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Research article

Some parameterized integral inequalities for p-convex mappings via the right Katugampola fractional integrals

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Abstract: We use the definition of a fractional integral operators, proposed by Katugampola, to establish a fractional Hermite–Hadamard's inequality for p-convex mappings and an identity with two parameters. We derive several parameterized integral inequalities associated with this identity, and provide three examples to illustrate the obtained results.

Keywords: Hermite–Hadamard's inequality; Simpson's inequality; Riemann–Liouville fractional integrals; Hadamard fractional integrals
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1. Introduction

If $f : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping and $a, b \in \mathcal{I}$ with $a \neq b$, then one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which is called as the Hermite-Hadamard's inequality in the literature.

The following inequality is named the Simpson's integral inequality:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4, \tag{1.2}$$

where $f : [a, b] \to \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty$.

For recent results concerning inequality (1.1) and (1.2), we refer the interested reader to [5, 6, 17, 21, 23, 25] and the references cited therein.

Let us review some concepts and the related results.

Definition 1. [30] *A function f defined on $\mathcal{I} \subseteq \mathbb{R}$ has a support at $x_0 \in \mathcal{I}$ if there exists an affine functions $A(x) = f(x_0) + \varrho(x - x_0)$ such that $A(x) \leq f(x)$ for all $x \in \mathcal{I}$. The graph of the support function A is called a line of support for f at x_0.*

Theorem 1. [30] *$f : (a, b) \to \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a, b)$.*

Definition 2. [11] *Let $\mathcal{I} \subseteq (0, \infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A mapping $g : \mathcal{I} \to \mathbb{R}$ is said to be a p-convex mapping, if*

$$g\Big(\big[tx^p + (1-t)y^p\big]^{\frac{1}{p}}\Big) \leq tg(x) + (1-t)g(y)$$

for all $x, y \in \mathcal{I}$ and $t \in [0, 1]$.

Many authors have worked in the inequalities and properties for p-convex mappings. For example, Zhang and Wan [39] presented some properties for p-convex mappings. Noor et al. [28] gave several Hermite–Hadamard's inequalities through p-convexity. İşcan et al. [12] established several Hermite–Hadamard's inequalities through p-quasi-convexity. Further inequalities of the Hermite–Hadamard type related to p-convexity in question with applications to fractional integrals can be found in [19, 34]. For more results related to p-convex mappings, please see [18, 29] and the references cited therein.

The following result is obvious.

Theorem 2. *If $g : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ and if we consider the mapping $f : [a^p, b^p] \to \mathbb{R}$, defined by $f(t) = g(t^{\frac{1}{p}})$, $p \in \mathbb{R} \setminus \{0\}$, then g is p-convex on $[a, b]$ if and only if f is convex on $[a^p, b^p]$, $p > 0$ (or $[b^p, a^p]$, $p < 0$).*

In [27], Noor et al. presented the following lemma to obtain Simpson type inequalities.

Lemma 1. *Let $g : [a, b] \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on (a, b). If $g' \in L^1([a, b])$ and $p \in \mathbb{R} \setminus \{0\}$, then*

$$\begin{aligned}
&\frac{1}{6}\left[g(a) + 4g\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) + g(b)\right] - \frac{p}{(b^p - a^p)} \int_a^b \frac{g(x)}{x^{1-p}} \mathrm{d}x \\
&= \frac{(b^p - a^p)}{p} \int_0^1 \frac{\mu(t)}{\left[(1-t)a^p + tb^p\right]^{1-\frac{1}{p}}} g'\Big(\left[(1-t)a^p + tb^p\right]^{\frac{1}{p}}\Big) \mathrm{d}t,
\end{aligned}$$

where

$$\mu(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}), \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

In [7], by introducing two parameters, Du et al. established the following lemma.

Lemma 2. *Let $g : [a,b] \subseteq \mathbb{R}^+ \to \mathbb{R}$ be differentiable on (a,b). For some fixed $m \in (0,1]$, if $g' \in L^1([a,b])$ and $k, t \in \mathbb{R}$, then for each $x \in [ma,b]$ the following equality holds:*

$$tg(ma) + (1-k)g(b) + (k-t)g\Big(\frac{b+ma}{2}\Big) - \frac{1}{b-ma}\int_{ma}^{b} g(x)\mathrm{d}x$$
$$= (b-ma)\Big[\int_0^{\frac{1}{2}} (x-t)g'(xb+m(1-x)a)\mathrm{d}x + \int_{\frac{1}{2}}^{1} (x-k)g'(xb+m(1-x)a)\mathrm{d}x\Big].$$

Next, we recall certain fractional integral operators that are of importance to our work.

Definition 3. *Let $f \in L^1([a,b])$. The Riemann-Liouville integrals $\mathcal{J}^{\mu}_{a^+}f$ and $\mathcal{J}^{\mu}_{b^-}f$ of order $\mu > 0$ are defined as*

$$\mathcal{J}^{\mu}_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)\mathrm{d}t$$

and

$$\mathcal{J}^{\mu}_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1}f(t)\mathrm{d}t,$$

with $a < x < b$ and $\Gamma(\cdot)$ is the gamma function defined by $\Gamma(\mu) = \int_0^\infty e^{-t}t^{\mu-1}\mathrm{d}t, \mathrm{Re}(\mu) > 0$.

In [20], Kunt et al. proved the following right Riemann-Liouville fractional Hermite–Hadamard type inequality.

Theorem 3. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a convex mapping. If $f \in L^1([a,b])$, then the following inequality for the right Riemann–Liouville fractional integral holds:*

$$f\left(\frac{a+\mu b}{\mu+1}\right) \le \frac{\Gamma(\mu+1)}{(b-a)^\mu}\mathcal{J}^{\mu}_{b^-}f(a) \le \frac{f(a)+\mu f(b)}{\mu+1} \tag{1.3}$$

with $\mu > 0$.

For recent results related to Riemann-Liouville fractional integrals, the interested reader is referred, for example, to [9, 31, 32, 37] and the references therein.

Definition 4. *Let $f \in L^1([a,b])$, then the left-sided and right-sided Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ are defined as*

$$\mathcal{H}^{\alpha}_{a^+}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\ln\Big(\frac{t}{\tau}\Big)\Big)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}$$

and

$$\mathcal{H}^{\alpha}_{b^-}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_t^b \Big(\ln\Big(\frac{\tau}{t}\Big)\Big)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau},$$

with $a < x < b$.

For more information about the Hadamard fractional integrals and related results, the interested reader is referred, for example, to [1, 36] and the references therein.

In 2011, Katugampola [14] presented a class of fractional integral operator, which generalizes Riemann-Liouville and Hadamard fractional integral operators into a single form.

Definition 5. *Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $\alpha > 0$ of $g \in \chi_c^\sigma(a,b)$ $(c \in \mathbb{R}, 1 \le \sigma \le \infty)$ are defined by*

$$^{\rho}I_{a^+}^{\alpha} g(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} g(t)\mathrm{d}t$$

and

$$^{\rho}I_{b^-}^{\alpha} g(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} g(t)\mathrm{d}t$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Theorem 4. [14] *Let $\alpha > 0$ and $\rho > 0$. Then for $x < b$,*
(i) $\lim_{\rho\to1} {}^{\rho}I_{b^-}^{\alpha} g(x) = \mathcal{J}_{b^-}^{\alpha} g(x)$;
(ii) $\lim_{\rho\to0^+} {}^{\rho}I_{b^-}^{\alpha} g(x) = \mathcal{H}_{b^-}^{\alpha} g(x)$.
Similar results also hold for left-sided operators.

The great influence of Katugampola fractional integrals in pure science and applied science is undeniable. Recently, the study of some well-known integral inequalities for the Katugampola fractional integrals has been carried out by some authors, including Chen and Katugampola [3] and Jleli et al. [13] in the study of Hermite–Hadamard type inequalities for convex mappings, Kermausuor [15] in the study of the generalized Ostrowski type inequalities for strong (s,m)-convex mappings, Mumcu et al. [26] in the Hermite–Hadamard type inequalities for harmonically convex mappings, Sousa and Capelas de Oliveira [35] in the study of a generalization of the reverse Minkowski's inequality. Moreover, some applications related to Katugampola fractional integrals can be found in [22, 38]. For more results related to the Katugampola fractional integral operators, the interested reader is directed to [2, 8, 10, 24, 33] and the references cited therein.

Motivated by the results in the papers above, especially the results developed in [7, 20] and [33], this work aims to investigate certain integral inequalities for p-convex mappings, which are related to the famous Hermite–Hadamard's inequality and Simpson's inequality. For this purpose, using only the right Katugampola fractional integrals, we establish a fractional Hermite–Hadamard type inequalities for p-convex mappings. Also, we present a fractional integral identity with two parameters. Using this integral identity, we derive certain parameterized integral inequalities, which unifies the Hermite–Hadamard type inequalities, the Simpson type inequality, the averaged midpoint-trapezoid inequality, as well as the trapezoid inequality. This is the main contribution of this work.

2. The right fractional Hermite–Hadamard's inequality

Using the right Katugampola fractional integrals, we have the following Hermite–Hadamard's inequalities for p-convex functions.

Theorem 5. *Let $g : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a p-convex mapping, $p > 0$ and $a < b$. If $g \in L^1([a,b])$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$g\left(\left[\frac{a^p + \alpha b^p}{\alpha + 1}\right]^{\frac{1}{p}}\right) \le \frac{p^\alpha \Gamma(\alpha+1)}{(b^p - a^p)^\alpha} {}^{p}I_{b^-}^{\alpha} g(a) \le \frac{g(a) + \alpha g(b)}{\alpha + 1}. \tag{2.1}$$

Proof. Suppose g is a p-convex mapping on $[a,b]$. Let $f(x)=g(x^{\frac{1}{p}})$, by Theorem 2, we can get f is a convex mapping on $[a^p,b^p]$ with $f\in L^1([a^p,b^p])$. Using Theorem 1, there is least one line of support

$$A(x)=f\Big(\frac{a^p+\alpha b^p}{\alpha+1}\Big)+\varrho\Big(x-\frac{a^p+\alpha b^p}{\alpha+1}\Big)\le f(x)$$

for all $x\in[a^p,b^p]$ and $\varrho\in\Big[f'_-\big(\frac{a^p+\alpha b^p}{\alpha+1}\big),f'_+\big(\frac{a^p+\alpha b^p}{\alpha+1}\big)\Big]$, that is,

$$g\bigg(\Big[\frac{a^p+\alpha b^p}{\alpha+1}\Big]^{\frac{1}{p}}\bigg)+\varrho\Big(x-\frac{a^p+\alpha b^p}{\alpha+1}\Big)\le g(x^{\frac{1}{p}}).$$

Using the change of variable $x=tb^p+(1-t)a^p$ for $t\in[0,1]$, we have

$$g\bigg(\Big[\frac{a^p+\alpha b^p}{\alpha+1}\Big]^{\frac{1}{p}}\bigg)+\varrho\Big(tb^p+(1-t)a^p-\frac{a^p+\alpha b^p}{\alpha+1}\Big)\le g\Big(\Big[tb^p+(1-t)a^p\Big]^{\frac{1}{p}}\Big). \tag{2.2}$$

Multiplying both sides of (2.2) by $\alpha t^{\alpha-1}$ and integrating over $[0,1]$ with respect to t, we obtain

$$\begin{aligned}
&\alpha\int_0^1 t^{\alpha-1}\mathrm{d}t\cdot g\bigg(\Big[\frac{a^p+\alpha b^p}{\alpha+1}\Big]^{\frac{1}{p}}\bigg)\\
&\quad+\varrho\alpha\bigg\{\int_0^1 t^{\alpha-1}\Big[tb^p+(1-t)a^p\Big]\mathrm{d}t-\int_0^1 t^{\alpha-1}\mathrm{d}t\cdot\frac{a^p+\alpha b^p}{\alpha+1}\bigg\}\\
&=g\bigg(\Big[\frac{a^p+\alpha b^p}{\alpha+1}\Big]^{\frac{1}{p}}\bigg)+\varrho\Big(\frac{a^p+\alpha b^p}{\alpha+1}-\frac{a^p+\alpha b^p}{\alpha+1}\Big)\\
&=g\bigg(\Big[\frac{a^p+\alpha b^p}{\alpha+1}\Big]^{\frac{1}{p}}\bigg)\\
&\le\alpha\int_0^1 t^{\alpha-1}g\Big(\Big[tb^p+(1-t)a^p\Big]^{\frac{1}{p}}\Big)\mathrm{d}t\\
&=\frac{\alpha p}{(b^p-a^p)^{\alpha}}\int_a^b\frac{\lambda^{p-1}}{(\lambda^p-a^p)^{1-\alpha}}g(\lambda)\mathrm{d}\lambda\\
&=\frac{p^{\alpha}\Gamma(\alpha+1)}{(b^p-a^p)^{\alpha}}\,{}^{p}I^{\alpha}_{b^-}g(a),
\end{aligned}\tag{2.3}$$

which proves the left part of inequality (2.1).

On the other hand, using the p-convexity of g on $[a,b]$, we have

$$\begin{aligned}
\frac{p^{\alpha}\Gamma(\alpha+1)}{(b^p-a^p)^{\alpha}}\,{}^{p}I^{\alpha}_{b^-}g(a)&=\alpha\int_0^1 t^{\alpha-1}g\Big(\Big[tb^p+(1-t)a^p\Big]^{\frac{1}{p}}\Big)\mathrm{d}t\\
&\le\alpha\int_0^1 t^{\alpha-1}\Big[tg(b)+(1-t)g(a)\Big]\mathrm{d}t\\
&=\frac{\alpha}{\alpha+1}g(b)+\frac{1}{\alpha+1}g(a),
\end{aligned}$$

which proves the right part of inequality (2.1). Thus, the proof is completed. □

Remark 1. *In Theorem 5, taking limit $p \to 1$ we obtain inequality* (1.3) *with $a > 0$.*

Remark 2. *The left Katugampola fractional Hermite–Hadamard type inequalities for p-convex mappings can be established based on the similar approach adopted in Theorem 5, and we omit the details.*

3. Main results

We prove the following lemma for our results.

Lemma 3. *Assume that $g : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable mapping on (a, b) satisfying $g' \in L^1([a, b])$, where $0 < a < b$. Then, for some fixed $p > 0$ and any $u, v \in \mathbb{R}, \alpha > 0$, the following equality holds:*

$$
\begin{aligned}
& ug(a) + (v-u)g\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big) + (1-v)g(b) - \frac{p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\,{}^pI^\alpha_{b-}g(a) \\
&= \frac{b^p-a^p}{p}\int_0^1 \mu(t)\big(tb^p+(1-t)a^p\big)^{\frac{1}{p}-1} g'\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\mathrm{d}t,
\end{aligned}
\tag{3.1}
$$

where

$$
\mu(t) = \begin{cases} t^\alpha - u, & t \in [0, \frac{1}{2}), \\ t^\alpha - v, & t \in [\frac{1}{2}, 1]. \end{cases}
$$

Proof. We note that

$$
\begin{aligned}
Q &= \int_0^1 \mu(t)\big(tb^p+(1-t)a^p\big)^{\frac{1}{p}-1} g'\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\mathrm{d}t \\
&= \int_0^{\frac{1}{2}} (t^\alpha-u)\big(tb^p+(1-t)a^p\big)^{\frac{1}{p}-1} g'\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^1 (t^\alpha-v)\big(tb^p+(1-t)a^p\big)^{\frac{1}{p}-1} g'\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\mathrm{d}t.
\end{aligned}
$$

Integrating by parts, we have the following identity:

$$
\begin{aligned}
Q &= \frac{p}{b^p-a^p}\Big\{\int_0^{\frac{1}{2}} (t^\alpha-u)\mathrm{d}g\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big) + \int_{\frac{1}{2}}^1 (t^\alpha-v)\mathrm{d}g\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\Big\} \\
&= \frac{p}{b^p-a^p}\Big\{t^\alpha g\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\Big|_0^1 - \alpha\int_0^1 t^{\alpha-1} g\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\mathrm{d}t \\
&\quad - ug\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\Big|_0^{\frac{1}{2}} - vg\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\Big|_{\frac{1}{2}}^1\Big\} \\
&= \frac{p}{b^p-a^p}\Big\{ug(a) + (v-u)g\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big) + (1-v)g(b)\Big\} \\
&\quad - \frac{\alpha p}{b^p-a^p}\int_0^1 t^{\alpha-1} g\Big(\sqrt[p]{tb^p+(1-t)a^p}\Big)\mathrm{d}t.
\end{aligned}
\tag{3.2}
$$

Using the change of variable $x^p = tb^p + (1-t)a^p$ for $t \in [0,1]$ and multiplying both sides of (3.2) by $\frac{b^p-a^p}{p}$, we obtain that

$$\begin{aligned}\frac{b^p-a^p}{p}Q &= ug(a) + (v-u)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) + (1-v)g(b) - \frac{\alpha p}{(b^p-a^p)^{\alpha}}\int_a^b \frac{x^{p-1}}{(x^p-a^p)^{1-\alpha}}g(x)\mathrm{d}x \\ &= ug(a) + (v-u)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) + (1-v)g(b) - \frac{p^{\alpha}\Gamma(\alpha+1)}{(b^p-a^p)^{\alpha}}{}^{p}I^{\alpha}_{b^-}g(a).\end{aligned} \tag{3.3}$$

Thus, the proof of identity in (3.1) is completed.

Remark 3. *Consider Lemma 3.*
(i) *Taking $p = 1$ and $\alpha = 1$, we have Lemma 2.1 for $m = 1$ presented by Du et al. in [7].*
(ii) *Taking $\alpha = 1, u = \frac{1}{6}$ and $v = \frac{5}{6}$, we have Lemma 0.4 for $p > 0$ presented by Noor et al. in [27].*

For the sake of simplicity, we denote

$$\mathcal{T}_g(\alpha, p, u, v; a, b) := ug(a) + (v-u)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) + (1-v)g(b) - \frac{p^{\alpha}\Gamma(\alpha+1)}{(b^p-a^p)^{\alpha}}{}^{p}I^{\alpha}_{b^-}g(a),$$

unless otherwise specified.

Using Lemma 3, we state the forthcoming theorem.

Theorem 6. *Let $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $g' \in L^1([a,b])$. Suppose that $|g'|^q$ is a p-convex mapping on $[a,b]$ for $p > 0, q \geq 1$, and given constants $\alpha > 0, 0 \leq u \leq 1, 0 \leq v \leq 1$.*
(i) *If $p \in [1,\infty)$, then the following inequality is true:*

$$\begin{aligned}&\left|\mathcal{T}_g(\alpha, p, u, v; a, b)\right| \\ &\leq \frac{a^{1-p}(b^p-a^p)}{p}\left\{\mathbb{A}_1^{1-\frac{1}{q}}(\alpha;u)\Big[\mathbb{A}_2(\alpha;u)|g'(b)|^q + \Big(\mathbb{A}_1(\alpha;u)-\mathbb{A}_2(\alpha;u)\Big)|g'(a)|^q\Big]^{\frac{1}{q}}\right. \\ &\quad \left. + \mathbb{B}_1^{1-\frac{1}{q}}(\alpha;v)\Big[\mathbb{B}_2(\alpha;v)|g'(b)|^q + \Big(\mathbb{B}_1(\alpha;v)-\mathbb{B}_2(\alpha;v)\Big)|g'(a)|^q\Big]^{\frac{1}{q}}\right\}.\end{aligned} \tag{3.4}$$

(ii) *If $p \in (0,1)$, then the following inequality is satisfied:*

$$\begin{aligned}&\left|\mathcal{T}_g(\alpha, p, u, v; a, b)\right| \\ &\leq \frac{b^{1-p}(b^p-a^p)}{p}\left\{\mathbb{A}_1^{1-\frac{1}{q}}(\alpha;u)\Big[\mathbb{A}_2(\alpha;u)|g'(b)|^q + \Big(\mathbb{A}_1(\alpha;u)-\mathbb{A}_2(\alpha;u)\Big)|g'(a)|^q\Big]^{\frac{1}{q}}\right. \\ &\quad \left. + \mathbb{B}_1^{1-\frac{1}{q}}(\alpha;v)\Big[\mathbb{B}_2(\alpha;v)|g'(b)|^q + \Big(\mathbb{B}_1(\alpha;v)-\mathbb{B}_2(\alpha;v)\Big)|g'(a)|^q\Big]^{\frac{1}{q}}\right\},\end{aligned} \tag{3.5}$$

where

$$\mathbb{A}_1(\alpha;u) = \begin{cases} \frac{2\alpha u^{1+\frac{1}{\alpha}}}{\alpha+1} - \frac{u}{2} + \frac{1}{(\alpha+1)2^{\alpha+1}}, & 0 \leq u \leq (\frac{1}{2})^{\alpha}, \\ \frac{u}{2} - \frac{1}{(\alpha+1)2^{\alpha+1}}, & (\frac{1}{2})^{\alpha} < u \leq 1, \end{cases}$$

$$\mathbb{B}_1(\alpha;v)=\begin{cases}-\frac{v}{2}+\frac{2^{\alpha+1}-1}{(\alpha+1)2^{\alpha+1}}, & 0\le v<\left(\frac{1}{2}\right)^{\alpha},\\ \frac{2\alpha v^{1+\frac{1}{\alpha}}}{\alpha+1}-\frac{3v}{2}+\frac{2^{\alpha+1}+1}{(\alpha+1)2^{\alpha+1}}, & \left(\frac{1}{2}\right)^{\alpha}\le v\le 1,\end{cases}$$

$$\mathbb{A}_2(\alpha;u)=\begin{cases}\frac{\alpha u^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{u}{8}+\frac{1}{(\alpha+2)2^{\alpha+2}}, & 0\le u\le\left(\frac{1}{2}\right)^{\alpha},\\ \frac{u}{8}-\frac{1}{(\alpha+2)2^{\alpha+2}}, & \left(\frac{1}{2}\right)^{\alpha}<u\le 1,\end{cases}$$

and

$$\mathbb{B}_2(\alpha;v)=\begin{cases}-\frac{3v}{8}+\frac{2^{\alpha+2}-1}{(\alpha+2)2^{\alpha+2}}, & 0\le v<\left(\frac{1}{2}\right)^{\alpha},\\ \frac{\alpha v^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{5v}{8}+\frac{2^{\alpha+2}+1}{(\alpha+2)2^{\alpha+2}}, & \left(\frac{1}{2}\right)^{\alpha}\le v\le 1.\end{cases}$$

Proof. Suppose that $p\in[1,\infty)$. By means of Lemma 3, we have that

$$\left|\mathcal{T}_g(\alpha,p,u,v;a,b)\right|\le\frac{b^p-a^p}{p}\int_0^1|\mu(t)|\left(tb^p+(1-t)a^p\right)^{\frac{1}{p}-1}\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|\mathrm{d}t. \tag{3.6}$$

Since $p\in[1,\infty)$, we deduce that

$$\left(tb^p+(1-t)a^p\right)^{\frac{1}{p}-1}\le(a^p)^{\frac{1}{p}-1}=a^{1-p},\qquad(0\le t\le 1). \tag{3.7}$$

Using inequality (3.7) in (3.6) and the Hölder inequality, we have that

$$\begin{aligned}&\left|\mathcal{T}_g(\alpha,p,u,v;a,b)\right|\\ &\le\frac{a^{1-p}(b^p-a^p)}{p}\left\{\int_0^{\frac{1}{2}}|t^\alpha-u|\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|\mathrm{d}t+\int_{\frac{1}{2}}^1|t^\alpha-v|\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|\mathrm{d}t\right\}\\ &\le\frac{a^{1-p}(b^p-a^p)}{p}\left\{\left(\int_0^{\frac{1}{2}}|t^\alpha-u|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|t^\alpha-u|\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &\quad\left.+\left(\int_{\frac{1}{2}}^1|t^\alpha-v|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|t^\alpha-v|\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Using the p-convexity of $|g'|^q$, we obtain that

$$\begin{aligned}&\left|\mathcal{T}_g(\alpha,p,u,v;a,b)\right|\\ &\le\frac{a^{1-p}(b^p-a^p)}{p}\left\{\left(\int_0^{\frac{1}{2}}|t^\alpha-u|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|t^\alpha-u|\left[t|g'(b)|^q+(1-t)|g'(a)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &\quad\left.+\left(\int_{\frac{1}{2}}^1|t^\alpha-v|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|t^\alpha-v|\left[t|g'(b)|^q+(1-t)|g'(a)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

The desired inequality yields from the above by noting that

$$\int_0^{\frac{1}{2}} \left|t^\alpha - u\right| \mathrm{d}t = \begin{cases} \frac{2\alpha u^{1+\frac{1}{\alpha}}}{\alpha+1} - \frac{u}{2} + \frac{1}{(\alpha+1)2^{\alpha+1}}, & 0 \le u \le (\frac{1}{2})^\alpha, \\ \frac{u}{2} - \frac{1}{(\alpha+1)2^{\alpha+1}}, & (\frac{1}{2})^\alpha < u \le 1, \end{cases}$$

$$\int_{\frac{1}{2}}^1 \left|t^\alpha - v\right| \mathrm{d}t = \begin{cases} -\frac{v}{2} + \frac{2^{\alpha+1}-1}{(\alpha+1)2^{\alpha+1}}, & 0 \le v < (\frac{1}{2})^\alpha, \\ \frac{2\alpha v^{1+\frac{1}{\alpha}}}{\alpha+1} - \frac{3v}{2} + \frac{2^{\alpha+1}+1}{(\alpha+1)2^{\alpha+1}}, & (\frac{1}{2})^\alpha \le v \le 1, \end{cases}$$

$$\int_0^{\frac{1}{2}} \left|t^\alpha - u\right| t\mathrm{d}t = \begin{cases} \frac{\alpha u^{1+\frac{2}{\alpha}}}{\alpha+2} - \frac{u}{8} + \frac{1}{(\alpha+2)2^{\alpha+2}}, & 0 \le u \le (\frac{1}{2})^\alpha, \\ \frac{u}{8} - \frac{1}{(\alpha+2)2^{\alpha+2}}, & (\frac{1}{2})^\alpha < u \le 1, \end{cases}$$

and

$$\int_{\frac{1}{2}}^1 \left|t^\alpha - v\right| t\mathrm{d}t = \begin{cases} -\frac{3v}{8} + \frac{2^{\alpha+2}-1}{(\alpha+2)2^{\alpha+2}}, & 0 \le v < (\frac{1}{2})^\alpha, \\ \frac{\alpha v^{1+\frac{2}{\alpha}}}{\alpha+2} - \frac{5v}{8} + \frac{2^{\alpha+2}+1}{(\alpha+2)2^{\alpha+2}}, & (\frac{1}{2})^\alpha \le v \le 1. \end{cases}$$

Thus, this completes the proof for case $p \in [1, \infty)$.

To prove (*ii*), suppose that $p \in (0, 1)$, then we obtain the required inequality in (3.5) by applying the fact that

$$\left(ta^p + (1-t)b^p\right)^{\frac{1}{p}-1} \le (b^p)^{\frac{1}{p}-1} = b^{1-p}. \tag{3.8}$$

Now, we state some special cases of Theorem 6.

Corollary 1. *In Theorem 6, if we take $q = 1$, then we have that*

$$\left|\mathcal{T}_g(\alpha, p, u, v; a, b)\right| \le \frac{\zeta^{1-p}(b^p - a^p)}{p}\Big\{\Big(\mathbb{A}_1(\alpha; u) + \mathbb{B}_1(\alpha; v) - \mathbb{A}_2(\alpha; u) - \mathbb{B}_2(\alpha; v)\Big)|g'(a)| + \Big(\mathbb{A}_2(\alpha; u) + \mathbb{B}_2(\alpha; v)\Big)|g'(b)|\Big\},$$

where

$$\zeta = \begin{cases} a, & 1 \le p < \infty, \\ b, & 0 < p < 1. \end{cases} \tag{3.9}$$

Remark 4. *Consider Corollary 1.*
(*i*) *For $u = 0, v = 1$, we have the midpoint inequality:*

$$\left|g\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) - \frac{p^\alpha\Gamma(\alpha+1)}{(b^p - a^p)^\alpha}\,{}^pI^\alpha_{b-}g(a)\right| \le \frac{\zeta^{1-p}(b^p - a^p)}{p}\Big\{\Big[\frac{1}{(\alpha+1)(\alpha+2)}\Big(\alpha^2 + 3\alpha + 1 + \frac{\alpha+3}{2^{\alpha+1}}\Big) - \frac{7}{8}\Big]|g'(a)| + \Big(\frac{1}{(\alpha+2)2^{\alpha+1}} + \frac{\alpha+1}{\alpha+2} - \frac{5}{8}\Big)|g'(b)|\Big\}.$$

(*ii*) *For* $u=\frac{1}{6}, v=\frac{5}{6}$, *we have the Simpson type inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[g(a)+4g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+g(b)\right]-\frac{p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}{}^pI^\alpha_{b-}g(a)\right|\\&\le\frac{\zeta^{1-p}(b^p-a^p)}{p}\left\{\left[\mathbb{A}_1\left(\alpha;\frac{1}{6}\right)+\mathbb{B}_1\left(\alpha;\frac{5}{6}\right)-\mathbb{A}_2\left(\alpha;\frac{1}{6}\right)-\mathbb{B}_2\left(\alpha;\frac{5}{6}\right)\right]|g'(a)|\right.\\&\left.+\left[\mathbb{A}_2\left(\alpha;\frac{1}{6}\right)+\mathbb{B}_2\left(\alpha;\frac{5}{6}\right)\right]|g'(b)|\right\}.\end{aligned}$$

(*iii*) *For* $u=\frac{1}{4}, v=\frac{3}{4}$, *we have the averaged midpoint-trapezoid type inequality:*

$$\begin{aligned}&\left|\frac{1}{4}\left[g(a)+2g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+g(b)\right]-\frac{p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}{}^pI^\alpha_{b-}g(a)\right|\\&\le\frac{\zeta^{1-p}(b^p-a^p)}{p}\left\{\left[\mathbb{A}_1\left(\alpha;\frac{1}{4}\right)+\mathbb{B}_1\left(\alpha;\frac{3}{4}\right)-\mathbb{A}_2\left(\alpha;\frac{1}{4}\right)-\mathbb{B}_2\left(\alpha;\frac{3}{4}\right)\right]|g'(a)|\right.\\&\left.+\left[\mathbb{A}_2\left(\alpha;\frac{1}{4}\right)+\mathbb{B}_2\left(\alpha;\frac{3}{4}\right)\right]|g'(b)|\right\}.\end{aligned}$$

(*iv*) *For* $u=v=\frac{1}{2}$, *we have the trapezoid inequality:*

$$\begin{aligned}&\left|\frac{g(a)+g(b)}{2}-\frac{p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}{}^pI^\alpha_{b-}g(a)\right|\\&\le\frac{\zeta^{1-p}(b^p-a^p)}{p}\left\{\left[\frac{\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{2}\right)^{\frac{2}{\alpha}}+\frac{1}{(\alpha+1)(\alpha+2)}-\frac{1}{4}\right]|g'(a)|\right.\\&\left.+\left[\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{2}\right)^{\frac{2}{\alpha}}+\frac{1}{\alpha+2}-\frac{1}{4}\right]|g'(b)|\right\}.\end{aligned}$$

Corollary 2. *For* $0\le u\le\frac{1}{2}$ *and* $\frac{1}{2}\le v\le 1$, *if we take* $p=1$ *and* $\alpha=1$ *in Theorem 6, then we have*

$$\begin{aligned}&\left|ug(a)+(v-u)g\left(\frac{a+b}{2}\right)+(1-v)g(b)-\frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right|\\&\le(b-a)\left\{\left(u^2-\frac{u}{2}+\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{u^3}{3}-\frac{u}{8}+\frac{1}{24}\right)|g'(b)|^q+\left(-\frac{u^3}{3}+u^2-\frac{3u}{8}+\frac{1}{12}\right)|g'(a)|^q\right]^{\frac{1}{q}}\right.\\&\left.+\left(v^2-\frac{3v}{2}+\frac{5}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{v^3}{3}-\frac{5v}{8}+\frac{3}{8}\right)|g'(b)|^q+\left(-\frac{v^3}{3}+v^2-\frac{7v}{8}+\frac{1}{4}\right)|g'(a)|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Remark 5. *Consider Corollary 2.*
(*i*) *For* $u=0, v=1$, *we obtain the midpoint inequality:*

$$\begin{aligned}&\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right|\\&\le(b-a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{24}|g'(b)|^q+\frac{1}{12}|g'(a)|^q\right)^{\frac{1}{q}}+\left(\frac{1}{12}|g'(b)|^q+\frac{1}{24}|g'(a)|^q\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Specially, when we take $q = 1$, we obtained Theorem 2.2 established by Kirmaci in [16].
(ii) *For $u = \frac{1}{6}, v = \frac{5}{6}$, we obtain the Simpson type inequality:*

$$\left|\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right|$$
$$\leq (b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left\{\left(\frac{29}{1296}|g'(b)|^q + \frac{61}{1296}|g'(a)|^q\right)^{\frac{1}{q}} + \left(\frac{61}{1296}|g'(b)|^q + \frac{29}{1296}|g'(a)|^q\right)^{\frac{1}{q}}\right\}.$$

which is Corollary 2.9 for $m = 1$ established by Du in [7].
(iii) *For $u = \frac{1}{4}, v = \frac{3}{4}$, we obtain the averaged midpoint-trapezoid type inequality:*

$$\left|\frac{1}{4}\left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right|$$
$$\leq (b-a)\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{64}|g'(b)|^q + \frac{3}{64}|g'(a)|^q\right)^{\frac{1}{q}} + \left(\frac{3}{64}|g'(b)|^q + \frac{1}{64}|g'(a)|^q\right)^{\frac{1}{q}}\right\}.$$

(iv) *For $u = v = \frac{1}{2}$, we obtain the trapezoid inequality:*

$$\left|\frac{g(a) + g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right|$$
$$\leq (b-a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{48}|g'(b)|^q + \frac{5}{48}|g'(a)|^q\right)^{\frac{1}{q}} + \left(\frac{5}{48}|g'(b)|^q + \frac{1}{48}|g'(a)|^q\right)^{\frac{1}{q}}\right\}.$$

Specially, when we take $q = 1$, we obtained Theorem 2.2 established by Dragomir in [4].

In the next theorem, we will use the following functions.
(1) The beta function,

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x, y > 0.$$

(2) The incomplete beta function,

$$\beta(a; x, y) = \int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t, \ \ 0 < a < 1, \ \ x, y > 0.$$

(3) The hypergeometric function,

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \ \ c > b > 0, |z| < 1.$$

Theorem 7. *Let $g : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $g' \in L^1([a, b])$. In addition, suppose that $|g'|^q$ is a p-convex mapping on $[a, b]$ for $p > 0$, and given constants $\alpha > 0, 0 \leq u \leq 1, 0 \leq v \leq 1$, and $r, q > 1$ such that $\frac{1}{r} + \frac{1}{q} = 1$.*
(i) *If $p \in [1, \infty)$, then the following inequality is true:*

$$\left|\mathcal{T}_g(\alpha, p, u, v; a, b)\right|$$
$$\leq \frac{a^{1-p}(b^p - a^p)}{p}\left\{\mathbb{A}_3^{\frac{1}{r}}(\alpha; u, r)\left(\frac{|g'(b)|^q + 3|g'(a)|^q}{8}\right)^{\frac{1}{q}} + \mathbb{B}_3^{\frac{1}{r}}(\alpha; v, r)\left(\frac{3|g'(b)|^q + |g'(a)|^q}{8}\right)^{\frac{1}{q}}\right\}. \tag{3.10}$$

(*ii*) *If* $p \in (0,1)$*, then the following inequality is satisfied:*

$$\begin{aligned}
&\left|\mathcal{T}_g(\alpha,p,u,v;a,b)\right| \\
&\le \frac{b^{1-p}(b^p-a^p)}{p}\left\{\mathbb{A}_3^{\frac{1}{r}}(\alpha;u,r)\left(\frac{|g'(b)|^q+3|g'(a)|^q}{8}\right)^{\frac{1}{q}}+\mathbb{B}_3^{\frac{1}{r}}(\alpha;v,r)\left(\frac{3|g'(b)|^q+|g'(a)|^q}{8}\right)^{\frac{1}{q}}\right\},
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
&\mathbb{A}_3(\alpha;u,r) \\
&=\begin{cases}
\frac{1}{\alpha r+1}(\frac{1}{2})^{\alpha r+1}, & u=0,\\
\left[\begin{array}{c}\frac{u^{r+\frac{1}{\alpha}}}{\alpha}\beta(\frac{1}{\alpha},r+1)+\frac{1}{\alpha(r+1)}(\frac{1}{2})^{1-\alpha}\left((\frac{1}{2})^{\alpha}-u\right)^{r+1}\\ \times{}_2F_1(1-\frac{1}{\alpha},1;r+2;1-2^{\alpha}u)\end{array}\right], & 0<u<(\frac{1}{2})^{\alpha},\\
\frac{1}{2\alpha}(\frac{1}{2})^{\alpha r}\beta(\frac{1}{\alpha},r+1), & u=(\frac{1}{2})^{\alpha},\\
\frac{1}{\alpha}u^{r+\frac{1}{\alpha}}\beta(\frac{1}{2^{\alpha}u};\frac{1}{\alpha},r+1), & (\frac{1}{2})^{\alpha}<u\le 1,
\end{cases}
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\mathbb{B}_3(\alpha;v,r) \\
&=\begin{cases}
\frac{1}{\alpha r+1}\left[1-(\frac{1}{2})^{\alpha r+1}\right], & v=0,\\
\left[\begin{array}{c}\frac{(1-v)^{r+1}}{\alpha(r+1)}\cdot{}_2F_1\left(1-\frac{1}{\alpha},1;r+2;1-v\right)\\ -\frac{1}{\alpha(r+1)}(\frac{1}{2})^{1-\alpha}\left((\frac{1}{2})^{\alpha}-v\right)^{r+1}\cdot{}_2F_1(1-\frac{1}{\alpha},1;r+2;1-2^{\alpha}v)\end{array}\right], & 0<v<(\frac{1}{2})^{\alpha},\\
\frac{(1-v)^{r+1}}{\alpha(r+1)}\cdot{}_2F_1\left(1-\frac{1}{\alpha},1;r+2;1-(\frac{1}{2})^{\alpha}\right), & v=(\frac{1}{2})^{\alpha},\\
\left[\begin{array}{c}\frac{v^{r+\frac{1}{\alpha}}}{\alpha}\left[\beta\left(r+1,\frac{1}{\alpha}\right)-\beta(\frac{1}{2^{\alpha}v};\frac{1}{\alpha},r+1)\right]\\ +\frac{(1-v)^{r+1}}{\alpha(r+1)}\cdot{}_2F_1\left(1-\frac{1}{\alpha},1;r+2;1-v\right)\end{array}\right], & (\frac{1}{2})^{\alpha}<v\le 1.
\end{cases}
\end{aligned} \tag{3.13}$$

Proof. Suppose that $p \in [1,\infty)$. If we use Lemma 3, inequality (3.7) and the Hölder inequality, then we have

$$\begin{aligned}
&\left|\mathcal{T}_g(\alpha,p,u,v;a,b)\right| \\
&\le \frac{a^{1-p}(b^p-a^p)}{p}\left\{\int_0^{\frac{1}{2}}\left|t^{\alpha}-u\right|\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|\mathrm{d}t+\int_{\frac{1}{2}}^1\left|t^{\alpha}-v\right|\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|\mathrm{d}t\right\} \\
&\le \frac{a^{1-p}(b^p-a^p)}{p}\left\{\left(\int_0^{\frac{1}{2}}\left|t^{\alpha}-u\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left|t^{\alpha}-v\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^1\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{3.14}$$

Since $|g'|^q$ is p-convex, we get

$$\int_0^{\frac{1}{2}}\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|^q\mathrm{d}t \le \int_0^{\frac{1}{2}}\left[t|g'(b)|^q+(1-t)|g'(a)|^q\right]\mathrm{d}t=\frac{|g'(b)|^q+3|g'(a)|^q}{8} \tag{3.15}$$

and

$$\int_{\frac{1}{2}}^{1} \left|g'\left(\sqrt[p]{tb^p + (1-t)a^p}\right)\right|^q \mathrm{d}t \le \int_{\frac{1}{2}}^{1} \left[t|g'(b)|^q + (1-t)|g'(a)|^q\right]\mathrm{d}t = \frac{3|g'(b)|^q + |g'(a)|^q}{8}. \tag{3.16}$$

Also,

$$\begin{aligned} &\int_0^{\frac{1}{2}} \left|t^\alpha - u\right|^r \mathrm{d}t \\ &= \begin{cases} \frac{1}{\alpha r+1}(\frac{1}{2})^{\alpha r+1}, & u = 0, \\ \left[\begin{array}{c} \frac{u^{r+\frac{1}{\alpha}}}{\alpha}\beta(\frac{1}{\alpha}, r+1) + \frac{1}{\alpha(r+1)}(\frac{1}{2})^{1-\alpha}\left((\frac{1}{2})^\alpha - u\right)^{r+1} \\ \times {}_2F_1(1-\frac{1}{\alpha}, 1; r+2; 1-2^\alpha u) \end{array}\right], & 0 < u < (\frac{1}{2})^\alpha, \\ \frac{1}{2\alpha}(\frac{1}{2})^{\alpha r}\beta(\frac{1}{\alpha}, r+1), & u = (\frac{1}{2})^\alpha, \\ \frac{1}{\alpha}u^{r+\frac{1}{\alpha}}\beta(\frac{1}{2^\alpha u}; \frac{1}{\alpha}, r+1), & (\frac{1}{2})^\alpha < u \le 1, \end{cases} \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\int_0^{1} \left|t^\alpha - v\right|^r \mathrm{d}t \\ &= \begin{cases} \frac{1}{\alpha r+1}, & v = 0, \\ \frac{1}{\alpha}\left[\begin{array}{c} v^{r+\frac{1}{\alpha}}\beta\left(r+1, \frac{1}{\alpha}\right) \\ +\frac{(1-v)^{r+1}}{r+1} \cdot {}_2F_1\left(1-\frac{1}{\alpha}, 1; r+2; 1-v\right) \end{array}\right], & 0 < v \le 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\int_{\frac{1}{2}}^{1} \left|t^\alpha - v\right|^r \mathrm{d}t \\ &= \int_0^1 \left|t^\alpha - v\right|^r \mathrm{d}t - \int_0^{\frac{1}{2}} \left|t^\alpha - v\right|^r \mathrm{d}t \\ &= \begin{cases} \frac{1}{\alpha r+1}\left[1-(\frac{1}{2})^{\alpha r+1}\right], & v = 0, \\ \left[\begin{array}{c} \frac{(1-v)^{r+1}}{\alpha(r+1)} \cdot {}_2F_1\left(1-\frac{1}{\alpha}, 1; r+2; 1-v\right) \\ -\frac{1}{\alpha(r+1)}(\frac{1}{2})^{1-\alpha}\left((\frac{1}{2})^\alpha - v\right)^{r+1} \cdot {}_2F_1(1-\frac{1}{\alpha}, 1; r+2; 1-2^\alpha v) \end{array}\right], & 0 < v < (\frac{1}{2})^\alpha, \\ \frac{(1-v)^{r+1}}{\alpha(r+1)} \cdot {}_2F_1\left(1-\frac{1}{\alpha}, 1; r+2; 1-(\frac{1}{2})^\alpha\right), & v = (\frac{1}{2})^\alpha, \\ \left[\begin{array}{c} \frac{v^{r+\frac{1}{\alpha}}}{\alpha}\left[\beta\left(r+1, \frac{1}{\alpha}\right) - \beta(\frac{1}{2^\alpha v}; \frac{1}{\alpha}, r+1)\right] \\ +\frac{(1-v)^{r+1}}{\alpha(r+1)} \cdot {}_2F_1\left(1-\frac{1}{\alpha}, 1; r+2; 1-v\right) \end{array}\right], & (\frac{1}{2})^\alpha < v \le 1. \end{cases} \end{aligned} \tag{3.18}$$

Using (3.15)-(3.18) in (3.14), we obtain the desired inequality in (3.10). Thus, this completes the proof for case $p \in (1, \infty)$.

Case (*ii*) can be proved by utilizing the above process with relation (3.8). Thus, the proof is completed.

If we next use the well-known Young's inequality

$$XY \leq \frac{X^r}{r} + \frac{Y^q}{q}, \quad \forall\, X, Y \geq 0, \quad r, q > 1, \quad \frac{1}{r} + \frac{1}{q} = 1 \tag{3.19}$$

in Theorem 7, then we have the following corollary.

Corollary 3. *Under all assumptions of Theorem 7, we have*

$$\left|\mathcal{T}_g(\alpha, p, u, v; a, b)\right| \leq \frac{\zeta^{1-p}(b^p - a^p)}{p}\left[\frac{\mathbb{A}_3(\alpha; u, r) + \mathbb{B}_3(\alpha; v, r)}{r} + \frac{|g'(b)|^q + |g'(a)|^q}{2q}\right],$$

where $\zeta, \mathbb{A}_3(\alpha; u, r)$ *and* $\mathbb{B}_3(\alpha; v, r)$ *are defined by* (3.9), (3.12) *and* (3.13)*, respectively.*

Some special cases of Theorem 7 are stated as follows.

Corollary 4. *In Theorem 7, if we take* $p = 1$*, then we have*

$$\begin{aligned}&\left|ug(a) + (v-u)g\Big(\frac{a+b}{2}\Big) + (1-v)g(b) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathcal{J}^\alpha_{b^-}g(a)\right| \\ &\leq (b-a)\left\{\mathbb{A}_3^{\frac{1}{r}}(\alpha; u, r)\Big(\frac{|g'(b)|^q + 3|g'(a)|^q}{8}\Big)^{\frac{1}{q}} + \mathbb{B}_3^{\frac{1}{r}}(\alpha; v, r)\Big(\frac{3|g'(b)|^q + |g'(a)|^q}{8}\Big)^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 5. *For* $0 \leq u \leq \frac{1}{2}$ *and* $\frac{1}{2} \leq v \leq 1$*, if we take* $\alpha = 1$ *in Corollary 4, then we have*

$$\begin{aligned}&\left|ug(a) + (v-u)g\Big(\frac{a+b}{2}\Big) + (1-v)g(b) - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \\ &\leq (b-a)\Big(\frac{1}{r+1}\Big)^{\frac{1}{r}}\left\{\left[u^{r+1} + \Big(\frac{1}{2} - u\Big)^{r+1}\right]^{\frac{1}{r}}\Big(\frac{|g'(b)|^q + 3|g'(a)|^q}{8}\Big)^{\frac{1}{q}}\right. \\ &\quad \left. + \left[\Big(v - \frac{1}{2}\Big)^{r+1} + (1-v)^{r+1}\right]^{\frac{1}{r}}\Big(\frac{3|g'(b)|^q + |g'(a)|^q}{8}\Big)^{\frac{1}{q}}\right\}.\end{aligned}$$

Remark 6. *Consider Corollary 5.*
(*i*) *For* $u = 0, v = 1$*, we obtain the midpoint inequality:*

$$\begin{aligned}&\left|g\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \\ &\leq \frac{b-a}{4}\Big(\frac{1}{r+1}\Big)^{\frac{1}{r}}\left\{\Big(\frac{|g'(b)|^q + 3|g'(a)|^q}{4}\Big)^{\frac{1}{q}} + \Big(\frac{3|g'(b)|^q + |g'(a)|^q}{4}\Big)^{\frac{1}{q}}\right\},\end{aligned}$$

which is Theorem 2.3 established by Kirmaci in [16].
(*ii*) *For* $u = \frac{1}{6}, v = \frac{5}{6}$*, we obtain the Simpson type inequality:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \\
&\le \frac{b-a}{12}\left(\frac{1+2^{r+1}}{3(r+1)}\right)^{\frac{1}{r}}\left\{\left(\frac{|g'(b)|^q+3|g'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|g'(b)|^q+|g'(a)|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

(*iii*) *For $u=\frac{1}{4}, v=\frac{3}{4}$, we obtain the averaged midpoint-trapezoid type inequality:*

$$
\begin{aligned}
&\left|\frac{1}{4}\left[g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \\
&\le \frac{b-a}{8}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left\{\left(\frac{|g'(b)|^q+3|g'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|g'(b)|^q+|g'(a)|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

(*iv*) *For $u=v=\frac{1}{2}$, we obtain the trapezoid inequality:*

$$
\begin{aligned}
&\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \\
&\le \frac{b-a}{4}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left\{\left(\frac{|g'(b)|^q+3|g'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|g'(b)|^q+|g'(a)|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Theorem 8. *Under all assumptions of Theorem 7, we have*

$$
\begin{aligned}
&\left|\mathcal{T}_g(\alpha,p,u,v;a,b)\right| \\
&\le \frac{b^{1-p}(b^p-a^p)}{p}\Big(\mathbb{A}_3(\alpha;u,r)+\mathbb{B}_3(\alpha;v,r)\Big)^{\frac{1}{r}}\left\{\frac{1}{2}\cdot{}_2F_1\left(q-\frac{q}{p},1;3;\frac{b^p-a^p}{b^p}\right)|g'(b)|^q\right. \\
&\left.+\left[{}_2F_1\left(q-\frac{q}{p},1;2;\frac{b^p-a^p}{b^p}\right)-\frac{1}{2}\cdot{}_2F_1\left(q-\frac{q}{p},1;3;\frac{b^p-a^p}{b^p}\right)\right]|g'(a)|^q\right\}^{\frac{1}{q}},
\end{aligned} \tag{3.20}
$$

where $\mathbb{A}_3(\alpha;u,r)$ and $\mathbb{B}_3(\alpha;v,r)$ are defined by (3.12) *and* (3.13)*, respectively.*

Proof. If we use Lemma 3, the Hölder inequality and the p-convexity of $|g'|^q$, then we have that

$$
\begin{aligned}
&\left|\mathcal{T}_g(\alpha,p,u,v;a,b)\right| \\
&\le \frac{b^p-a^p}{p}\int_0^1 |\mu(t)|\Big(tb^p+(1-t)a^p\Big)^{\frac{1}{p}-1}\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|\mathrm{d}t \\
&\le \frac{b^p-a^p}{p}\left(\int_0^1 |\mu(t)|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^1 \Big(tb^p+(1-t)a^p\Big)^{(\frac{q}{p}-q)}\left|g'\left(\sqrt[p]{tb^p+(1-t)a^p}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{b^p-a^p}{p}\left(\int_0^1 |\mu(t)|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^1 \Big(tb^p+(1-t)a^p\Big)^{(\frac{q}{p}-q)}\Big[t|g'(b)|^q+(1-t)|g'(a)|^q\Big]\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{b^p-a^p}{p}\left(\int_0^1 |\mu(t)|^r\mathrm{d}t\right)^{\frac{1}{r}}\left\{|g'(b)|^q\int_0^1 t\Big(tb^p+(1-t)a^p\Big)^{(\frac{q}{p}-q)}\mathrm{d}t\right. \\
&\left.+|g'(a)|^q\int_0^1 (1-t)\Big(tb^p+(1-t)a^p\Big)^{(\frac{q}{p}-q)}\mathrm{d}t\right\}^{\frac{1}{q}}.
\end{aligned} \tag{3.21}
$$

The desired inequality yields from the above by noting that

$$\int_0^1 \left|\mu(t)\right|^r \mathrm{d}t = \int_0^{\frac{1}{2}} \left|t^\alpha - u\right|^r \mathrm{d}t + \int_{\frac{1}{2}}^1 \left|t^\alpha - v\right|^r \mathrm{d}t$$
$$= \mathbb{A}_3(\alpha; u, r) + \mathbb{B}_3(\alpha; v, r),$$

$$\int_0^1 t\Big(tb^p + (1-t)a^p\Big)^{(\frac{q}{p}-q)} \mathrm{d}t = \frac{b^{(1-p)q}}{2} \cdot {}_2F_1\Big(q - \frac{q}{p}, 1; 3; \frac{b^p - a^p}{b^p}\Big)$$

and

$$\int_0^1 (1-t)\Big(tb^p + (1-t)a^p\Big)^{(\frac{q}{p}-q)} \mathrm{d}t$$
$$= \frac{b^{(1-p)q}}{2}\Bigg[2 \cdot {}_2F_1\Big(q - \frac{q}{p}, 1; 2; \frac{b^p - a^p}{b^p}\Big) - {}_2F_1\Big(q - \frac{q}{p}, 1; 3; \frac{b^p - a^p}{b^p}\Big)\Bigg].$$

Thus, the proof is completed.

Similar to Corollary 3, using Young's inequality, we get the following result.

Corollary 6. *Under all assumptions of Theorem 8, we have*

$$\begin{aligned}
&\left|\mathcal{T}_g(\alpha, p, u, v; a, b)\right| \\
&\leq \frac{b^{1-p}(b^p - a^p)}{p}\Bigg\{\frac{\mathbb{A}_3(\alpha; u, r) + \mathbb{B}_3(\alpha; v, r)}{r} + \frac{1}{2}q^{-1} \cdot {}_2F_1\Big(q - \frac{q}{p}, 1; 3; \frac{b^p - a^p}{b^p}\Big)|g'(b)|^q \\
&+ q^{-1}\Bigg[{}_2F_1\Big(q - \frac{q}{p}, 1; 2; \frac{b^p - a^p}{b^p}\Big) - \frac{1}{2} \cdot {}_2F_1\Big(q - \frac{q}{p}, 1; 3; \frac{b^p - a^p}{b^p}\Big)\Bigg]|g'(a)|^q\Bigg\},
\end{aligned} \tag{3.22}$$

where $\mathbb{A}_3(\alpha; u, r)$ *and* $\mathbb{B}_3(\alpha; v, r)$ *are defined by (3.12) and (3.13), respectively.*

Some special cases of Theorem 8 are stated as follows.

Corollary 7. *In Theorem 8, if we take* $p = 1$*, then we have*

$$\left|ug(a) + (v-u)g\Big(\frac{a+b}{2}\Big) + (1-v)g(b) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathcal{J}_{b^-}^\alpha g(a)\right|$$
$$\leq (b-a)\Big(\mathbb{A}_3(\alpha; u, r) + \mathbb{B}_3(\alpha; v, r)\Big)^{\frac{1}{r}}\left[\frac{|g'(b)|^q + |g'(a)|^q}{2}\right]^{\frac{1}{q}}.$$

Corollary 8. *For* $0 \leq u \leq \frac{1}{2}$ *and* $\frac{1}{2} \leq v \leq 1$*, if we take* $\alpha = 1$ *in Corollary 7, then we have*

$$\left|ug(a) + (v-u)g\Big(\frac{a+b}{2}\Big) + (1-v)g(b) - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right|$$
$$\leq (b-a)\Big(\frac{1}{r+1}\Big)^{\frac{1}{r}}\left[u^{r+1} + \Big(\frac{1}{2} - u\Big)^{r+1} + \Big(v - \frac{1}{2}\Big)^{r+1} + (1-v)^{r+1}\right]^{\frac{1}{r}}\left[\frac{|g'(b)|^q + |g'(a)|^q}{2}\right]^{\frac{1}{q}}.$$

Remark 7. *Consider Corollary 8.*
(i) *For $u = 0, v = 1$, we obtain the midpoint inequality:*

$$\left|g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \leq \frac{b-a}{2}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\frac{|g'(b)|^q + |g'(a)|^q}{2}\right]^{\frac{1}{q}}.$$

(ii) *For $u = \frac{1}{6}, v = \frac{5}{6}$, we obtain the Simpson type inequality:*

$$\left|\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \leq \frac{b-a}{6}\left(\frac{1+2^{r+1}}{3(r+1)}\right)^{\frac{1}{r}}\left[\frac{|g'(b)|^q + |g'(a)|^q}{2}\right]^{\frac{1}{q}}.$$

(iii) *For $u = \frac{1}{4}, v = \frac{3}{4}$, we obtain the averaged midpoint-trapezoid type inequality:*

$$\left|\frac{1}{4}\left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \leq \frac{b-a}{4}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\frac{|g'(b)|^q + |g'(a)|^q}{2}\right]^{\frac{1}{q}}.$$

(iv) *For $u = v = \frac{1}{2}$, we obtain the trapezoid inequality:*

$$\left|\frac{g(a) + g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t\right| \leq \frac{b-a}{2}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\frac{|g'(b)|^q + |g'(a)|^q}{2}\right]^{\frac{1}{q}}.$$

4. Examples

In this section, we present three examples to illustrate our main results.

Example 1. Let $a = 1, b = 2, \alpha = p = 2, g(x) = x^{-2}$. Then all the assumptions in Theorem 5 are satisfied. Clearly,

$$g\left(\left[\frac{a^p + \alpha b^p}{\alpha+1}\right]^{\frac{1}{p}}\right) = g(\sqrt{3}) = \frac{1}{3},$$

$$\frac{p^\alpha \Gamma(\alpha+1)}{(b^p - a^p)^\alpha}\,{}^pI^\alpha_{b^-}g(a) = \frac{4}{9}\int_1^2 t(t^2-1)g(t)\mathrm{d}t = \frac{4}{9}\left(\frac{3}{2} - \ln 2\right) \approx 0.3586,$$

and

$$\frac{g(a) + \alpha g(b)}{\alpha+1} = \frac{g(1) + 2g(2)}{3} = \frac{1}{2}.$$

It is clear that $\frac{1}{3} < 0.3586 < \frac{1}{2}$, which demonstrates the result described in Theorem 5.

Example 2. For $p > 1$, let $g(x) = \frac{1}{1-p}x^{1-p}$ for $x \in (0, \infty)$. Then $|g'(x)| = x^{-p}$ is a p-convex mapping. If we take $a = 2$, $b = 4$, $u = 0.5$, $v = 0.5$, $\alpha = 1$, $p = 2$ and $q = 1$, then all the assumptions in Theorem 6 are satisfied.
The left-hand side term of (3.4) is:

$$\left|\mathcal{T}_g(\alpha, p, u, v; a, b)\right| = \left|\frac{g(2) + g(4)}{2} - \frac{1}{6}\int_2^4 tg(t)\mathrm{d}t\right| = \frac{1}{24}.$$

The right-hand side term of (3.4) is:

$$\frac{a^{1-p}(b^p - a^p)}{p}\Big[(\mathbb{A}_1(\alpha;u) + \mathbb{B}_1(\alpha;v) - \mathbb{A}_2(\alpha;u) - \mathbb{B}_2(\alpha;v))|g'(a)|$$
$$+ (\mathbb{A}_2(\alpha;u) + \mathbb{B}_2(\alpha;v))|g'(b)|\Big]$$
$$= 3\Big[(\mathbb{A}_1(1;0.5) + \mathbb{B}_1(1;0.5) - \mathbb{A}_2(1;0.5) - \mathbb{B}_2(1;0.5))|g'(2)|$$
$$+ (\mathbb{A}_2(1;0.5) + \mathbb{B}_2(1;0.5))|g'(4)|\Big]$$
$$= \frac{15}{128}.$$

It is clear that $\frac{1}{24} < \frac{15}{128}$, which demonstrates the first result described in Theorem 6.

Example 3. For $p < 1$, let $g(x) = \frac{1}{2}x^2$ for $x \in (0, \infty)$. Then $|g'(x)| = x$ is a p-convex mapping. If we take $a = 3$, $b = 4$, $u = \frac{1}{6}$, $v = \frac{5}{6}$, $\alpha = 1.3$, $p = 0.5$ and $q = 1$, then all the assumptions in Theorem 6 are satisfied.

The left-hand side term of (3.5) is:

$$\begin{aligned}&\left|\mathcal{T}_g(\alpha, p, u, v; a, b)\right| \\ &= \left|\frac{1}{6}\left[g(3) + 4g\left(\left[\frac{3^{0.5} + 4^{0.5}}{2}\right]^{\frac{1}{0.5}}\right) + g(4)\right] - \frac{0.5^{\alpha}\Gamma(2.3)}{(4^{0.5} - 3^{0.5})^{\alpha}}\,{}^{0.5}I^{1.3}_{4^-}g(3)\right| \\ &\approx 0.2243.\end{aligned} \tag{4.1}$$

The right-hand side term of (3.5) is:

$$\frac{b^{1-p}(b^p - a^p)}{p}\Big[(\mathbb{A}_1(\alpha;u) + \mathbb{B}_1(\alpha;v) - \mathbb{A}_2(\alpha;u) - \mathbb{B}_2(\alpha;v))|g'(a)|$$
$$+ (\mathbb{A}_2(\alpha;u) + \mathbb{B}_2(\alpha;v))|g'(b)|\Big]$$
$$= \frac{4^{1-0.5}(4^{0.5} - 3^{0.5})}{0.5}\left\{\left[\mathbb{A}_1\left(1.3;\frac{1}{6}\right) + \mathbb{B}_1\left(1.3;\frac{5}{6}\right) - \mathbb{A}_2\left(1.3;\frac{1}{6}\right) - \mathbb{B}_2\left(1.3;\frac{5}{6}\right)\right]|g'(3)|\right.$$
$$\left. + \left[\mathbb{A}_2\left(1.3;\frac{1}{6}\right) + \mathbb{B}_2\left(1.3;\frac{5}{6}\right)\right]|g'(4)|\right\}$$
$$\approx 0.5443.$$

It is clear that $0.2243 < 0.5443$, which demonstrates the second result described in Theorem 6.

5. Conclusion

Using the right Katugampola fractional integrals, four main results associated with the Hermite–Hadamard type and Simpson type inequalities involving p-convex mappings are obtained. Several interesting results can be derived by considering different values for the parameters u, v, p and α as well. It is worth mentioning that certain results proved in this article generalize parts of the results provided by Du et al. [7] Kirmaci [16] and Noor et al. [27]. With these techniques and the ideas developed in this work, it is possible to explore further estimations of other type integral inequalities for Katugampola fractional integrals which involve other related classes of mappings.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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