] = H^{1,r}_{r,s+1}\Big[-z\Big|\begin{array}{c} (1-\gamma_1,\gamma_1'),\cdots,(1-\gamma_r,\gamma_r') \\ (0,1),(1-l_1,l_1'),\cdots,(1-l_s,l_s') \end{array}\Big], $$

where $H^{1,r}_{r,s+1}[z]$ stands for the Fox's H–function, which definition is given *via* Mellin–Barnes type complex integral, see [6].

The generalized multi-index Bessel function was introduced in a power series form by Nisar *et al.* [19] as

$$ \mathcal{J}^{(\alpha_j)_m,\tau,c}_{(\beta_j)_m,\kappa,b}(z) = \sum_{n\geq 0} \frac{(\tau)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j+\frac{1+b}{2})}\frac{(cz)^n}{n!}, \qquad (m\in\mathbb{N}), \tag{1.12} $$

where α_j, $j=1,2,\cdots,m$, $\tau,\delta,b,c\in\mathbb{C}$; $\sum_{j=1}^m \Re(\alpha_j) > \max\{0,\Re(\kappa)-1\}$; $\Re(\beta_j)>0$, $j=1,2,\cdots,m$, $\Re(\tau)>0$; $\Re(\delta)>0$ and $(\lambda)_n$ denotes the familiar Pochhammer symbol *viz.*

$$ (\lambda)_\mu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu=0) \\ \lambda(\lambda+1),....,(\lambda+n-1) & (\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-;\mu\in\mathbb{N})\,, \end{cases} $$

while $(0)_0$ is *conventionally* taken to be unity and $\mathbb{Z}_0^-$ signifies the set of non–positive integers.

Consider some special cases of $\mathcal{J}^{(\alpha_j)_m,\tau,c}_{(\beta_j)_m,\kappa,b}(z)$.

(i) If we put $\kappa=0, b=c=m=1$, $\alpha_1=1$, $\beta_1=\nu$ and replace z by $z^2/4$ in (1.12) we arrive at the Bessel function of the first kind [5, p.7.2, Eq.(2)]

$$ \mathcal{J}^{1,\tau,1}_{\nu,0,1}(z^2/4) = \left(\frac{2}{z}\right)^\nu J_\nu(z), \qquad (z,\nu\in\mathbb{C};\mathbb{R}(\nu)>0). $$

(ii) For $b=-1, c=1$ (1.12) reduces to the multi–index Mittag–Leffler function [23]

$$ \mathcal{J}^{(\alpha_j)_m,\tau,1}_{(\beta_j)_m,\kappa,-1}(z) = E^{(\alpha_j)_m,\tau}_{(\beta_j)_m,\kappa}(z). \tag{1.13} $$

(iii) A connection to Fox–Wright function ${}_r\Psi_s$ is

$$\mathcal{J}^{(\alpha_j)_m,\tau,c}_{(\beta_j)_m,k,b}(z) = \frac{\Gamma(\delta)}{\Gamma(\tau)}\, {}_2\Psi_{m+1}\Big[\begin{matrix} (\tau,\kappa) \\ (\beta_1+\frac{1+b}{2},\alpha_1),\cdots,(\beta_m+\frac{1+b}{2},\alpha_m) \end{matrix} \Big| cz \Big].$$

(iv) In turn, the Fox's H–function representation becomes

$$\mathcal{J}^{(\alpha_j)_m,\tau,c}_{(\beta_j)_m,k,b}(z) = \frac{1}{\Gamma(\tau)}\, H^{0,1}_{1,m+1}\Big[-cz \Big| \begin{matrix} (1-\tau,\kappa) \\ (0,1),(\beta_1+\frac{1-b}{2},\alpha_1),\cdots,(\beta_m+\frac{1-b}{2},\alpha_m) \end{matrix} \Big].$$

In the sequel we shall need the definition of the multivariable Srivastava polynomials introduced by Srivastava and Garg [28, p. 2, Eq. (1.4)] as the s–triple series

$$S_d^{p_1,p_2,\cdots,p_s}(z_1,\cdots,z_s) = \sum_{k_1,\ldots,k_s=0}^{p_1k_1+\cdots+p_sk_s\le d} (-d)_{p_1k_1+\cdots+p_sk_s} A(d,k_1,\cdots,k_s)\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_s^{k_s}}{k_s!}, \tag{1.14}$$

where $(d,p_1,\cdots,p_s)\in\mathbb{N}_0^{s+1}$ and the coefficients $A(d,k_1,\cdots,k_s)\in\mathbb{C}$. Evidently, the case $s=1$ corresponds to the polynomial of the form [27]:

$$S_d^p(z) = \sum_{k=0}^{[d/p]}(-d)_{pk}A(d,k)\frac{z^k}{k!}, \qquad (d\in\mathbb{N}_0).$$

2. Fractional integral formulae

In this section we present fractional integral formulae involving the product of multivariable Srivastava polynomials and multi-index Bessel function using left and right Marichev-Saigo-Maeda operators, which are expressed in terms of Fox-Wright function under the above specified conditions of (1.11).

Theorem 1. *For all* $\mu,\mu',\varepsilon,\varepsilon',\gamma,\tau,\alpha_j,\beta_j,\rho\in\mathbb{C},(j=1,2,\cdots,m)$ *which satisfy* $\Re(\beta_j)>0$, $\sum_{j=1}^m\Re(\alpha_j)>\max\{0;\Re(\kappa)-1\}$, $\Re(\gamma)>0$, $\Re(\rho)>\max\{0,\Re(\mu+\mu'+\varepsilon-\gamma),\Re(\mu'-\varepsilon')\}\in\mathbb{R}^+$. *Then we have the left fractional integral formula*

$$\begin{aligned}
\Big(I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\Big(t^{\rho-1}S_d^{p_1,\cdots,p_s}(y_1t^{\lambda_1},\cdots,y_st^{\lambda_s})\mathcal{J}^{(\alpha_j)_m,\tau,c}_{(\beta_j)_m,\kappa,b}(zt^{\nu})\Big)\Big)(u) &= u^{\rho+\gamma-\mu-\mu'-1}\\
\sum_{k_1,\cdots,k_s=0}^{p_1k_1+\cdots+p_sk_s\le d}(-d)_{p_1k_1+\cdots+p_sk_s}A(d,k_1,\cdots,k_s)\frac{(y_1)^{k_1}}{k_1!},\cdots,\frac{(y_s)^{k_s}}{k_s!}u^{\sum_{j=1}^s\lambda_jk_j}\\
\times{}_4\Psi_{m+3}\Big[\begin{matrix}(\rho+\gamma-\mu-\mu'-\varepsilon+\sum_{j=1}^s\lambda_jk_j,\nu),(\rho+\varepsilon'-\mu'+\sum_{j=1}^s\lambda_jk_j,\ \nu),\\ (\rho+\gamma-\mu-\mu'+\sum_{j=1}^s\lambda_jk_j,\ \nu),\ (\rho+\varepsilon'+\sum_{j=1}^s\lambda_jk_j,\ \nu),\\ (\rho+\sum_{j=1}^s\lambda_jk_j,\nu),(\tau,\kappa)\\ (\rho+\gamma+\varepsilon'-\mu'+\sum_{j=1}^s\lambda_jk_j,\ \nu),(\beta_j+\frac{b+1}{2},\alpha_j)_{j=1}^m\end{matrix}\Big| zcu^{\nu}\Big].
\end{aligned} \tag{2.1}$$

Proof. Put the multi–index Bessel function (1.12) and the multivariable S_d polynomial (1.14) into

$$\mathscr{I}_1 := \Big(I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}t^{\rho-1}S_d^{p_1,\cdots,p_s}(y_1t^{\lambda_1},\cdots,y_st^{\lambda_s})\mathcal{J}^{(\alpha_j)_m,\tau,c}_{(\beta_j)_m,q,b}(zt^{\nu})\Big)(u)$$

and by the legitimate changing of the order of integration and summation we have

$$\mathscr{I}_1 = \sum_{k_1,...,k_s=0}^{p_1k_1+\cdots+p_sk_s\le d} (-d)_{p_1k_1+\cdots+p_sk_s} A(d,k_1,...,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}$$
$$\times \sum_{n\ge 0} \frac{c^n(\tau)_{\kappa n}}{\Gamma(\alpha_j n+\beta_j+\frac{b+1}{2})}\frac{z^n}{n!}\left(I_{0,+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma} t^{\rho+\sum_{j=1}^s \lambda_j k_j+\nu n-1}\right)(u),$$

Applying (1.9) we conclude

$$\mathscr{I}_1 = \frac{1}{\Gamma(\tau)}\sum_{k_1,...,k_s=0}^{p_1k_1+\cdots+p_sk_s\le d} (-d)_{p_1k_1+\cdots+p_sk_s} A(d,k_1,...,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}\sum_{n\ge 0}\frac{c^n\Gamma(\tau+\kappa n)}{\Gamma(\alpha_j n+\beta_j+\frac{b+1}{2})}$$
$$\times \frac{\Gamma(\rho+\sum_{j=1}^s\lambda_jk_j+\nu n)\Gamma(\rho+\sum_{j=1}^s\lambda_jk_j+\nu n+\gamma-\mu-\mu'-\varepsilon)}{\Gamma(\rho+\sum_{j=1}^s\lambda_jk_j+\nu n+\varepsilon')\Gamma(\rho+\gamma-\mu-\mu'+\sum_{j=1}^s\lambda_jk_j+\nu n)}$$
$$\times \frac{\Gamma(\rho+\varepsilon'-\mu+\sum_{j=1}^s\lambda_jk_j+\nu n)}{\Gamma(\rho+\sum_{j=1}^s\lambda_jk_j+\nu n+\gamma-\mu'-\varepsilon)}\frac{z^n}{n!}u^{\rho+\gamma-\mu-\mu'+\sum_{j=1}^s\lambda_jk_j+\nu n-1}.$$

Reducing the expression in view of (1.11) we achieve the required result (2.1). □

In view of the relation (1.3), we deduce a first consequence of Theorem 1.

Corollary 2.1. *For all $\mu,\mu',\varepsilon,\varepsilon',\gamma,\tau,\alpha_j,\beta_j,\rho\in\mathbb{C}$, $j=1,\cdots,m$ which satisfy $\Re(\beta_j)>0$, $\sum_{j=1}^m\Re(\alpha_j)>\max\{0,\Re(\kappa)-1\}$, $\Re(\gamma)>0$, $\Re(\rho)>\max\{0,\Re(\varepsilon-\gamma)\}$. Then we have*

$$\left(I_{0+}^{\mu,\varepsilon,\gamma}\left(t^{\rho-1}S_d^{p_1,\cdots,p_s}(y_1t^{\lambda_1},\cdots,y_st^{\lambda_s})\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^\nu)\right)\right)(u)$$
$$= u^{\rho+\gamma-\mu-\mu'-1}\sum_{k_1,...,k_s=0}^{p_1k_1+\cdots+p_sk_s\le d}(-d)_{p_1k_1+\cdots+p_sk_s}A(d,k_1,...,k_s)\frac{(y_1)^{k_1}}{k_1!},\cdots,\frac{(y_s)^{k_s}}{k_s!}u^{\sum_{j=1}^s\lambda_jk_j}$$
$$\times {}_3\Psi_{m+2}\left[\begin{matrix}(\rho+\gamma-\varepsilon+\sum_{j=1}^s\lambda_jk_j,\nu),\ (\rho+\sum_{j=1}^s\lambda_jk_j,\nu),\ (\tau,\kappa)\\ (\rho+\mu+\gamma+\sum_{j=1}^s\lambda_jk_j,\ \nu),(\rho-\varepsilon+\sum_{j=1}^s\lambda_jk_j,\ \nu),(\beta_j+\frac{b+1}{2},\alpha_j)_{j=1}^m\end{matrix}\middle| zcu^\nu\right], \tag{2.2}$$

where $(\theta_j)_{j=1}^m$ stands for the sequence $\theta_1,\cdots,\theta_m$.

Theorem 2. *For all $\mu,\mu',\varepsilon,\varepsilon',\gamma,\tau,\alpha_j,\beta_j,\rho\in\mathbb{C}$ $(j=1,\cdots,m)$ which satisfy $\Re(\beta_j)>0$, $\sum_{j=1}^m\Re(\alpha_j)>\max\{0,\Re(\kappa)-1\}$; $\Re(\gamma)>0$, $\Re(\rho)<1+\min\{\Re(-\varepsilon),\Re(\mu+\mu'-\gamma),\Re(\mu-\varepsilon'-\gamma)\}$ we have*

$$\left(I_{-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{-\gamma-\rho}S_d^{p_1,\cdots,p_s}(y_1t^{\lambda_1},\cdots,y_st^{\lambda_s})\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{-\nu})\right)\right)(u)=u^{-\rho-\mu-\mu'-1}$$
$$\times\sum_{k_1,\cdots,k_s=0}^{p_1k_1+\cdots+p_sk_s\le d}(-d)_{p_1k_1+\cdots+p_sk_s}A(d,k_1,\cdots,k_2)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}u^{\sum_{j=1}^s\lambda_jk_j}$$
$$\times {}_4\Psi_{m+3}\left[\begin{matrix}(\rho+\mu+\mu'-\sum_{j=1}^s\lambda_jk_j,\nu),(\rho+\mu+\varepsilon'-\sum_{j=1}^s\lambda_jk_j,\ \nu),\\ (\rho+\gamma-\sum_{j=1}^s\lambda_jk_j,\ \nu),(\rho+\mu+\mu'+\varepsilon'-\gamma-\rho-\sum_{j=1}^s\lambda_jk_j,\nu),\end{matrix}\right.$$
$$\left.\begin{matrix}(\rho-\varepsilon+\gamma-\sum_{j=1}^s\lambda_jk_j,\nu),(\tau,\kappa)\\ (\rho+\mu-\varepsilon+\gamma-\sum_{j=1}^s\lambda_jk_j,\nu),\ (\beta_j+\frac{b+1}{2},\alpha_j)_{j=1}^m\end{matrix}\middle| zcu^\nu\right]. \tag{2.3}$$

Proof. To establish the stated prove that above result, using (1.12) and (1.14) as series form, and than arranging the order of integration and summation (which is valid under the given condition of Theorem 2), left hand side of (2.3) becomes

$$\left(I_{0-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho-1}\, S_{d}^{p_1,p_2,\cdots,p_s}(y_1t^{\lambda_1},..,y_st^{\lambda_s})\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{-\nu})\right)\right)(u) \tag{2.4}$$

$$= \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s\leqslant d}(-d)_{p_1k_1,\cdots,p_sk_s}A(d;k_1,\cdots,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}$$

$$\times\sum_{n=0}^{\infty}\frac{c^n(\tau)_{\kappa n}}{\Gamma(\alpha_jn+\beta_j+\frac{b+1}{2})}\frac{z^n}{n!}\left(I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}(t^{-\rho-\gamma+\sum_{j=1}^{s}\lambda_jk_j-\nu n})\right)(u).$$

Now, applying the relation (1.10), we have

$$= \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s\leqslant d}(-d)_{p_1k_1,\cdots,p_sk_s}A(d;k_1,\cdots,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}\sum_{n=0}^{\infty}\frac{c^n\Gamma(\tau+\kappa n)}{\Gamma(\alpha_jn+\beta_j+\frac{b+1}{2})}$$

$$\times\frac{\Gamma(\rho+\mu+\mu'-\sum_{j=1}^{s}\lambda_jk_j+\nu n)\Gamma(\rho+\mu+\varepsilon'-\sum_{j=1}^{s}\lambda_jk_j+\nu n)}{\Gamma(\rho+\gamma-\sum_{j=1}^{s}\lambda_jk_j+\nu n)\Gamma(\rho+\mu+\mu'+\varepsilon'-\sum_{j=1}^{s}\lambda_jk_j+\nu n)}$$

$$\times\frac{1}{\Gamma(\tau)}\frac{\Gamma(\rho+\mu+\varepsilon'-\sum_{j=1}^{s}\lambda_jk_j+\nu n)}{\Gamma(1+\mu-\varepsilon'-(\rho+\sum_{j=1}^{s}\lambda_jk_j+\nu n))}\frac{z^n}{n!}u^{-\rho-\mu-\mu'+\sum_{j=1}^{s}\lambda_jk_j+\nu n}.$$

Finally, solving the above expression with the help of (1.11), we achieve the required result (2). □

In view of the relation (1.4), we get the following consequence of Theorem 2.

Corollary 2.2. *For all* $\mu,\varepsilon,\gamma,\tau,\alpha_j,\beta_j,\rho\in\mathbb{C}$ *(j=1,2, $\cdots$, m) which satisfy* $\Re(\beta_j)>0$, $\sum_{j=1}^{m}\Re(\alpha_j)>max\{0;\Re(\kappa)-1\}$, $\Re(\mu)>0$, $\Re(1-\gamma-\rho)<1+min\{\Re(-\varepsilon),\ \Re(-\gamma)\}$ *then following integral formula holds true:*

$$\left(I_{0-}^{\mu,\varepsilon,\gamma}\left(t^{-\gamma-\rho}S_{d}^{p_1,p_2,\cdots,p_s}(y_1t^{\lambda_1},\cdots,y_st^{\lambda_s})\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{-\nu})\right)\right)(u)=u^{-\rho-\mu-\mu'-1}$$

$$\times\sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s\leqslant d}(-d)_{p_1k_1,\cdots,p_sk_s}A(d;k_1,\cdots,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}u^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\times{}_3\Psi_{m+2}\left[\begin{matrix}(\rho+\mu+\delta-\sum_{j=1}^{s}\lambda_jk_j,\nu),(\tau,\kappa),\\(\rho+\mu-\sum_{j=1}^{s}\lambda_jk_j,\ \nu),(\beta_j+\frac{b+1}{2},\alpha_j)_{j=1}^{m},\end{matrix}\right.$$

$$\left.\begin{matrix}(\rho+\mu+\varepsilon-\sum_{j=1}^{s}\lambda_jk_j,\ \nu)\\(\rho+2\mu+\varepsilon+\gamma-\sum_{j=1}^{s}\lambda_jk_j,\nu)\end{matrix}\right|zcu^{\nu}\Bigg]. \tag{2.5}$$

3. Fractional derivative formulae

Here, we compute fractional derivative formulae involving the product of multivariable Srivastava polynomials and multi-index Bessel function using left and right Marichecv-Saigo-Maeda operators, which are expressed in terms of Fox-Wright function under the given conditions of (1.11).

Theorem 3. *For all* $\mu, \mu', \varepsilon, \varepsilon', \gamma, \tau, \alpha_j, \beta_j, \rho \in \mathbb{C}$ *(j=1,2, $\cdots$, m) be such that* $\Re(\beta_j) > 0$, $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(\kappa) - 1\}$, $\Re(\gamma) > 0$, $\Re(\rho) > max\{0, \Re(\gamma - \mu - \mu' - \varepsilon'),\ \Re(\mu - \varepsilon)\}$. *Then left-sided fractional derivative formula holds true:*

$$\left(D_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho-1} S_d^{p_1,p_2,\cdots,p_s}(y_1 t^{\lambda_1}, \cdots, y_s t^{\lambda_s}) \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{\nu})\right)\right)(u) = u^{\rho+\gamma-\mu-\mu'-1}$$

$$\times \sum_{k_1,\ldots,k_s=0}^{p_1k_1,\ldots,p_sk_s \leqslant d} (-d)_{p_1k_1,\ldots,p_sk_s} A(d; k_1, \ldots, k_s) \frac{y_1^{k_1}}{k_1!}, \ldots., \frac{y_s^{k_s}}{k_s!} u^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$\times {}_4\Psi_{m+3}\left[\begin{array}{c} (\rho - \gamma + \mu + \mu' + \varepsilon' + \sum_{j=1}^{s} \lambda_j k_j, \nu), (\rho - \varepsilon + \mu + \sum_{j=1}^{s} \lambda_j k_j,\ \nu), \\ (\rho - \gamma + \mu - \varepsilon' + \sum_{j=1}^{s} \lambda_j k_j,\ \nu),\ (\rho - \varepsilon + \sum_{j=1}^{s} \lambda_j k_j,\ \nu), \end{array}\right.$$

$$\left.\begin{array}{c} (\rho + \sum_{j=1}^{s} \lambda_j k_j, \nu), (\tau, \kappa) \\ (\rho - \gamma + \mu - \mu' + \sum_{j=1}^{s} \lambda_j k_j,\ \nu), (\beta_j + \frac{b+1}{2}, \alpha_j)_{j=1}^{m} \end{array} \middle| zcu^{\nu}\right]. \tag{3.1}$$

Proof. In order to prove that above result, using (1.12) and (1.14) as series form, and than arranging the order of integration and summation (which is valid under the given condition of Theorem 3), left hand side of (3.1) becomes

$$\left(D_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho-1}\ S_d^{p_1,p_2,\cdots,p_s}(y_1 t^{\lambda_1}, \cdots, y_s t^{\lambda_s}) \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{\nu})\right)\right)(u)$$

$$= \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\ldots,p_sk_s \leqslant d} (-d)_{p_1k_1,\cdots,p_sk_s} A(d; k_1, \cdots, k_s) \frac{y_1^{k_1}}{k_1!}, \cdots, \frac{y_s^{k_s}}{k_s!}$$

$$\times \sum_{n=0}^{\infty} \frac{c^n (\tau)_{\kappa n}}{\Gamma(\alpha_j n + \beta_j + \frac{b+1}{2})} \frac{z^n}{n!} \left(D_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho+\sum_{j=1}^{s} \lambda_j k_j + \nu n - 1}\right)\right)(u).$$

Applying the result (1.5) and (1.9), we get

$$= \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s \leqslant d} (-d)_{p_1k_1,\cdots,p_sk_s} A(d; k_1, \cdots, k_s) \frac{y_1^{k_1}}{k_1!}, \cdots, \frac{y_s^{k_s}}{k_s!} \sum_{n=0}^{\infty} \frac{c^n \Gamma(\tau + \kappa n)}{\Gamma(\alpha_j n + \beta_j + \frac{b+1}{2})}$$

$$\times \frac{\Gamma(\rho + \sum_{j=1}^{s} \lambda_j k_j + \nu n) \Gamma(\rho + \sum_{j=1}^{s} \lambda_j k_j + \nu n - \gamma + \mu + \mu' - \varepsilon')}{\Gamma(\rho + \sum_{j=1}^{s} \lambda_j k_j + \nu n - \varepsilon) \Gamma(\rho - \gamma + \mu + \mu' + \sum_{j=1}^{s} \lambda_j k_j + \nu n)}$$

$$\times \frac{1}{\Gamma(\tau)} \frac{\Gamma(\rho - \varepsilon + \mu + \sum_{j=1}^{s} \lambda_j k_j + \nu n)}{\Gamma(\rho + \sum_{j=1}^{s} \lambda_j k_j + \nu n - \gamma + \mu' - \varepsilon')} \frac{z^n}{n!} u^{\rho-\gamma+\mu+\mu'+\sum_{j=1}^{s} \lambda_j k_j + \nu n - 1}.$$

Finally, using definition (1.11), we achieve the desired result (3.1). □

In view of the relation (1.7), we get the following consequence of Theorem 3.

Corollary 3.1. *For all $\mu, \varepsilon, \gamma, \tau, \alpha_j, \beta_j, \rho \in \mathbb{C}$ (j=1,2, $\cdots$, m) which satisfy $\Re(\beta_j) > 0$, $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(\kappa) - 1\}$, $\Re(\gamma) > 0$ $\Re(\rho) > max\{0, \Re(\varepsilon - \gamma)\}$, then left-sided fractional derivative formula holds true:*

$$\left(D_{0+}^{\mu,\varepsilon,\gamma}\left(t^{\rho-1} S_d^{p_1,p_2,\cdots,p_s}(y_1 t^{\lambda_1}, \cdots, y_s t^{\lambda_s}) \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{\nu})\right)\right)(u) = u^{\rho+\gamma-\mu-1}$$

$$\times \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s \leqslant d} (-d)_{p_1k_1,\cdots,p_sk_s} A(d; k_1, \cdots, k_s) \frac{y_1^{k_1}}{k_1!}, \cdots, \frac{y_s^{k_s}}{k_s!} u^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$\times {}_3\Psi_{m+2}\left[\begin{array}{c} (\rho+\gamma+\mu+\varepsilon+\sum_{j=1}^{s}\lambda_j k_j, \nu), (\rho+\sum_{j=1}^{s}\lambda_j k_j,\ \nu), \\ (\rho+\gamma+\sum_{j=1}^{s}\lambda_j k_j,\ \nu),\ (\rho+\varepsilon+\sum_{j=1}^{s}\lambda_j k_j,\ \nu), \end{array}\right.$$

$$\left.\begin{array}{c} (\tau,\kappa) \\ (\beta_j+\frac{b+1}{2}, \alpha_j)_{j=1}^{m} \end{array} \middle| zcu^{\nu}\right]. \tag{3.2}$$

Theorem 4. *For all $\mu, \mu', \varepsilon, \varepsilon', \tau, \gamma, \alpha_j, \beta_j, \rho \in \mathbb{C}$ (j=1,2, $\cdots$, m) which satisfy $\Re(\beta_j) > 0$, $\sum_{j=1}^{m} \Re(\alpha_j) > max\{0; \Re(\kappa) - 1\}$, $\Re(\gamma) > 0$, $\Re(\rho) < 1 + min\{\Re(-\varepsilon'), \Re(\mu + \mu' - \gamma), \Re(\mu + \varepsilon - \gamma)\}$. Then right-sided fractional derivative formula holds true:*

$$\left(D_{0-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\gamma-\rho} S_d^{p_1,p_2,\cdots,p_s}(y_1 t^{\lambda_1}, \cdots, y_s t^{\lambda_s}) \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{-\nu})\right)\right)(u) = u^{\rho+\gamma-\mu-\mu'-1}$$

$$\times \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s \leqslant d} (-d)_{p_1k_1,\cdots,p_sk_s} A(d; k_1, \cdots, k_s) \frac{y_1^{k_1}}{k_1!}, \cdots, \frac{y_s^{k_s}}{k_s!} u^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$\times {}_4\Psi_{m+3}\left[\begin{array}{c} (\rho-\mu-\mu'-\sum_{j=1}^{s}\lambda_j k_j, \nu), (\rho-\mu-\mu'-\sum_{j=1}^{s}\lambda_j k_j,\ \nu), \\ (\rho-\mu-\mu'-\varepsilon-\sum_{j=1}^{s}\lambda_j k_j,\ \nu),\ (\rho-\gamma-\sum_{j=1}^{s}\lambda_j k_j,\ \nu), \end{array}\right.$$

$$\left.\begin{array}{c} (\rho-\mu'-\varepsilon-\sum_{j=1}^{s}\lambda_j k_j, \nu), (\tau,\kappa) \\ (\rho-\gamma+\varepsilon'-\mu-\sum_{j=1}^{s}\lambda_j k_j,\ \nu), (\beta_j+\frac{b+1}{2}, \alpha_j)_{j=1}^{m} \end{array} \middle| zcu^{-\nu}\right]. \tag{3.3}$$

Proof. Applying (1.12) and (1.14), and then arranging the order of integration and summation (which is valid under the condition of Theorem 4), left hand side of (3.3) can be write as

$$\left(D_{0-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\gamma-\rho}\ S_d^{p_1,p_2,\cdots,p_s}\left(y_1 t^{\lambda_1}, \cdots, y_s t^{\lambda_s}\right) \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{\nu})\right)\right)(u)$$

$$= \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s \leqslant d} (-d)_{p_1k_1,\cdots,p_sk_s} A(d; k_1, \cdots, k_s) \frac{y_1^{k_1}}{k_1!}, \cdots, \frac{y_s^{k_s}}{k_s!}$$

$$\times \sum_{n=0}^{\infty} \frac{c^n (\tau)_{\kappa n}}{\Gamma(\alpha_j n + \beta_j + \frac{b+1}{2})} \frac{z^n}{n!} \left(D_{0-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}(t^{\rho+\sum_{j=1}^{s}\lambda_j k_j - \nu n - 1})\right)(u).$$

Now in view of (1.6) and (1.10) we obtain the following expression

$$= \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s \leqslant d} (-d)_{p_1k_1,\cdots,p_sk_s} A(d; k_1, \cdots, k_s) \frac{y_1^{k_1}}{k_1!}, \cdots, \frac{y_s^{k_s}}{k_s!} \sum_{n=0}^{\infty} \frac{c^n \Gamma(\tau + \kappa n)}{\Gamma(\alpha_j n + \beta_j + \frac{b+1}{2})}$$

$$\times \frac{\Gamma(\rho-\mu-\mu'-\sum_{j=1}^{s}\lambda_j k_j+\nu n)\Gamma(\rho-\mu'-\varepsilon+\sum_{j=1}^{s}\lambda_j k_j+\nu n)}{\Gamma(\rho-\mu-\mu'-\varepsilon+\sum_{j=1}^{s}\lambda_j k_j+\nu n)\Gamma(\rho-\gamma-\mu+\varepsilon'-\sum_{j=1}^{s}\lambda_j k_j+\nu n)}$$

$$\times \frac{1}{\Gamma(\tau)}\frac{\Gamma(\rho+\varepsilon'-\gamma-\sum_{j=1}^{s}\lambda_j k_j+\nu n)}{\Gamma(\rho-\gamma-\sum_{j=1}^{s}\lambda_j k_j+\nu n)}\frac{z^n}{n!}u^{-\rho+\gamma-\mu-\mu'-\sum_{j=1}^{s}\lambda_j k_j+\nu n-1}.$$

Solving the above expression with the help of (1.1), we achieve the desired result (3.3). □

In view of the relation (1.8), we get the following consequence of Theorem 4.

Corollary 3.2. *For all $\mu, \varepsilon, \gamma, \tau, \alpha_j, \beta_j, \rho \in \mathbb{C}$ (j=1,2, $\cdots$, m) which satisfy $\Re(\beta_j) > 0$, $\sum_{j=1}^{m}\Re(\alpha_j) > max\{0; \Re(\kappa)-1\}$, $\Re(\gamma) > 0$, $\Re(1-\gamma-\rho) < 1 + min\{\Re(-\varepsilon), \Re(-\gamma)\}$. Then right-sided fractional derivative formula holds true:*

$$\left\{D_{0-}^{\mu,\varepsilon,\gamma}\left(t^{\gamma-\rho}S_d^{p_1,p_2,\cdots,p_s}(y_1t^{\lambda_1},\cdots,y_st^{\lambda_s})\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^{-\nu})\right)\right\}(u) = u^{\rho+\gamma-\mu-1}$$

$$\times \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s\leqslant d}(-d)_{p_1k_1,\cdots,p_sk_s}A(d;k_1,\cdots,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}u^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$\times {}_3\Psi_{m+2}\left[\begin{array}{c}(\rho-\mu-\varepsilon-\sum_{j=1}^{s}\lambda_j k_j,\nu),(\rho-\gamma-\sum_{j=1}^{s}\lambda_j k_j,\ \nu),\\(\rho-\mu-\varepsilon-\sum_{j=1}^{s}\lambda_j k_j,\ \nu),\ (\rho-\mu-\sum_{j=1}^{s}\lambda_j k_j,\ \nu),\end{array}\right.$$

$$\left.\begin{array}{c}(\tau,\kappa)\\(\beta_j+\frac{b+1}{2},\alpha_j)_{j=1}^{m}\end{array}\middle|zcu^{-\nu}\right]. \tag{3.4}$$

4. Interesting special cases

Here we make the further special cases of theorems and its corollaries.

(1) In view of the relation (1.13), we arrive at the following particular cases of Theorem 1, Theorem 2, Theorem 3, and Theorem 4, respectively.

Corollary 4.1. *Under stated the given conditions in Theorem 1, then left-sided fractional integral identity holds true*

$$\left(I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho-1}S_d^{p_1,p_2,\cdots,p_s}(y_1t^{\lambda_1},\cdots,y_st^{\lambda_s})\mathbb{E}_{(\beta_j)_m,\kappa,}^{(\alpha_j)_m,\tau,}(zt^{\nu})\right)\right)(u) = u^{\rho+\gamma-\mu-\mu'-1}$$

$$\times \sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s\leqslant d}(-d)_{p_1k_1,\cdots,p_sk_s}A(d;k_1,\cdots,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}u^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$\times {}_4\Psi_{m+3}\left[\begin{array}{c}(\rho+\gamma-\mu-\mu'-\varepsilon+\sum_{j=1}^{s}\lambda_j k_j,\nu),(\rho+\varepsilon'-\mu'+\sum_{j=1}^{s}\lambda_j k_j,\ \nu),\\(\rho+\gamma-\mu-\mu'+\sum_{j=1}^{s}\lambda_j k_j,\ \nu),\ (\rho+\varepsilon'+\sum_{j=1}^{s}\lambda_j k_j,\ \nu),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho+\sum_{j=1}^{s}\lambda_j k_j,\nu),(\tau,\kappa)\\(\rho+\gamma+\varepsilon'-\mu'+\sum_{j=1}^{s}\lambda_j k_j,\ \nu),(\beta_j,\alpha_j)_{j=1}^{m}\end{array}\middle|zu^{\nu}\right]. \tag{4.1}$$

Corollary 4.2. *Under stated the given assumptions in Theorem 2, then right-sided fractional integral identity holds true.*

$$\left(I_{0-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho-1}S_{d}^{p_1,p_2,\cdots,p_s}(y_1t^{\lambda_1},...,y_st^{\lambda_s})\mathbb{E}_{(\beta_j)_m,\kappa,}^{(\alpha_j)_m,\tau,}(zt^{-\nu})\right)\right)(u)=u^{-\rho+\gamma-\mu-\mu'-1}$$

$$\times\sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s\leqslant d}(-d)_{p_1k_1,\cdots,p_sk_s}A(d;k_1,\cdots,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}u^{(\sum_{j=1}^{s}\lambda_jk_j)}$$

$$\times{}_4\Psi_{m+3}\left[\begin{array}{c}(\rho+\mu+\mu'-\sum_{j=1}^{s}\lambda_jk_j,\nu),(\rho+\mu+\varepsilon'-\sum_{j=1}^{s}\lambda_jk_j,\ \nu),\\(\rho+\gamma-\sum_{j=1}^{s}\lambda_jk_j,\ \nu),(\rho+\mu+\mu'+\varepsilon'-\gamma-\rho-\sum_{j=1}^{s}\lambda_jk_j,\nu),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho-\varepsilon+\gamma-\sum_{j=1}^{s}\lambda_jk_j,\nu),(\tau,\kappa)\\(\rho+\mu-\varepsilon+\gamma-\sum_{j=1}^{s}\lambda_jk_j,\nu),\ (\beta_j,\alpha_j)_{j=1}^{m}\end{array}\middle|zu^{\nu}\right].\tag{4.2}$$

Corollary 4.3. *Under stated the given conditions in Theorem 3, then left-sided fractional derivative identity holds true.*

$$\left(D_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho-1}S_{d}^{p_1,p_2,\kappa,p_s}(y_1t^{\lambda_1},\kappa,y_st^{\lambda_s})\mathbb{E}_{(\beta_j)_m,\kappa}^{(\alpha_j)_m,\tau,}(zt^{\nu})\right)\right)(u)=u^{\rho+\gamma-\mu-\mu'-1}$$

$$\times\sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s\leqslant d}(-d)_{p_1k_1,\cdots,p_sk_s}A(d;k_1,\cdots,k_s)\frac{\omega_1^{k_1}}{k_1!},\cdots,\frac{\omega_s^{k_s}}{k_s!}u^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\times{}_4\Psi_{m+3}\left[\begin{array}{c}(\rho-\gamma+\mu+\mu'+\varepsilon'+\sum_{j=1}^{s}\lambda_jk_j,\nu),(\rho-\varepsilon+\mu+\sum_{j=1}^{s}\lambda_jk_j,\ \nu),\\(\rho-\gamma+\mu-\varepsilon'+\sum_{j=1}^{s}\lambda_jk_j,\ \nu),\ (\rho-\varepsilon+\sum_{j=1}^{s}\lambda_jk_j,\ \nu),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho+\sum_{j=1}^{s}\lambda_jk_j,\nu),(\tau,\kappa)\\(\rho-\gamma+\mu-\mu'+\sum_{j=1}^{s}\lambda_jk_j,\ \nu),(\beta_j,\alpha_j)_{j=1}^{m}\end{array}\middle|zu^{\nu}\right].\tag{4.3}$$

Corollary 4.4. *Under stated the given conditions in Theorem 4, then following right-sided fractional derivative identity holds true.*

$$\left(D_{0-}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho-1}S_{d}^{p_1,p_2,\cdots,p_s}(y_1t^{\lambda_1},\cdots,y_st^{\lambda_s})\mathbb{E}_{(\beta_j)_m,\kappa}^{(\alpha_j)_m,\tau}(zt^{\nu})\right)\right)(u)=u^{\rho+\gamma-\mu-\mu'-1}$$

$$\times\sum_{k_1,\cdots,k_s=0}^{p_1k_1,\cdots,p_sk_s\leqslant d}(-d)_{p_1k_1,\cdots,p_sk_s}A(d;k_1,\cdots,k_s)\frac{y_1^{k_1}}{k_1!},\cdots,\frac{y_s^{k_s}}{k_s!}u^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\times{}_4\Psi_{m+3}\left[\begin{array}{c}(\rho-\mu-\mu'-\sum_{j=1}^{s}\lambda_jk_j,\nu),(\rho-\mu-\mu'-\sum_{j=1}^{s}\lambda_jk_j,\ \nu),\\(\rho-\mu-\mu'-\varepsilon-\sum_{j=1}^{s}\lambda_jk_j,\ \nu),\ (\rho-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\ \nu),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho-\mu'-\varepsilon-\sum_{j=1}^{s}\lambda_jk_j,\nu),(\tau,\kappa)\\(\rho-\gamma+\varepsilon'-\mu-\sum_{j=1}^{s}\lambda_jk_j,\ \nu),(\beta_j,\alpha_j)_{j=1}^{m}\end{array}\middle|zu^{-\nu}\right].\tag{4.4}$$

(ii) Taking s=1 and $A_{d,k}$=1 in (1.14), we have multivarible Srivastava polynomials reduces to the Gould-Hoper polynomials (see [28, p.8]) i.e,

$$S_d^p[y]\to(-1)^d\left(\frac{y}{h}\right)^{y/p}H_y^p\left[-\left(\frac{h}{y}\right)^{1/d},h\right],$$

our integral formula (2.1) readily yields the following special cases:

Corollary 4.5. *Under stated the given conditions in Theorem 1, then left sided fractional integral formula involving the product of Gould-Hopper polynomials and multi–index Bessel function is given as:*

$$\left(I_{0+}^{\mu,\mu',\varepsilon,\varepsilon',\gamma}\left(t^{\rho-1}H_d^p(yt^\lambda)\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\tau,c}(zt^\nu)\right)\right)(u) = u^{\rho+\gamma-\mu-\mu'-1}$$

$$\times\sum_{k=0}^{y/p}\left(\begin{array}{c} d \\ pk \end{array}\right)\frac{(pk)!}{(k)!}h^k y^{d-pk}u^{\lambda k}$$

$$\times\,{}_4\Psi_{m+3}\left[\begin{array}{c}(\rho+\gamma-\mu-\mu'-\varepsilon+\lambda k,\nu),(\rho+\varepsilon'-\mu'+\lambda k,\ \nu),\\ (\rho+\gamma-\mu-\mu'+\lambda k,\ \nu)\ (\rho+\varepsilon'+\lambda k,\ \nu),\end{array}\right.$$

$$\left.\begin{array}{c}(\rho+\lambda k,\nu),(\tau,\kappa)\\ (\rho+\gamma+\varepsilon'-\mu'+\lambda k,\ \nu),(\beta_j+\frac{b+1}{2},\alpha_j)_{j=1}^m\end{array}\middle|zcu^\nu\right]. \tag{4.5}$$

Further, consequences of integral formulas (2.3), (3.1), (3.3) including above polynomials, can be similarly deduced.

(iii) It can be easily seen that setting b=1, c=-1, d=0, $A_{0,0}$=1 then $S_0^p[y]\to 1$ in resulting identities [(2.1), (2.3)], respectively yields the corresponding known results in Agarwal *et al.* [2, P.296, Eqs.(3.2)(4.1)].

(iv) We have for m=1, multi–index Mittag-Leffler function reduces to four parameters Mittag-Leffler function and s=1, $S_d^{p_1\cdots p_s}[y]\to S_d^p[y]$. Therefore, applying these values in our results [(4.1), (4.2), (4.3), (4.4)], respectively, then we can easily obtain the known results investigated by Mishra *et al.* [18, p.4,8, Eqs.(16),(23),(29),(4)].

(v) On setting s=1, d=0, $A_{0,0}$=1 then $S_0^p[y]\to 1$ in obtained results [(2.2), (2), (3.2), (3.4)], then we can easily deduce the known results given by Suthar *et al.* [32, p.28,Eqs.(2.9),(2.11), (2.13),(2.15)].

(vi) Further, if we set multivariable Srivastava polynomials $S_d^{p_1\cdots p_s}[y]$ to unity with some suitable parametric replacements in resulting identities yields the corresponding known integral and derivative formulas in Agarwal and Nieto [1, p.540-541,Eqs.(14)(18)], Ahmed [3, p.2-4,Eqs.(3.1)(4.1)(5.1)(6.1)], Saxena *et al.* [26, p.17,20,Eqs.(2.4)(4.8)].

5. Discussion and conclusion

In this manuscript, we have investigated four image formulas of generalized fractional hypergeometric (of Marichev-Saigo-Maeda) operators involving the product of multivariable Srivastava polynomials and multi–index Bessel function, which are expressed in terms of Fox- Wright function. The results presented in this article are extensions of the known results given by various authors (see, e.g, [2, 3, 18, 19, 26, 32]). Moreover, the results derived in this paper also correspondence to Saigo hypergeometric fractional calculus operators as special cases and it can be easily seen that, if we set ε=-μ and ε=0 in (1.3) and (1.4), they yields the Erdelyi-Kober, the Riemann-Liouville, and the Weyl fractional integral and derivative operators. Thereby, the results presented here can also be obtained corresponding to the above well known fractional operators. Further, by suitably specializing the coefficients $A_{n,p}$ of the polynomials $S_d^{p_1\cdots p_s}[\omega]$, our results can be deduced to the classical orthogonal polynomials such as the Hermite polynomials $H_n[\omega]$, the Jacobi polynomials

$J_n^{(p,q)}[\omega]$, and Laguerre polynomials $L_n^{(p)}[\omega]$, and Bessel polynomials $Y_n[\omega, p, q]$. Therefore, the results derived in this article would at once give way a large number of results involving a many diversity of special functions occurring in the problems of mathematical physics, science, and engineering, etc.

Acknowledgment

The author K.S. Nisar thanks to Prince Sattam bin Abdulaziz University, Saudi Arabia for providing facilities and support.

Conflict of interest

The authors declare there is no conflicts of interest in this paper.

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