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Research article

Nonlinear multi-term fractional differential equations with Riemann-Stieltjes integro-multipoint boundary conditions

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Abstract: In this paper, we consider a nonlinear multi-term Caputo fractional differential equation with nonlinearity depending on the unknown function together with its lower-order Caputo fractional derivatives and equipped with Riemann-Stieltjes integro multipoint boundary conditions. The given problem is transformed to an equivalent fixed point problem, which is then solved with the aid of standard fixed point theorems to establish the existence and uniqueness results for the problem at hand. Examples are constructed for the illustration of the obtained results.

Keywords: Caputo derivative; Riemann-Stieltjes integral; multipoint boundary conditions; existence of solutions; fixed point
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1. Introduction

Fractional differential equations frequently appear in the mathematical modelling of many physical and engineering problems. One can find the potential application of fractional-order operators in malaria and HIV/AIDS model [1], bioengineering [2], ecology [3], viscoelasticity [4], fractional dynamical systems [5, 6] and so forth. Influenced by the practical applications of fractional calculus tools, many researchers turned to the further development of this branch of mathematical analysis. For the theoretical background of fractional derivatives and integrals, we refer the reader to the texts [7], while a detailed account of fractional differential equations can be found in [8, 9, 10]. In a recent monograph [11], the authors presented several results on initial and boundary value problems of Hadamard-type fractional differential equations and inclusions.

Fractional-order differential equations equipped with a variety of boundary conditions have been studied in the last few decades. The literature on the topic includes the existence and uniqueness results related to classical, periodic/anti-periodic, nonlocal, multi-point, and integral boundary conditions; for instance, see [12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and the references therein.

Recently, in [22], Ahmad *et al.* considered a boundary value problem involving sequential fractional derivatives given by

$$({}^cD^q + \mu {}^cD^{q-1})x(t) = f(t, x(t), {}^cD^\kappa x(t)),\ \mu > 0,\ \ 0 < \kappa < 1,\ 1 < q \le 2,\ t \in (0, 1), \tag{1.1}$$

supplemented with nonlocal integro-multipoint boundary conditions:

$$\begin{cases} \rho_1 x(0) + \rho_2 x(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i x(\sigma_i) + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} x(s)ds, \\ \rho_3 x'(0) + \rho_4 x'(1) = \displaystyle\sum_{i=1}^{m-2} \delta_i x'(\sigma_i) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} x'(s)ds, \\ 0 < \sigma_1 < \sigma_2 < \ldots < \sigma_{m-2} < \ldots < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \ldots < \xi_{p-2} < \eta_{p-2} < 1, \end{cases} \tag{1.2}$$

where ${}^cD^q$, ${}^cD^\kappa$ denote the Caputo fractional derivative of order q and κ respectively (for the definition of Caputo fractional derivative, see Definition 2.2), f is a given continuous function, $\rho_p (p = 1, 2, 3, 4)$ are real constants and α_i, δ_i $(i = 1, 2, \ldots, m-2)$, r_j, γ_j $(j = 1, 2, \ldots, p-2)$, are positive real constants. Existence and uniqueness results for the problem (1.2) were proved by using the fixed point theorems due to Banach and Krasnoselskii.

In [23], Ahmad *et al.* studied the existence and uniqueness of solutions for a new class of boundary value problems for multi-term fractional differential equations supplemented with four-point boundary conditions

$$\begin{cases} \lambda\, {}^CD^\alpha x(t) + {}^CD^\beta x(t) = f(t, x(t)), \quad t \in J := (0, T), \\ x'(\xi) = \nu\, {}^CD^\gamma x(\eta), \qquad x(T) = \mu I^\delta x(\theta), \quad 0 < \xi, \eta, \theta < T, \end{cases} \tag{1.3}$$

where ${}^CD^\chi$ is Caputo fractional derivatives of order $\chi \in \{\alpha, \beta, \gamma\}$, $\lambda, \nu, \mu \in \mathbb{R}$, $1 < \alpha \le 2$, $1 < \beta < \alpha$, $0 \le \gamma < \alpha - \beta < 1$, $\delta > 0$, I^δ is the Riemann-Liouville fractional integral of order δ, and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [24], the authors studied the existence of solutions for a nonlinear Liouville-Caputo-type fractional differential equation on an arbitrary domain:

$${}^cD_a^q x(t) = f(t, x(t)),\ 3 < q \le 4,\ t \in (a, b), \tag{1.4}$$

supplemented with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions of the form:

$$x(a) = \sum_{i=1}^{n-2} \alpha_i x(\eta_i) + \int_a^b x(s)dA(s),\ x'(a) = 0,\ x(b) = 0,\ x'(b) = 0, \tag{1.5}$$

where ${}^cD_a^q$ denotes the Caputo fractional derivative of order q, $a < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < b$, $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, A is a function of bounded variation, and $\alpha_i \in \mathbb{R}$, $i = 1, 2, \cdots, n-2$.

In the present paper, we investigate the existence of solutions for an abstract nonlinear multi-term Caputo fractional differential equation with nonlinearity depending on the unknown function together with its lower-order Caputo fractional derivatives given by

$$\mu\, {}^cD_a^q x(t) + \xi\, {}^cD_a^r x(t) = f(t, x(t), {}^cD_a^p x(t), {}^cD_a^{p+1} x(t)),\ 3 < q \le 4,\ 0 < p, r \le 1,\ t \in (a, b), \tag{1.6}$$

supplemented with Riemann-Stieltjes integro-multipoint boundary conditions

$$x(a) = \sum_{i=1}^{n-2} \alpha_i x(\eta_i) + \int_a^b x(s)dA(s),\ x'(a) = 0,\ x(b) = 0,\ x'(b) = 0, \tag{1.7}$$

where ${}^cD_a^\theta$ denotes the Caputo fractional differential operator of order θ with $\theta = q, r, p$, $a < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < b$, $f : [a,b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a given continuous function, A is a function of bounded variation, and μ $(\mu \neq 0), \xi, \alpha_i \in \mathbb{R},\ i = 1, 2, \cdots, n-2$.

The rest of the paper is arranged as follows. In section 2, we prove a basic result related to the linear variant of the problem (1.6)-(1.7), which plays a key role in the forthcoming analysis. We also recall some basic concepts of fractional calculus. The existence result is presented in Section 3, while the uniqueness result in Section 4. Examples illustrating the obtained results are also presented. The paper concludes with Section 5 with some interesting observations.

2. Basic result

Before presenting an auxiliary lemma, we recall some basic definitions of fractional calculus [8].

Definition 2.1. *The Riemann-Liouville fractional integral of order σ with lower limit a for function ϕ is defined as*

$$I_a^\sigma \phi(t) = \frac{1}{\Gamma(\sigma)} \int_a^t (t-s)^{\sigma-1} \phi(s) ds,\ \sigma > a,$$

provided the integral exists.

Definition 2.2. *For (n-1)-times absolutely continuous function $\phi : (a, \infty) \longrightarrow \mathbb{R}$, the Caputo derivative of fractional order σ is defined as*

$${}^cD_a^\sigma \phi(t) = \frac{1}{\Gamma(n-\sigma)} \int_a^t (t-s)^{n-\sigma-1} \phi^{(n)}(s) ds,\ n-1 < \sigma \leq n,\ n = [\sigma] + 1,$$

where $[\sigma]$ denotes the integer part of the real number σ.

In passing we remark that we write ${}^cD_a^\sigma$ and I_a^σ as ${}^cD^\sigma$ and I^σ respectively when $a = 0$.

Lemma 2.1. *[8] For $n-1 < q < n$, the general solution of the fractional differential equation ${}^cD_a^q x(t) = 0,\ t \in (a,b)$, is*

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_{n-1}(t-a)^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0, 1, \ldots, n-1$. Furthermore,

$$I_a^q\, {}^cD_a^q x(t) = x(t) + \sum_{i=0}^{n-1} c_i (t-a)^i.$$

Lemma 2.2. *Let $\psi \in C([a,b])$. Then the unique solution of the linear multi-term fractional differential equation*

$$\mu\, {}^cD_a^q x(t) + \xi\, {}^cD_a^r x(t) = \psi(t),\ 3 < q \leq 4,\ 0 < r < 1,\ t \in (a,b), \tag{2.1}$$

subject to the boundary conditions (1.7) is given by

$$x(t) = \frac{-\xi}{\mu} I_a^{q-r} x(t) + \frac{1}{\mu} I_a^q \psi(t) + \frac{1}{\mu}\Big[\phi_1(t)\Big(\xi I_a^{q-r} x(b) - I_a^q \psi(b)\Big) + \phi_2(t)\Big(\xi I_a^{q-r-1} x(b) - I_a^{q-1}\psi(b)\Big)$$
$$+\phi_3(t)\Big(\xi \sum_{i=1}^{n-2} \alpha_i I_a^{q-r} x(\eta_i) - \sum_{i=1}^{n-2} \alpha_i I_a^q \psi(\eta_i) + \xi \int_a^b I_a^{q-r} x(s) dA(s) - \int_a^b I_a^q \psi(s) dA(s)\Big)\Big], \quad (2.2)$$

where

$$\phi_i(t) = (t-a)^3\sigma_i + (t-a)^2\delta_i + \lambda_i, \ i = 1,2,3, \quad (2.3)$$
$$\lambda_1 = 1 - (b-a)^3\sigma_1 - (b-a)^2\delta_1, \ \lambda_j = -(b-a)^3\sigma_j - (b-a)^2\delta_j, \ j = 2,3, \quad (2.4)$$
$$\delta_1 = \frac{-3(b-a)\sigma_1}{2}, \ \delta_2 = \frac{1 - 3(b-a)^2\sigma_2}{2(b-a)}, \ \delta_3 = \frac{-3(b-a)\sigma_3}{2}, \quad (2.5)$$
$$\sigma_1 = \frac{-2A_1}{\gamma_1}, \ \sigma_2 = \frac{2\gamma_2}{\gamma_1}, \ \sigma_3 = \frac{2}{\gamma_1}, \quad (2.6)$$
$$\gamma_1 = (b-a)^3 A_1 - 3(b-a)A_2 + 2A_3, \ \gamma_2 = \frac{(b-a)^2 A_1 - A_2}{2(b-a)}, \quad (2.7)$$
$$A_1 = \sum_{i=1}^{n-2} \alpha_i + \int_a^b dA(s) - 1, \ A_2 = \sum_{i=1}^{n-2} \alpha_i(\eta_i - a)^2 + \int_a^b (s-a)^2 dA(s), \quad (2.8)$$
$$A_3 = \sum_{i=1}^{n-2} \alpha_i(\eta_i - a)^3 + \int_a^b (s-a)^3 dA(s), \quad (2.9)$$

and it is assumed that $\gamma_1 \neq 0$.

Proof. Applying the integral operator I_a^q on both sides of fractional differential equation (2.1) and using Lemma 2.1, we get

$$x(t) = \frac{-\xi}{\mu} I_a^{q-r} x(t) + \frac{1}{\mu} I_a^q \psi(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3, \quad (2.10)$$
$$x'(t) = \frac{-\xi}{\mu} I_a^{q-r-1} x(t) + \frac{1}{\mu} I_a^{q-1} \psi(s) + c_1 + 2c_2(t-a) + 3c_3(t-a)^2, \quad (2.11)$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,3$ are unknown arbitrary constants. Using the boundary conditions (1.7) in (2.10) and (2.11), we obtain $c_1 = 0$ and

$$c_0 + (b-a)^2 c_2 + (b-a)^3 c_3 = J_1, \quad (2.12)$$

$$2(b-a)c_2 + 3(b-a)^2 c_3 = J_2, \quad (2.13)$$

$$A_1 c_0 + A_2 c_2 + A_3 c_3 = J_3, \quad (2.14)$$

where A_i $(i = 1,2,3)$ are given by (2.8), (2.9) and

$$J_1 = \frac{1}{\mu}\Big(\xi I_a^{q-r} x(b) - I_a^q \psi(b)\Big), \ J_2 = \frac{1}{\mu}\Big(\xi I_a^{q-r-1} x(b) - I_a^{q-1}\psi(b)\Big),$$

$$J_3 = \frac{1}{\mu}\Big(\xi \sum_{i=1}^{n-2} \alpha_i I_a^{q-r} x(\eta_i) - \sum_{i=1}^{n-2} \alpha_i I_a^{q} \psi(\eta_i) + \xi \int_a^b I_a^{q-r} x(s) dA(s) - \int_a^b I_a^{q} \psi(s) dA(s)\Big). \quad (2.15)$$

Eliminating c_0 from (2.12) and (2.14), we get

$$(A_2 - (b-a)^2 A_1)c_2 + (A_3 - (b-a)^3 A_1)c_3 = J_3 - A_1 J_1. \quad (2.16)$$

Solving (2.13) and (2.16), we find that

$$c_2 = \delta_1 J_1 + \delta_2 J_2 + \delta_3 J_3, \quad (2.17)$$

$$c_3 = \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3, \quad (2.18)$$

where δ_i and σ_i ($i = 1, 2, 3$) are defined by (2.5) and (2.6) respectively. Using (2.17) and (2.18) in (2.12), we get

$$c_0 = \lambda_1 J_1 + \lambda_2 J_2 + \lambda_3 J_3, \quad (2.19)$$

where λ_i ($i = 1, 2, 3$) are given by (2.4). Inserting the values of c_0, c_1, c_2 and c_3 in (2.10) together with notations (2.3), we obtain the solution (2.2). The converse of the lemma follows by direct computation. □

Now we recall some preliminary concepts from functional analysis related to our work.

Definition 2.3. *Let Ω be a bounded set in metric space (Y, d). The Kuratowski measure of noncompactness, $\alpha(\Omega)$, is defined as*

$$\inf\{\varepsilon : \Omega \text{ covered by a finitely many sets such that the diameter of each set } \leq \varepsilon\}.$$

Definition 2.4. *[25] Let $\mathfrak{J} : \mathfrak{D}(\mathfrak{J}) \subseteq Y \longrightarrow Y$ be a bounded and continuous operator on Banach space Y. Then $\mathfrak{J}$ is called a condensing map if $\alpha(\mathfrak{J}(A)) < \alpha(A)$ for all bounded sets $A \subset \mathfrak{D}(\mathfrak{J})$, where α denotes the Kuratowski measure of noncompactness.*

Lemma 2.3. *[26] The map $F + G$ is a k-set contraction with $0 \leq k < 1$, and thus also condensing, if the following conditions hold:*

(*i*) *$F, G : \mathfrak{D} \subseteq Y \longrightarrow Y$ are operators on the Banach space Y;*
(*ii*) *F is k-contractive, that is, for all $x, y \in \mathfrak{D}$ and a fixed $k \in [0, 1)$, $||Fx - Fy|| \leq k||x - y||$;*
(*iii*) *G is compact.*

Lemma 2.4. *(Sadovskii Theorem [27]) Let A be a convex, bounded and closed subset of a Banach space Y and $\mathfrak{J} : A \longrightarrow A$ be a condensing map. Then $\mathfrak{J}$ has a fixed point.*

3. Existence of solutions

For $0 < p \leq 1$, let $\mathcal{A} = \{x : x, {}^cD_a^p x(t), {}^cD_a^{p+1} x(t) \in C([a, b], \mathbb{R})\}$ denote the Banach space of all continuous functions from $[a, b] \longrightarrow \mathbb{R}$ endowed with the norm defined by

$$||x||^* = \sup_{t\in[a,b]} \{|x(t)| + |{}^cD_a^p x(t)| + |{}^cD_a^{p+1} x(t)|\}. \quad (3.1)$$

In view of Lemma 2.2, we transform the problem (1.6)-(1.7) into an equivalent fixed point problem as

$$x = \mathcal{F}x, \tag{3.2}$$

where $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$\begin{aligned}
&(\mathcal{F}x)(t)\\
=\;& \frac{-\xi}{\mu} I_a^{q-r} x(t) + \frac{1}{\mu} I_a^q \widehat{f}(x(t)) + \frac{\phi_1(t)}{\mu}\Big[\xi \int_a^b \frac{(b-s)^{q-r-1}}{\Gamma(q-r)} x(s)ds - \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} \widehat{f}(x(s))ds\Big]\\
&+\frac{\phi_2(t)}{\mu}\Big[\xi \int_a^b \frac{(b-s)^{q-r-2}}{\Gamma(q-r-1)} x(s)ds - \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} \widehat{f}(x(s))ds\Big]\\
&+\frac{\phi_3(t)}{\mu}\Big[\xi \sum_{i=1}^{n-2} \alpha_i \int_a^{\eta_i} \frac{(\eta_i-s)^{q-r-1}}{\Gamma(q-r)} x(s)ds - \sum_{i=1}^{n-2} \alpha_i \int_a^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} \widehat{f}(x(s))ds\\
&+\xi \int_a^b \Big(\int_a^s \frac{(s-u)^{q-r-1}}{\Gamma(q-r)} x(u)du\Big)dA(s) - \int_a^b \Big(\int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} \widehat{f}(x(u))du\Big)dA(s)\Big],
\end{aligned} \tag{3.3}$$

where $\phi_i(t),\ i = 1, 2, 3$ are defined by (2.3) and $\widehat{f}(x(t)) = f(t, x(t), {}^cD_a^p x(t), {}^cD_a^{p+1} x(t))$.

From (3.3), we have

$$\begin{aligned}
&({}^cD_a^p \mathcal{F}x)(t)\\
=\;& \frac{-\xi}{\mu} I_a^{q-p-r} x(t) + \frac{1}{\mu} I_a^{q-p} \widehat{f}(x(t)) + \frac{\omega_1(t)}{\mu}\Big[\xi \int_a^b \frac{(b-s)^{q-r-1}}{\Gamma(q-r)} x(s)ds\\
&- \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)} \widehat{f}(x(s))ds\Big] + \frac{\omega_2(t)}{\mu}\Big[\xi \int_a^b \frac{(b-s)^{q-r-2}}{\Gamma(q-r-1)} x(s)ds - \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)} \widehat{f}(x(s))ds\Big]\\
&+\frac{\omega_3(t)}{\mu}\Big[\xi \sum_{i=1}^{n-2} \alpha_i \int_a^{\eta_i} \frac{(\eta_i-s)^{q-r-1}}{\Gamma(q-r)} x(s)ds - \sum_{i=1}^{n-2} \alpha_i \int_a^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} \widehat{f}(x(s))ds\\
&+\xi \int_a^b \Big(\int_a^s \frac{(s-u)^{q-r-1}}{\Gamma(q-r)} x(u)du\Big)dA(s) - \int_a^b \Big(\int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)} \widehat{f}(x(u))du\Big)dA(s)\Big],
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\omega_1(t) &= {}^cD_a^p \phi_1(t) = {}^cD_a^p((t-a)^3\sigma_1 + (t-a)^2\delta_1 + \lambda_1) = 6\sigma_1 \frac{(t-a)^{3-p}}{\Gamma(4-p)} + 2\delta_1 \frac{(t-a)^{2-p}}{\Gamma(3-p)},\\
\omega_2(t) &= {}^cD_a^p \phi_2(t) = {}^cD_a^p((t-a)^3\sigma_2 + (t-a)^2\delta_2 + \lambda_2) = 6\sigma_2 \frac{(t-a)^{3-p}}{\Gamma(4-p)} + 2\delta_2 \frac{(t-a)^{2-p}}{\Gamma(3-p)},\\
\omega_3(t) &= {}^cD_a^p \phi_3(t) = {}^cD_a^p((t-a)^3\sigma_3 + (t-a)^2\delta_3 + \lambda_3) = 6\sigma_3 \frac{(t-a)^{3-p}}{\Gamma(4-p)} + 2\delta_3 \frac{(t-a)^{2-p}}{\Gamma(3-p)},
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&({}^cD^{p+1} \mathcal{F}x)(t)\\
=\;& \frac{-\xi}{\mu} I_a^{q-p-r-1} x(t) + \frac{1}{\mu} I_a^{q-p-1} \widehat{f}(x(t)) + \frac{\nu_1(t)}{\mu}\Big[\xi \int_a^b \frac{(b-s)^{q-r-1}}{\Gamma(q-r)} x(s)ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)}\widehat{f}(x(s))ds\Big] + \frac{\nu_2(t)}{\mu}\Big[\xi\int_a^b \frac{(b-s)^{q-r-2}}{\Gamma(q-r-1)}x(s)ds - \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)}\widehat{f}(x(s))ds\Big]\\
&+\frac{\nu_3(t)}{\mu}\Big[\xi\sum_{i=1}^{n-2}\alpha_i\int_a^{\eta_i}\frac{(\eta_i-s)^{q-r-1}}{\Gamma(q-r)}x(s)ds - \sum_{i=1}^{n-2}\alpha_i\int_a^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\widehat{f}(x(s))ds\\
&+\xi\int_a^b\Big(\int_a^s \frac{(s-u)^{q-r-1}}{\Gamma(q-r)}x(u)du\Big)dA(s) - \int_a^b\Big(\int_a^s\frac{(s-u)^{q-1}}{\Gamma(q)}\widehat{f}(x(u))du\Big)dA(s)\Big], \qquad (3.6)
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_1(t) &= {}^cD_a^{p+1}\phi_1(t) = {}^cD_a^{p+1}((t-a)^3\sigma_1 + (t-a)^2\delta_1 + \lambda_1) = 6\sigma_1\frac{(t-a)^{2-p}}{\Gamma(3-p)} + 2\delta_1\frac{(t-a)^{1-p}}{\Gamma(2-p)},\\
\nu_2(t) &= {}^cD_a^{p+1}\phi_2(t) = {}^cD_a^{p+1}((t-a)^3\sigma_2 + (t-a)^2\delta_2 + \lambda_2) = 6\sigma_2\frac{(t-a)^{2-p}}{\Gamma(3-p)} + 2\delta_2\frac{(t-a)^{1-p}}{\Gamma(2-p)}, \qquad (3.7)\\
\nu_3(t) &= {}^cD_a^{p+1}\phi_3(t) = {}^cD_a^{p+1}((t-a)^3\sigma_3 + (t-a)^2\delta_3 + \lambda_3) = 6\sigma_3\frac{(t-a)^{2-p}}{\Gamma(3-p)} + 2\delta_3\frac{(t-a)^{1-p}}{\Gamma(2-p)}.
\end{aligned}
$$

For the sake of computational convenience, we introduce

$$
\begin{aligned}
\Lambda_1 &= \frac{|\xi|}{|\mu|}\Big[\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_1\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_2\frac{(b-a)^{q-r-1}}{\Gamma(q-r)} + \bar{\phi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^{q-r}}{\Gamma(q-r+1)}\\
&\quad + \int_a^b\frac{(s-a)^{q-r}}{\Gamma(q-r+1)}dA(s)\Big)\Big],\\
\bar{\Lambda}_1 &= \frac{1}{|\mu|}\Big[\frac{(b-a)^{q}}{\Gamma(q+1)} + \bar{\phi}_1\frac{(b-a)^{q}}{\Gamma(q+1)} + \bar{\phi}_2\frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{\phi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^{q}}{\Gamma(q+1)}\\
&\quad + \int_a^b\frac{(s-a)^{q}}{\Gamma(q+1)}dA(s)\Big)\Big], \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_2 &= \frac{|\xi|}{|\mu|}\Big[\frac{(b-a)^{q-p-r}}{\Gamma(q-p-r+1)} + \bar{\omega}_1\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\omega}_2\frac{(b-a)^{q-r-1}}{\Gamma(q-r)} + \bar{\omega}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^{q-r}}{\Gamma(q-r+1)}\\
&\quad + \int_a^b\frac{(s-a)^{q-r}}{\Gamma(q-r+1)}dA(s)\Big)\Big],\\
\bar{\Lambda}_2 &= \frac{1}{|\mu|}\Big[\frac{(b-a)^{q-p}}{\Gamma(q-p+1)} + \bar{\omega}_1\frac{(b-a)^{q}}{\Gamma(q+1)} + \bar{\omega}_2\frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{\omega}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^{q}}{\Gamma(q+1)}\\
&\quad + \int_a^b\frac{(s-a)^{q}}{\Gamma(q+1)}dA(s)\Big)\Big], \qquad (3.9)
\end{aligned}
$$

$$
\Lambda_3 = \frac{|\xi|}{|\mu|}\Big[\frac{(b-a)^{q-p-r-1}}{\Gamma(q-p-r)} + \bar{\nu}_1\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\nu}_2\frac{(b-a)^{q-r-1}}{\Gamma(q-r)} + \bar{\nu}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^{q-r}}{\Gamma(q-r+1)}
$$

$$+\int_a^b \frac{(s-a)^{q-r}}{\Gamma(q-r+1)}dA(s)\Big)\Big],$$

$$\bar{\Lambda}_3 = \frac{1}{|\mu|}\Big[\frac{(b-a)^{q-p-1}}{\Gamma(q-p)} + \bar{\nu}_1\frac{(b-a)^q}{\Gamma(q+1)} + \bar{\nu}_2\frac{(b-a)^{q-1}}{\Gamma(q)} + \bar{\nu}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^q}{\Gamma(q+1)} + \int_a^b \frac{(s-a)^q}{\Gamma(q+1)}dA(s)\Big)\Big], \tag{3.10}$$

where $\bar{\phi}_i = \sup_{t\in[a,b]}|\phi_i(t)|,\ \bar{\omega}_i = \sup_{t\in[a,b]}|\omega_i(t)|,\ \bar{\nu}_i = \sup_{t\in[a,b]}|\nu_i(t)|,\ i=1,2,3,$

$$\Delta = \max\{\Lambda_1,\ \Lambda_2,\ \Lambda_3\}, \tag{3.11}$$

$$\bar{\Delta} = max\{\bar{\Lambda}_1,\ \bar{\Lambda}_2,\ \bar{\Lambda}_3\}. \tag{3.12}$$

In the following result, we prove the existence of solutions for the problem (1.6)-(1.7) by applying Lemma 2.4.

Theorem 3.1. *Let $f : [a,b]\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ be a continuous function. Assume that:*

(O_1) *there exists a function $\rho \in C([a,b],\mathbb{R}_+)$ such that*

$$|f(t,x_1,x_2,x_3)| \le \rho(t),\ for\ t\in[a,b],\ and\ each\ x_i\in\mathbb{R},\ i=1,2,3;$$

(O_2) *$\kappa < 1$, where $\kappa = 3\Delta$ and Δ is defined by (3.11).*

Then problem (1.6)-(1.7) has at least one solution on $[a,b]$.

Proof. Consider a closed bounded and convex ball $B_\tau = \{x\in\mathcal{A} : \|x\|^* \le \tau\} \subseteq \mathcal{A}$, where τ is a fixed constant. Let us define $\mathcal{F}_1,\mathcal{F}_2 : B_\tau \longrightarrow B_\tau$ by

$$(\mathcal{F}_1x)(t) = \frac{-\xi}{\mu}I_a^{q-r}x(t) + \frac{\xi\,\phi_1(t)}{\mu}\int_a^b\frac{(b-s)^{q-r-1}}{\Gamma(q-r)}x(s)ds + \frac{\xi\,\phi_2(t)}{\mu}\int_a^b\frac{(b-s)^{q-r-2}}{\Gamma(q-r-1)}x(s)ds + \frac{\xi\,\phi_3(t)}{\mu}\Big[\sum_{i=1}^{n-2}\alpha_i\int_a^{\eta_i}\frac{(\eta_i-s)^{q-r-1}}{\Gamma(q-r)}x(s)ds + \int_a^b\Big(\int_a^s\frac{(s-u)^{q-r-1}}{\Gamma(q-r)}x(u)du\Big)dA(s)\Big],\ t\in[a,b]$$

$$(\mathcal{F}_2x)(t) = \frac{1}{\mu}I_a^q\widehat{f}(x(t)) - \frac{\phi_1(t)}{\mu}\int_a^b\frac{(b-s)^{q-1}}{\Gamma(q)}\widehat{f}(x(s))ds - \frac{\phi_2(t)}{\mu}\int_a^b\frac{(b-s)^{q-2}}{\Gamma(q-1)}\widehat{f}(x(s))ds - \frac{\phi_3(t)}{\mu}\Big[\sum_{i=1}^{n-2}\alpha_i\int_a^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\widehat{f}(x(s))ds + \int_a^b\Big(\int_a^s\frac{(s-u)^{q-1}}{\Gamma(q)}\widehat{f}(x(u))du\Big)dA(s)\Big],\ t\in[a,b].$$

Observe that,

$$(\mathcal{F}x)(t) = (\mathcal{F}_1x)(t) + (\mathcal{F}_2x)(t),\ t\in[a,b].$$

Now we show that $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy all the conditions of Lemma 2.4. The proof will be given in several steps.

Step 1. $\mathcal{F}B_\tau \subset B_\tau$.

Let us choose $\tau \geq \dfrac{3\|\rho\|\bar{\Delta}}{1-\kappa}$, where κ is defined in (O_2) and $\bar{\Delta}$ is given by (3.12). For $x \in B_\tau$, we have

$$\begin{aligned} |\mathcal{F}x(t)| &\leq \|x\|^* \frac{|\xi|}{|\mu|}\Big[\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_1 \frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_2 \frac{(b-a)^{q-r-1}}{\Gamma(q-r)} + \bar{\phi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^{q-r}}{\Gamma(q-r+1)} \\ &+ \int_a^b \frac{(s-a)^{q-r}}{\Gamma(q-r+1)}dA(s)\Big)\Big] + \frac{\|\rho\|}{|\mu|}\Big[\frac{(b-a)^q}{\Gamma(q+1)} + \bar{\phi}_1\frac{(b-a)^q}{\Gamma(q+1)} + \bar{\phi}_2\frac{(b-a)^{q-1}}{\Gamma(q)} \\ &+ \bar{\phi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^q}{\Gamma(q+1)} + \int_a^b \frac{(s-a)^q}{\Gamma(q+1)}dA(s)\Big)\Big] \\ &\leq \tau\Lambda_1 + \|\rho\|\bar{\Lambda}_1 \leq \tau(\kappa/3) + \|\rho\|\bar{\Delta}, \end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned} |{}^cD_a^p\mathcal{F}x(t)| &\leq \tau\Lambda_2 + \|\rho\|\bar{\Lambda}_2 \leq \tau(\kappa/3) + \|\rho\|\bar{\Delta}, \\ |{}^cD_a^{p+1}\mathcal{F}x(t)| &\leq \tau\Lambda_3 + \|\rho\|\bar{\Lambda}_3 \leq \tau(\kappa/3) + \|\rho\|\bar{\Delta}. \end{aligned}$$

Hence

$$\begin{aligned} \|\mathcal{F}x\|^* &= \sup_{x\in[a,b]}\{|\mathcal{F}x(t)| + |{}^cD_a^p\mathcal{F}x(t)| + |{}^cD_a^{p+1}\mathcal{F}x(t)|\} \\ &\leq \tau\kappa + 3\|\rho\|\bar{\Delta} < \tau. \end{aligned}$$

Thus we get $\mathcal{F}B_\tau \subset B_\tau$.

Step 2. $\mathcal{F}_1$ is a κ-contractive.
For $x, y \in B_\tau$ and using the condition O_2, we have

$$\begin{aligned} &\Big|(\mathcal{F}_1x)(t) - (\mathcal{F}_1y)(t)\Big| \\ \leq\ & \frac{|\xi|}{|\mu|}I_a^{q-r}|x(t)-y(t)| + \frac{|\xi||\phi_1(t)|}{|\mu|}I_a^{q-r}|x(b)-y(b)| + \frac{|\xi||\phi_2(t)|}{|\mu|}I_a^{q-r-1}|x(b)-y(b)| \\ &+ \frac{|\xi||\phi_3(t)|}{|\mu|}\Big(\sum_{i=1}^{n-2}|\alpha_i|I_a^{q-r}|x(\eta_i)-y(\eta_i)| + \int_a^b I_a^{q-r}|x(s)-y(s)|dA(s)\Big) \\ \leq\ & \frac{|\xi|}{|\mu|}\Big[\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_1\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_2\frac{(b-a)^{q-r-1}}{\Gamma(q-r)} + \bar{\phi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^{q-r}}{\Gamma(q-r+1)} \\ &+ \int_a^b \frac{(s-a)^{q-r}}{\Gamma(q-r+1)}dA(s)\Big)\Big]\|x-y\| \\ =\ & \Lambda_1\|x-y\| \leq (\kappa/3)\|x-y\|. \end{aligned}$$

Similarly, we can obtain

$$|{}^cD_a^p\mathcal{F}_1x(t) -{}^c D_a^p\mathcal{F}_1y(t)| \leq \Lambda_2\|x-y\| \leq (\kappa/3)\|x-y\|,$$

$$|{}^cD_a^{p+1}\mathcal{F}_1x(t) -{}^c D_a^{p+1}\mathcal{F}_1y(t)| \leq \Lambda_3\|x-y\| \leq (\kappa/3)\|x-y\|.$$

Hence

$$\|\mathcal{F}_1 x - \mathcal{F}_1 y\|^* \le \kappa \|x - y\|,$$

which proves that $\mathcal{F}_1$ is κ-contractive.

Step 3. $\mathcal{F}_2$ is compact.
Continuity of f implies that the operator $\mathcal{F}_2$ is continuous. Also, $\mathcal{F}_2$ is uniformly bounded on B_τ as

$$|\mathcal{F}_2 x(t)| \le \|\rho\|\bar{\Lambda}_1, \ |{}^c D_a^p \mathcal{F}_2 x(t)| \le \|\rho\|\bar{\Lambda}_2, \quad \text{and } |{}^c D_a^{p+1} \mathcal{F}_2 x(t)| \le \|\rho\|\bar{\Lambda}_3,$$

which imply that $\|\mathcal{F}_2 x\|^* \le 3\|\rho\|\bar{\Delta}$.

Let $t_1,\ t_2 \in [a,b]$ with $t_1 < t_2$ and $x \in B_\tau$. We have

$$\begin{aligned}
& |\mathcal{F}_2 x(t_2) - \mathcal{F}_2 x(t_1)| \\
\le\ & \frac{1}{|\mu|\Gamma(q)}\Big[\int_a^{t_1} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}||\rho(s)|ds + \int_{t_1}^{t_2} |(t_2 - s)^{q-1}||\rho(s)|ds\Big] \\
& + \frac{|\phi_1(t_2) - \phi_1(t_1)|}{|\mu|}\int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)}|\rho(s)|ds + \frac{|\phi_2(t_2) - \phi_2(t_1)|}{|\mu|}\int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)}|\rho(s)|ds \\
& + \frac{|\phi_3(t_2) - \phi_3(t_1)|}{|\mu|}\Big[\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\eta_i}\frac{(\eta_i - s)^{q-1}}{\Gamma(q)}|\rho(s)|ds \\
& + \int_a^b \Big(\int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)}|\rho(u)|du\Big)dA(s)\Big] \\
\le\ & \frac{\|\rho\|}{|\mu|\Gamma(q+1)}\Big[|(t_2 - a)^q - (t_1 - a)^q| + 2(t_2 - t_1)^q\Big] \\
& + \frac{\|\rho\|(b-a)^q|\phi_1(t_2) - \phi_1(t_1)|}{|\mu|\Gamma(q+1)} + \frac{\|\rho\|(b-a)^{q-1}|\phi_2(t_2) - \phi_2(t_1)|}{|\mu|\Gamma(q)} \\
& + \frac{\|\rho\||\phi_3(t_2) - \phi_3(t_1)|}{|\mu|\Gamma(q+1)}\Big[\sum_{i=1}^{n-2}|\alpha_i|(\eta_i - a)^q + \int_a^b (s-a)^q dA(s)\Big],
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
& |{}^c D_a^p \mathcal{F}_2 x(t_2) - {}^c D_a^p \mathcal{F}_2 x(t_1)| \\
\le\ & \frac{\|\rho\|}{|\mu|\Gamma(q-p+1)}\Big[|(t_2 - a)^{q-p} - (t_1 - a)^{q-p}| + 2(t_2 - t_1)^{q-p}\Big] \\
& + \frac{\|\rho\|(b-a)^q|\omega_1(t_2) - \omega_1(t_1)|}{|\mu|\Gamma(q+1)} + \frac{\|\rho\|(b-a)^{q-1}|\omega_2(t_2) - \omega_2(t_1)|}{|\mu|\Gamma(q)} \\
& + \frac{\|\rho\||\omega_3(t_2) - \omega_3(t_1)|}{|\mu|\Gamma(q+1)}\Big[\sum_{i=1}^{n-2}|\alpha_i|(\eta_i - a)^q + \int_a^b (s-a)^q dA(s)\Big],
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
& |{}^c D_a^{p+1} \mathcal{F}_2 x(t_2) - {}^c D_a^{p+1} \mathcal{F}_2 x(t_1)| \\
\le\ & \frac{\|\rho\|}{|\mu|\Gamma(q-p)}\Big[|(t_2 - a)^{q-p-1} - (t_1 - a)^{q-p-1}| + 2(t_2 - t_1)^{q-p-1}\Big]
\end{aligned}$$

$$+\frac{\|\rho\|(b-a)^q|\nu_1(t_2)-\nu_1(t_1)|}{|\mu|\Gamma(q+1)}+\frac{\|\rho\|(b-a)^{q-1}|\nu_2(t_2)-\nu_2(t_1)|}{|\mu|\Gamma(q)}$$
$$+\frac{\|\rho\||\nu_3(t_2)-\nu_3(t_1)|}{|\mu|\Gamma(q+1)}\Big[\sum_{i=1}^{n-2}|\alpha_i|(\eta_i-a)^q+\int_a^b(s-a)^q dA(s)\Big]. \tag{3.15}$$

The right hand sides of the inequalities (3.13)-(3.15) tend to zero as $t_2 - t_1 \longrightarrow 0$ independent of x. Thus, $\mathcal{F}_2$ is equicontinuous on B_τ. Therefore, by Arzelá-Ascoli theorem, $\mathcal{F}_2$ is a relatively compact on B_τ.

Step 4. $\mathcal{F}$ is condensing. Since $\mathcal{F}_1$ is continuous, κ-contractive and $\mathcal{F}_2$ is compact, therefore, by Lemma 2.3, the operator $\mathcal{F}: B_\tau \longrightarrow B_\tau$, with $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ is a condensing map on B_τ.

Hence, by Lemma 2.4, the operator $\mathcal{F}$ has a fixed point. Therefore, the problem (1.6)-(1.7) has at least one solution on $[a, b]$. □

Example 3.1. *Consider the fractional boundary value problem.*

$$\begin{cases} 45\, {}^cD^{\frac{35}{9}}x(t) + 9\, {}^cD^{\frac{1}{8}}x(t) = f(t, x(t), {}^cD^{\frac{1}{4}}x(t), {}^cD^{\frac{5}{4}}x(t)),\ t \in (0,1), \\ x(0) = \sum_{i=1}^{4}\alpha_i x(\eta_i) + \int_0^1 x(s)dA(s),\ x'(0) = 0,\ x(1) = 0,\ x'(1) = 0, \end{cases} \tag{3.16}$$

where $q = \frac{35}{9}$, $r = \frac{1}{8}$, $p = \frac{1}{4}$, $a = 0$, $b = 1$, $\mu = 45$, $\xi = 9$, $\alpha_1 = \frac{-1}{300}$, $\alpha_2 = \frac{-1}{22}$, $\alpha_3 = \frac{1}{240}$, $\alpha_4 = \frac{2}{39}$, $\eta_1 = \frac{1}{17}$, $\eta_2 = \frac{2}{17}$, $\eta_3 = \frac{3}{17}$, $\eta_4 = \frac{4}{17}$, *and*

$$f(t, x(t), {}^cD^{\frac{1}{4}}x(t), {}^cD^{\frac{5}{4}}x(t)) = \frac{1}{\sqrt{e^{2t}+80}}\Big(\frac{|x(t)|}{1+|x(t)|} + \sin^2({}^cD^{\frac{1}{4}}x(t)) + \cos({}^cD^{\frac{5}{4}}x(t))\Big).$$

Let us take $A(s) = \frac{s^2}{2}$. Using the given data, we have that $A_1 \simeq -0.493339$, $A_2 \simeq 0.252328$, $A_3 \simeq 0.200616$, $\gamma_1 \simeq -0.849088$, $\gamma_2 \simeq -0.372834$, $\sigma_1 \simeq -1.16204$, $\sigma_2 \simeq 0.878198$, $\sigma_3 \simeq -2.35546$, $\delta_1 \simeq 1.74306$, $\delta_2 \simeq -0.817295$, $\delta_3 \simeq 3.53319$, $\lambda_1 \simeq 0.41898$, $\lambda_2 \simeq -0.060903$, $\lambda_3 \simeq -1.17773$, $\bar{\phi}_1 \simeq 1.00000$, $\bar{\phi}_2 \simeq 0.113338$, $\bar{\phi}_3 \simeq 1.17773$, $\bar{\omega}_1 \simeq 0.591136$, $\bar{\omega}_2 \simeq 0.175009$, $\bar{\omega}_3 \simeq 1.19823$, $\bar{\nu}_1 \simeq 0.541872$, $\bar{\nu}_2 \simeq 1.49758$, $\bar{\nu}_3 \simeq 1.09837$, $\Lambda_1 \simeq 0.031094$, $\Lambda_2 \simeq 0.034096$, $\Lambda_3 \simeq 0.134535$, $\bar{\Lambda}_1 \simeq 0.000700$, $\bar{\Lambda}_2 \simeq 0.002538$, $\bar{\Lambda}_3 \simeq 0.012289$.
Also, the conditions O_1 and O_2 are satisfied as we have,

$$|f(t, x(t), {}^cD^{\frac{1}{4}}x(t), {}^cD^{\frac{5}{4}}x(t))| \le \frac{3}{\sqrt{e^{2t}+80}} = \rho(t),$$

and $\kappa \approx 0.403605 < 1$, where κ is defined in (O_2). Hence, the conditions of Theorem (3.1) hold. Therefore, from conclusion of Theorem 3.1 the problem (3.16) has at least one solution on $[0, 1]$.

4. Uniqueness of solutions

Next, we prove the uniqueness of solutions for the problem (1.6)-(1.7) via Banach fixed point theorem.

Theorem 4.1. *Assume that $f : [a,b]\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function such that,*

$$|f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)| \le L\Big(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\Big),$$
$$L>0,\ \forall\, t\in[a,b],\ x_i,y_i\in\mathbb{R},\ i=1,2,3. \tag{4.1}$$

Then the problem (1.6)-(1.7) has a unique solution on $[a,b]$ if

$$\kappa + 3L\bar{\Delta} < 1, \tag{4.2}$$

where κ is defined in (O_2) and $\bar{\Delta}$ is given by (3.12).

Proof. Setting $\sup_{t\in[a,b]}|f(t,0,0,0)| = \mathcal{N} < \infty$, and selecting

$$r^* \ge \frac{3\mathcal{N}\bar{\Delta}}{1-\kappa-3L\bar{\Delta}},$$

we define $B_{r^*} = \{x\in\mathcal{A} : \|x\|^* \le r^*\}$, and show that $\mathcal{F}B_{r^*}\subset B_{r^*}$, where the operator $\mathcal{F}$ is defined by (3.3). For $x\in B_{r^*}$, we use (4.1) to find that

$$\begin{aligned}
|f(t,x(t),{}^cD_a^p x(t),{}^cD_a^{p+1}x(t))| &= |f(t,x(t),{}^cD_a^p x(t),{}^cD_a^{p+1}x(t)) - f(t,0,0,0) + f(t,0,0,0)|\\
&\le |f(t,x(t),{}^cD_a^p x(t),{}^cD_a^{p+1}x(t)) - f(t,0,0,0)| + |f(t,0,0,0)|\\
&\le L(|x(t)| + |{}^cD_a^p x(t)| + |{}^cD_a^{p+1}x(t)|) + \mathcal{N}\\
&\le L\|x\|^* + \mathcal{N} \le Lr^* + \mathcal{N},
\end{aligned}$$

where we used the norm given by (3.1).
Then, we have

$$\begin{aligned}
|\mathcal{F}x(t)| &\le r^*\frac{|\xi|}{|\mu|}\Bigg[\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_1\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_2\frac{(b-a)^{q-r-1}}{\Gamma(q-r)} + \bar{\phi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^{q-r}}{\Gamma(q-r+1)}\\
&\quad + \int_a^b \frac{(s-a)^{q-r}}{\Gamma(q-r+1)}dA(s)\Big)\Bigg] + \frac{(Lr^*+\mathcal{N})}{|\mu|}\Bigg[\frac{(b-a)^q}{\Gamma(q+1)} + \bar{\phi}_1\frac{(b-a)^q}{\Gamma(q+1)} + \bar{\phi}_2\frac{(b-a)^{q-1}}{\Gamma(q)}\\
&\quad + \bar{\phi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\eta_i-a)^q}{\Gamma(q+1)} + \int_a^b\frac{(s-a)^q}{\Gamma(q+1)}dA(s)\Big)\Bigg]\\
&\le r^*\Lambda_1 + (Lr^*+\mathcal{N})\bar{\Lambda}_1 \le (\kappa/3)r^* + (Lr^*+\mathcal{N})\bar{\Delta}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|{}^cD_a^p\mathcal{F}x(t)| &\le r^*\Lambda_2 + (Lr^*+\mathcal{N})\bar{\Lambda}_2 \le (\kappa/3)r^* + (Lr^*+\mathcal{N})\bar{\Delta}.\\
|{}^cD_a^{p+1}\mathcal{F}x(t)| &\le r^*\Lambda_3 + (Lr^*+\mathcal{N})\bar{\Lambda}_3 \le (\kappa/3)r^* + (Lr^*+\mathcal{N})\bar{\Delta}.
\end{aligned}$$

Hence we have

$$\|\mathcal{F}x\|^* \le \kappa r^* + 3(Lr^*+\mathcal{N})\bar{\Delta} < r^*.$$

Thus, $\mathcal{F}x \in B_{r^*}$ for any $x \in B_{r^*}$. Therefore, $\mathcal{F}B_{r^*} \subset B_{r^*}$. Now, we show that $\mathcal{F}$ is a contraction. For $x, y \in \mathcal{A}$ and $t \in [a, b]$, we obtain

$$\begin{aligned}
&\Big|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\Big| \\
\le\ & \frac{|\xi|}{|\mu|} I_a^{q-r}|x(t) - y(t)| + \frac{1}{|\mu|} I_a^q|\widehat{f}(x(t)) - \widehat{f}(y(t))| + \frac{1}{|\mu|}\Big[|\phi_1(t)|\Big(|\xi| \int_a^b \frac{(b-s)^{q-r-1}}{\Gamma(q-r)}|x(s) - y(s)|ds \\
&+ \int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)}|\widehat{f}(x(s)) - \widehat{f}(y(s))|ds\Big) + |\phi_2(t)|\Big(|\xi| \int_a^b \frac{(b-s)^{q-r-2}}{\Gamma(q-r-1)}|x(s) - y(s)|ds \\
&+ \int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)}|\widehat{f}(x(s)) - \widehat{f}(y(s))|ds\Big) + |\phi_3(t)|\Big(|\xi| \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\eta_i} \frac{(\eta_i - s)^{q-r-1}}{\Gamma(q-r)}|x(s) - y(s)|ds \\
&+ \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\eta_i} \frac{(\eta_i - s)^{q-1}}{\Gamma(q)}|\widehat{f}(x(s)) - \widehat{f}(y(s))|ds + |\xi| \int_a^b \Big(\int_a^s \frac{(s-u)^{q-r-1}}{\Gamma(q-r)}|x(u) - y(u)|du\Big) dA(s) \\
&+ \int_a^b \Big(\int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)}|\widehat{f}(x(u)) - \widehat{f}(y(u))|du\Big) dA(s)\Big)\Big) \\
\le\ & \frac{|\xi|}{|\mu|}\Big[\frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_1 \frac{(b-a)^{q-r}}{\Gamma(q-r+1)} + \bar{\phi}_2 \frac{(b-a)^{q-r-1}}{\Gamma(q-r)} + \bar{\phi}_3\Big(\sum_{i=1}^{n-2} |\alpha_i| \frac{(\eta_i - a)^{q-r}}{\Gamma(q-r+1)} \\
&+ \int_a^b \frac{(s-a)^{q-r}}{\Gamma(q-r+1)} dA(s)\Big)\Big]\|x - y\| + \frac{1}{|\mu|}\Big[\frac{(b-a)^q}{\Gamma(q+1)} + \bar{\phi}_1 \frac{(b-a)^q}{\Gamma(q+1)} + \bar{\phi}_2 \frac{(b-a)^{q-1}}{\Gamma(q)} \\
&+ \bar{\phi}_3\Big(\sum_{i=1}^{n-2} |\alpha_i| \frac{(\eta_i - a)^q}{\Gamma(q+1)} + \int_a^b \frac{(s-a)^q}{\Gamma(q+1)} dA(s)\Big)\Big] L\|x - y\| \\
=\ & (\Lambda_1 + L\bar{\Lambda}_1)\|x - y\| \le (\kappa/3 + L\bar{\Delta})\|x - y\|,
\end{aligned}$$

In a similar mannar, we have

$$\begin{aligned}
|^c D_a^p \mathcal{F}x(t) -^c D_a^p \mathcal{F}y(t)| &\le (\Lambda_2 + L\bar{\Lambda}_2)\|x - y\| \le (\kappa/3 + L\bar{\Delta})\|x - y\|, \\
|^c D_a^{p+1} \mathcal{F}x(t) -^c D_a^{p+1} \mathcal{F}y(t)| &\le (\Lambda_3 + L\bar{\Lambda}_3)\|x - y\| \le (\kappa/3 + L\bar{\Delta})\|x - y\|.
\end{aligned}$$

Consequently, we obtain $\|(\mathcal{F}x) - (\mathcal{F}y)\|^* \le (\kappa + 3L\bar{\Delta})\|x - y\|$, which in view of (4.2) implies that the operator $\mathcal{F}$ is a contraction. Therefore, $\mathcal{F}$ has a unique fixed point, which corresponds to a unique solution of the problem (1.6)-(1.7) on $[a, b]$. This completes the proof. □

Example 4.1. *Consider the following fractional differential equation*

$$45\, {}^c D^{\frac{35}{9}} x(t) + 9\, {}^c D^{\frac{1}{8}} x(t) = \frac{1}{6(t^2+2)}\Big(\tan^{-1} x(t) + \cos({}^c D^{\frac{1}{4}} x(t)) + \frac{|{}^c D^{\frac{5}{4}} x(t)|}{1 + |{}^c D^{\frac{5}{4}} x(t)|}\Big), \tag{4.3}$$

$t \in [0, 1]$, *supplemented with the boundary conditions of Example (3.1).*
Obviously

$$f(t, x(t), {}^c D^{\frac{1}{4}} x(t), {}^c D^{\frac{5}{4}} x(t)) = \frac{1}{6(t^2+2)}\Big(\tan^{-1} x(t) + \cos({}^c D^{\frac{1}{4}} x(t)) + \frac{|{}^c D^{\frac{5}{4}} x(t)|}{1 + |{}^c D^{\frac{5}{4}} x(t)|}\Big).$$

Using the given data, we find that $\Delta \simeq 0.134535$ and $\bar{\Delta} \simeq 0.012289$, where Δ and $\bar{\Delta}$ are respectively given by (3.11) and (3.12). By the following inequality

$$|f(t, x(t), {}^cD^{\frac{1}{4}}x(t), {}^cD^{\frac{5}{4}}x(t)) - f(t, y(t), {}^cD^{\frac{1}{4}}y(t), {}^cD^{\frac{5}{4}}y(t))|$$

$$\leq \frac{1}{12}\Big(|x-y| + |{}^cD^{\frac{1}{4}}x - {}^cD^{\frac{1}{4}}y| + |{}^cD^{\frac{5}{4}}x - {}^cD^{\frac{5}{4}}y|\Big) \leq \frac{1}{12}\|x-y\|,$$

we have $L = \frac{1}{12}$. Clearly $(\kappa + 3L\bar{\Delta}) \simeq 0.406677 < 1$. Therefore, the hypothesis of Theorem (4.1) is satisfied and consequently the problem (4.3) has a unique solution on $[0,1]$.

Remark 4.1. *Letting $\xi = 0$ and $\mu = 1$ in the results of this paper, we obtain the ones for the fractional differential equation of the form:*

$${}^cD^q x(t) = f(t, x(t), {}^cD^p x(t), {}^cD^{p+1}x(t)),\ 3 < q \leq 4,\ 0 < p \leq 1,\ t \in [a,b],$$

supplemented with Riemann-Stieltjes integro-multipoint boundary conditions (1.7). In this case fixed point operator takes the form:

$$\begin{aligned}(\mathcal{F}x)(t) &= I^q\widehat{f}(x(t)) - \phi_1(t)\int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)}\widehat{f}(x(s))ds - \phi_2(t)\int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)}\widehat{f}(x(s))ds \\ &\quad -\phi_3(t)\Big[\sum_{i=1}^{n-2}\alpha_i\int_a^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\widehat{f}(x(s))ds + \int_a^b\Big(\int_a^s \frac{(s-u)^{q-1}}{\Gamma(q)}\widehat{f}(x(u))du\Big)dA(s)\Big].\end{aligned}$$

5. Conclusions

We have proved the existence and uniqueness results for a multi-term Caputo fractional differential equation with nonlinearity depending upon the known function x together with its lower-order derivatives ${}^cD_a^p x$, ${}^cD_a^{p+1}x, 0 < p < 1$, complemented with Riemann-Stieltjes integro multipoint boundary conditions.

In Theorem 3.1, the existence of solutions for the given problem is established by means of Sadovskii fixed point theorem. The proof of this result is based on the idea of splitting the operator $\mathcal{F}$ into the sum of two operators $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F}_1$ is κ-contractive and $\mathcal{F}_2$ is compact. One can notice that the entire operator $\mathcal{F}$ is not required to be contractive. On the other hand, Theorem 4.1 deals with the existence of a unique solution of the given problem via Banach contraction mapping principle, in which the entire operator $\mathcal{F}$ is shown to be contractive. Thus, the linkage between contractive conditions imposed in Theorems 3.1 and 4.1 provides a precise estimate to pass onto a unique solution from the existence of a solution for the problem at hand.

As a special case, by letting $\xi = 0$ and $\mu = 1$ in the results of this paper, we obtain the ones for the fractional differential equation of the form:

$${}^cD_a^q x(t) = f(t, x(t), {}^cD_a^p x(t), {}^cD_a^{p+1}x(t)),\ 3 < q \leq 4,\ 0 < p \leq 1,\ t \in (a,b),$$

supplemented with Riemann-Stieltjes integro-multipoint boundary conditions (1.7). In this case, the fixed point operator (3.3) takes the following form:

$$(\mathcal{F}x)(t) = I_a^q\widehat{f}(x(t)) - \phi_1(t)\int_a^b \frac{(b-s)^{q-1}}{\Gamma(q)}\widehat{f}(x(s))ds - \phi_2(t)\int_a^b \frac{(b-s)^{q-2}}{\Gamma(q-1)}\widehat{f}(x(s))ds$$

$$-\phi_3(t)\Big[\sum_{i=1}^{n-2}\alpha_i\int_a^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\widehat{f}(x(s))ds+\int_a^b\Big(\int_a^s\frac{(s-u)^{q-1}}{\Gamma(q)}\widehat{f}(x(u))du\Big)dA(s)\Big].$$

In case we take $\alpha_i = 0$ for all $i = 1, \ldots, n-2$, then our results correspond to the integral boundary conditions:

$$x(a)=\int_a^b x(s)dA(s),\ x'(a)=0,\ x(b)=0,\ x'(b)=0.$$

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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