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Research article

Extended Bessel-Maitland function and its properties pertaining to integral transforms and fractional calculus

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Abstract: The aim of this paper is to establish an (presumably new) extension of generalized Bessel-Maitland function by using the extension of extended beta function. In addition, investigate several important properties namely integral representation, derivatives, recurrence relation, Beta transform and Mellin transform. Further, certain properties of the Riemann-Liouville fractional calculus associated with extended generalized Bessel-Maitland function are also investigated.

Keywords: extended Bessel-Maitland function; extended beta function; integral transform; Riemann-Liouville fractional calculus operators
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1. Introduction and preliminaries

In applied sciences, many important functions are defined via improper integrals or series (or finite products). The general name of these important functions knows as special functions. In special function, one of the most important function (Bessel function) has gained importance and popularity due to its applications in the problem of cylindrical coordinate system, wave propagation, heat conduction in cylindrical object and static potential etc. In the recent years, some generalizations (unification) and number of integral transforms of Bessel functions have been given by many mathematicians and physicist as well as engineers. The Bessel-Maitland function $J_{\vartheta}^{\tau}(z)$ is a generalization of Bessel function, defined in [7] through a series representation as:

$$J_{\vartheta}^{\tau}(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\tau n+\vartheta+1)\,n!} \tag{1.1}$$

In fact, the application of Bessel-Maitland function are found in the diverse field of mathematical physics, engineering, biological, chemical in the book of Watson [26].

Further, generalization of the generalized Bessel-Maitland function defined by Pathak [13] is as follow:

$$J_{\vartheta,q}^{\tau,\varsigma}(z)=\sum_{n=0}^{\infty}\frac{(\varsigma)_{nq}}{\Gamma(\tau n+\vartheta+1)}\frac{(-z)^n}{n!} \tag{1.2}$$

where $\tau,\vartheta,\varsigma\in\mathbb{C};\ \Re(\tau)\geq 0,\Re(\vartheta)\geq -1,\ \Re(\varsigma)\geq 0,\ q\in(0,1)\cup\mathbb{N}$.

Motivated by the established potential for application of these Bessel-Maitland functions, we introduce here another interesting extension of the generalized Bessel-Maitland function as follow:

$$J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p)=\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\omega}(\varsigma+nq,s-\varsigma;p)}{B(\varsigma,s-\varsigma)}\frac{(s)_{nq}}{\Gamma(\tau n+\vartheta+1)}\frac{(-z)^n}{n!} \tag{1.3}$$

where $\tau,\vartheta,\varsigma,\omega\in\mathbb{C};\ p>0,\ \Re(\tau)>0,\Re(\vartheta)\geq -1,\ \Re(\varsigma)>0,\Re(s)>0,\ q\in(0,1)\cup\mathbb{N}$, which will be known as extended generalized Bessel Maitland function (EGBMF).

Here, $\mathcal{B}_{\omega}(x,y;p)$ is an extension of extended beta function introduced by Parmar et al. [12] in the following way:

$$\mathcal{B}_{\omega}(x,y;p)=\sqrt{\tfrac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}K_{\omega+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)dt \tag{1.4}$$

where $K_{\omega+\frac{1}{2}}(.)$ is the modified Bessel's function. The special case of (1.4) corresponding to $\omega=0$ be easily seen to reduce to the extended beta function

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(\frac{-p}{t(1-t)}\right)dt \tag{1.5}$$

upon making use of ([9], Eq (10.39.2)). If $p=0$ in Eq (1.5), reduces in to the classical beta function. For a detailed account of various properties, generalizations and applications of Bessel-Maitland functions, the readers may refer to the recent work of the researchers [3, 15, 21–25] and the references cited therein.

2. Integral representation

Theorem 2.1. *The extended generalized Bessel Maitland function will be able to depict:*

$$J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p)=\frac{1}{B(\varsigma,\omega-\varsigma)}\sqrt{\tfrac{2p}{\pi}}\int_0^1 t^{\varsigma-\frac{3}{2}}(1-t)^{s-\varsigma-\frac{3}{2}}$$

$$\times K_{\omega+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)J_{\vartheta,q}^{\tau,s}(t^q z)\,dt \tag{2.1}$$

where $\tau,\vartheta,\varsigma,\omega\in\mathbb{C};\ p>0,\ \Re(\tau)>0,\Re(\vartheta)\geq -1,\ \Re(\varsigma)>0,\Re(s)>0,\ q\in(0,1)\cup\mathbb{N}$.

Proof. Using Eq (1.4) in Eq (1.3), we obtain

$$J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = \sum_{n=0}^{\infty} \left\{ \sqrt{\frac{2p}{\pi}} \int_0^1 t^{\varsigma+nq-\frac{3}{2}} (1-t)^{s-\varsigma-\frac{3}{2}} K_{\omega+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) dt \right\}$$

$$\times \frac{(s)_{nq} (-z)^n}{B(\varsigma, s-\varsigma) \Gamma(\tau n + \vartheta + 1)\, n!} \tag{2.2}$$

Reciprocate the order of summation and integration, that is surd under the presumption given in the description of Theorem 2.1, we get

$$J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = \frac{1}{B(\varsigma, s-\varsigma)} \sum_{n=0}^{\infty} \sqrt{\frac{2p}{\pi}} \int_0^1 t^{\varsigma+nq-\frac{3}{2}} (1-t)^{s-\varsigma-\frac{3}{2}}$$

$$\times K_{\omega+\frac{1}{2}} \left(\frac{p}{t(1-t)} \right) \frac{(s)_{nq} (-z)^n}{\Gamma(\tau n + \vartheta + 1)\, n!} dt \tag{2.3}$$

Using Eq (1.2) in Eq (2.3), we obtain the desired result Eq (2.1). □

Corollary 2.2. *Let the condition of Theorem 2.1 be satisfied, the following integral representation holds:*

$$J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = \frac{1}{B(\varsigma, s-\varsigma)} \sqrt{\frac{2p}{\pi}} \int_0^{\infty} r^{\varsigma-\frac{3}{2}} (1+r)^{2-s} K_{\omega+\frac{1}{2}} \left(\frac{-p(1+r)^2}{r} \right)$$

$$\times J_{\vartheta,q}^{\tau,s} \left(\left(\frac{r}{1+r} \right)^q z \right) dr. \tag{2.4}$$

Proof. By taking $t = \frac{r}{1+r}$ in Theorem 2.1, After simplification, we obtain the desired result Eq (2.4). □

Corollary 2.3. *Assume the state of Theorem 2.1 is satisfied, the following integral representation holds:*

$$J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = \frac{2}{B(\varsigma, s-\varsigma)} \sqrt{\frac{2p}{\pi}} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2(\varsigma-1)} (\cos\theta)^{2(\omega-\varsigma-1)}$$

$$\times K_{\omega+\frac{1}{2}} \left(\frac{-p}{\sin^2\theta \cos^2\theta} \right) J_{\vartheta,q}^{\tau,s} \left(z \sin^{2q}\theta \right) d\theta . \tag{2.5}$$

Proof. If we set $t = \sin^2\theta$ in Theorem 2.1, we acquire the above result. □

3. Recurrence relation

Theorem 3.1. *Let* $\omega, \varsigma, \tau, \vartheta, \in \mathbb{C}$; $\Re(\tau) > 0, \Re(\vartheta) \geq -1$, $\Re(\varsigma) > 0, \Re(s) > 0$, $p > 0$, $q \in (0,1) \cup \mathbb{N}$, *then the recurrence relation holds true:*

$$J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = (\vartheta+1) J_{\vartheta+1,q}^{\tau,\varsigma,s;\omega}(z;p) + \tau z \frac{d}{dz} J_{\vartheta+1,q}^{\tau,\varsigma,s;\omega}(z;p) \tag{3.1}$$

Proof. Employing Eq (1.3) in right hand side of Eq (3.1), we obtain

$$\begin{aligned}
& (\vartheta+1)\, J_{\vartheta+1,q}^{\tau,\varsigma,s;\omega}(z;p) \;+\; \tau\, z \frac{d}{dz} J_{\vartheta+1,q}^{\tau,\varsigma,s;\omega}(z;p) \\
= \;& (\vartheta+1) \sum_{n=0}^{\infty} \frac{\mathcal{B}_\omega(\varsigma+\; nq, s-\varsigma; p)}{B(\varsigma, s-\varsigma)} \frac{(s)_{nq}(-z)^n}{\Gamma(\tau n+\vartheta+2)\, n!} \\
+ \;& \tau\, z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{\mathcal{B}_\omega(\varsigma+\; nq, s-\varsigma; p)}{B(\varsigma, s-\varsigma)} \frac{(s)_{nq}(-z)^n}{\Gamma(\tau n+\vartheta+2)\, n!} \\
= \;& \sum_{n=0}^{\infty} \frac{\mathcal{B}_\omega(\varsigma+\; nq, s-\varsigma; p)}{B(\varsigma, s-\varsigma)} \frac{(s)_{nq}(\tau n+\vartheta+1)(-z)^n}{\Gamma(\tau n+\vartheta+2)\, n!} \\
= \;& J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p).
\end{aligned}$$

□

4. Derivative formulae

Theorem 4.1. *For the extended generalized Bessel Maitland function, we have the following higher derivative formula:*

$$\frac{d^n}{dz^n} J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = (s)_q\,(s+q)_q \ldots (s+(n-1)\,q)_q\, J_{\vartheta+n\tau,q}^{\tau,\varsigma+nq,s+nq;\omega}(z;p). \tag{4.1}$$

Proof. Taking the derivative with respect to z in Eq (2.1), we get

$$\frac{d}{dz} J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = (s)_q\, J_{\vartheta+\tau,q}^{\tau,\varsigma+q,s+q;\omega}(z;p) \tag{4.2}$$

Again taking the derivative with respect to z in Eq (6.5), we get

$$\frac{d^2}{dz^2} J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = (s)_q\,(s+q)_q\, J_{\vartheta+2\tau,q}^{\tau,\varsigma+2q,s+2q;\omega}(z;p) \tag{4.3}$$

Ongoing the repetition of this technique n times, we get the desired result Eq (4.1). □

Theorem 4.2. *For the extended generalized Bessel Maitland function, the following differentiation holds:*

$$\frac{d^n}{dz^n} \left\{ z^{\vartheta} J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(\sigma\, z^{\tau}; p) \right\} = z^{\vartheta-n} J_{\vartheta-n,q}^{\tau,\varsigma,s;\omega}(\sigma\, z^{\tau}; p). \tag{4.4}$$

Proof. Replace z by σz^{τ} in Eq (2.1) and take its product with z^{ϑ}, then taking z-derivative n times. We obtain our result. □

5. Beta transform

Definition 5.1. *The Beta transform [19] of a function $f(z)$ is defined as:*

$$\mathfrak{B}\{f(z);\; a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1}\; f(z)\, dz \tag{5.1}$$

$$(a,b\in\mathbb{C},\; \Re(a)>0,\; \Re(b)>0).$$

Theorem 5.2. *Let $\omega, \varsigma, \tau, \vartheta, \in \mathbb{C}$; $\Re(\tau) > 0, \Re(\vartheta) \geq -1$, $\Re(\varsigma) > 0, \Re(s) > 0$, $p > 0$, $q \in (0,1) \cup \mathbb{N}$, Then the Beta transform of extended generalized Bessel Maitland function holds true:*

$$\mathfrak{B}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(\lambda z^{\tau}; p)\ ; \vartheta + 1, 1\right\} = J_{\vartheta+1,q}^{\tau,\varsigma,s;\omega}(\lambda; p). \tag{5.2}$$

Proof. By definition of Beta transform (5.1) and (1.3), we get

$$\mathfrak{B}\left(J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z^{\tau}; p)\ ; \vartheta + 1, 1\right)$$

$$= \int_0^1 z^{\vartheta}(1-z)\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\omega}(\varsigma + nq, s-\varsigma; p)}{B(\varsigma, s-\varsigma)}\frac{(s)_{nq}}{\Gamma(\tau n + \vartheta + 1)}\frac{(-\lambda z^{\tau})^n}{n!}dz, \tag{5.3}$$

Upon interchanging the order of summation and integration in Eq (5.3), which can easily verified by uniform convergence under the constraint with Theorem 5.2, we get

$$\mathfrak{B}\left(J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z^{\tau}; p)\ ; \vartheta + 1, 1\right) = \sum_{n=0}^{\infty}\frac{\mathcal{B}_{\omega}(\varsigma + nq, s-\varsigma; p)}{B(\varsigma, s-\varsigma)}\frac{(s)_{nq}}{\Gamma(\tau n + \vartheta + 1)}\frac{(-\lambda)^n}{n!}$$

$$\times \int_0^1 z^{\vartheta + \tau n}(1-z)\,dz,$$

Using the familiar definition of beta function, and interpreting with Eq (1.3), we get the desired representation Eq (5.2). □

6. Mellin transform

Definition 6.1. *The Mellin transform [19] of the function $f(z)$ is defined as*

$$\mathfrak{M}(f(z)\ ; \xi) = \int_0^{\infty} z^{\xi-1} f(z)\,dz = f^*(\xi), \quad (\Re(\xi) > 0) \tag{6.1}$$

then inverse Mellin transform

$$f(z) = \mathfrak{M}^{-1}[f^*(\xi)\ ; x] = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} f^*(\xi)\ x^{-\xi} d\xi. \tag{6.2}$$

In the next theorem, we give Mellin transform of the extended generalized Bessel Maitland function. Therefore, we require the definition of Wright generalized hypergeometric function [20] as:

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{l}(c_1, C_1), (c_2, C_2), \ldots, (c_p, C_p);\\ (d_1, D_1), (d_2, D_2), \ldots, (d_q, D_q);\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(c_i, C_i n) z^n}{\prod_{j=1}^{q}\Gamma(d_i, D_i n)\, n!} \tag{6.3}$$

where the coefficients $C_i\ (i = 1, 2, \ldots, p)$ and $D_j\ (j = 1, 2, \ldots, q)$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} D_j - \sum_{i=1}^{p} C_i \geq 0.$$

Theorem 6.2. *The Mellin transform of the extended generalized Bessel Maitland function is given by*

$$\mathfrak{M}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p);\xi\right\} = \frac{2^{\xi-1}\Gamma(\xi+s-\varsigma)}{\sqrt{\pi}\,\Gamma(\varsigma)\,\Gamma(s-\varsigma)}\Gamma\left(\frac{\xi-\omega}{2}\right)\Gamma\left(\frac{\xi+\omega+1}{2}\right)$$

$$\times\,{}_2\psi_2\left[\begin{array}{c}(s,q),\,(\varsigma+\xi,q)\,;\\(\vartheta+1,\tau),\,(s+2\xi,q)\,;\end{array}\;-z\right] \tag{6.4}$$

where $\omega,\varsigma,\tau,\vartheta,\xi,\in\mathbb{C};\ \Re(\tau)>0, \Re(\vartheta)\geq-1,\ \Re(\varsigma)>0, \Re(s)>0, \Re(\xi)>0,\ p>0,\ q\in(0,1)\cup\mathbb{N}$, *and* ${}_2\psi_2$ *is the Wright generalized hypergeometric function.*

Proof. Using the definition of Melllin transform (6.1) and (1.3), we obtain

$$\mathfrak{M}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p);\xi\right\} = \frac{1}{B(\varsigma,s-\varsigma)}\int_0^\infty p^{\xi-1}\left\{\sqrt{\tfrac{2p}{\pi}}\int_0^1 t^{\varsigma-\frac{3}{2}}(1-t)^{s-\varsigma-\frac{3}{2}}\right.$$

$$\left.\times K_{\omega+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)J_{\vartheta,q}^{\tau,s}(t^q z)\,dt\right\}dp, \tag{6.5}$$

Interchanging the order of integration in Eq (6.5), which is admittable because of the conditions in the statement of the Theorem 3.4, we get

$$\mathfrak{M}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p);\xi\right\} = \frac{1}{B(\varsigma,s-\varsigma)}\sqrt{\tfrac{2}{\pi}}\int_0^1 t^{\varsigma-\frac{3}{2}}(1-t)^{s-\varsigma-\frac{3}{2}}J_{\vartheta,q}^{\tau,s}(t^q z)\,dt,$$

$$\times\left\{\int_0^\infty p^{\xi-\frac{1}{2}}K_{\omega+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)dp\right\}dt \tag{6.6}$$

Now taking $u=\frac{p}{t(1-t)}$ in Eq (6.6), we get

$$\mathfrak{M}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p);\xi\right\} = \frac{1}{B(\varsigma,s-\varsigma)}\sqrt{\tfrac{2}{\pi}}\int_0^1 t^{\varsigma+\xi-1}(1-t)^{s-\varsigma+\xi-1}J_{\vartheta,q}^{\tau,s}(t^q z)\,dt$$

$$\times\int_0^\infty u^{\xi-\frac{1}{2}}K_{\omega+\frac{1}{2}}(u)\,du, \tag{6.7}$$

From Olver et al. [9]:

$$\int_0^\infty u^{\xi-\frac{1}{2}}K_{\omega+\frac{1}{2}}(u)\,du = 2^{\xi-\frac{3}{2}}\Gamma\left(\frac{\xi-\omega}{2}\right)\Gamma\left(\frac{\xi+\omega+1}{2}\right), \tag{6.8}$$

Applying Eq (6.8) in Eq (6.7), we obtain

$$\mathfrak{M}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p);\xi\right\} = \frac{1}{B(\varsigma,s-\varsigma)}\frac{2^{\xi-1}}{\sqrt{\pi}}\int_0^1 t^{\varsigma+\xi-1}(1-t)^{s-\varsigma+\xi-1}J_{\vartheta,q}^{\tau,s}(t^q z)\,dt$$

$$\times\Gamma\left(\frac{\xi-\omega}{2}\right)\Gamma\left(\frac{\xi+\omega+1}{2}\right),$$

Using Eq (1.2), and interchanging the order of summation and integration which is valid for $\Re(\tau) > 0, \Re(\vartheta) > 0, \Re(s) > 0, \Re(s) > \Re(\varsigma) > 0, \Re(s + \xi - \varsigma) > 0$, we obtain

$$\mathfrak{M}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p);\xi\right\} = \frac{2^{\xi-1}}{B(\varsigma, s-\varsigma)\ \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(s)_{nq}}{\Gamma(\tau n + \vartheta + 1)} \frac{(-z)^n}{n!}$$

$$\times \Gamma\left(\frac{\xi - \omega}{2}\right) \Gamma\left(\frac{\xi + \omega + 1}{2}\right) \int_0^1 t^{\varsigma+\xi+nq-1} (1-t)^{\xi+s-\varsigma-1} \, dt, \tag{6.9}$$

Using the relation between Beta function and Gamma function, we obtain

$$\mathfrak{M}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p);\xi\right\} = \frac{2^{\xi-1}}{B(\varsigma, s-\varsigma)\ \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(s)_{nq}}{\Gamma(\tau n + \vartheta + 1)} \Gamma\left(\frac{\xi - \omega}{2}\right)$$

$$\times \Gamma\left(\frac{\xi + \omega + 1}{2}\right) \frac{\Gamma(\varsigma + \xi + nq)\,\Gamma(s + \xi - \varsigma)}{\Gamma(s + 2\xi + nq)} \frac{(-z)^n}{n!},$$

After simplification, we obtain

$$\mathfrak{M}\left\{J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p);\xi\right\} = \frac{2^{\xi-1}\Gamma(\xi + s - \varsigma)\,\Gamma\left(\frac{\xi-\omega}{2}\right)\Gamma\left(\frac{\xi+\omega+1}{2}\right)}{\sqrt{\pi}\,\Gamma(\varsigma)\,\Gamma(s-\varsigma)}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(s + nq)\,\Gamma(\varsigma + \xi + nq)\,(-z)^n}{\Gamma(\tau n + \vartheta + 1)\ \Gamma(s + 2\xi + nq)\, n!}, \tag{6.10}$$

In view of Eq (6.3), we arrived at our result Eq (6.4). □

Corollary 6.3. *Taking $\xi = 1$ in Theorem 6.2, we get*

$$J_{\vartheta,q}^{\tau,\varsigma,s;\omega}(z;p) = \frac{\Gamma(1 + s - \varsigma)}{\sqrt{\pi}\,\Gamma(\varsigma)\,\Gamma(s-\varsigma)} \Gamma\left(\frac{1-\omega}{2}\right) \Gamma\left(\frac{\omega+2}{2}\right)$$

$$\times {}_2\psi_2\left[\begin{array}{ll} (s,q)\,,\ (\varsigma+1,q)\ ; & \\ (\vartheta+1,\tau)\,,\ (s+2,q)\ ; & -z \end{array}\right]. \tag{6.11}$$

7. Fractional calculus approach

In recent years, the fractional calculus has become a significant instrument for the modeling analysis and assumed a significant role in different fields, for example, material science, science, mechanics, power, science, economy and control theory. In addition, research on fractional differential equations (ordinary or partial) and other analogous topics is very active and extensive around the world. One may refer to the books [28, 31], and the recent papers [1, 2, 6, 16, 18, 27, 29, 30, 32–35] on the subject. In this portion, we derive a slight interesting properties of EMBMF associated with the right hand sided of Riemann-Liouville (R-L) fractional integral operator I_{a+}^{ζ} and the right sided of R-L fractional derivative operatorD_{a+}^{ζ}, which are defined for $\zeta \in \mathbb{C}, (\Re(\zeta) > 0), x > 0$ (See, for details [5, 17]):

$$\left(I_{a+}^{\zeta} f\right)(x) = \frac{1}{\Gamma(\zeta)} \int_a^x \frac{f(t)}{(x-t)^{1-\zeta}} dt, \tag{7.1}$$

and

$$\left(D_{a+}^{\zeta}f\right)(x) = \left(\frac{d}{dx}\right)^{\ell}\left(I_{a+}^{\ell-\zeta}f\right)(x) \quad \ell = [\Re(\zeta)+1]. \tag{7.2}$$

where $[\Re(\zeta)]$ is the integral part of $\Re(\zeta)$.

A generalization of R-L fractional derivative operator (7.2) by introducing a right hand sided R-L fractional derivative operator $D_{a+}^{\zeta,\sigma}$ of order $0<\zeta<1$ and $0\leq\sigma\leq 1$with respect to x by Hilfer [4] is as follows:

$$\left(D_{a+}^{\zeta,\sigma}f\right)(x) = \left(I_{a+}^{\sigma(1-\zeta)}\frac{d}{dx}\right)\left(I_{a+}^{(1-\sigma)(1-\zeta)}f\right)(x). \tag{7.3}$$

The generalization Eq (7.3) yields the R-L fractional derivative operator D_{a+}^{ζ} when $\sigma=0$ and moreover, in its special case when $\sigma=1$, the definition (7.3) would reduce to the familiar Caputo fractional derivative operator [5].

Theorem 7.1. *Let $\zeta,\ \lambda,\tau,\ \vartheta,\varsigma,s\ \in\mathbb{C}$ be such that $\Re(\zeta)>0$, $p\geq 0$ and the conditions given in Eq* (1.3) *is satisfied, for $x>a$,the following relation holds:*

$$\left(I_{a+}^{\zeta}\left\{(z-a)^{\vartheta}\ J_{\vartheta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(z-a)^{\tau};p\right)\right\}\right)(x)$$
$$= (x-a)^{\zeta+\vartheta}\ J_{\vartheta+\zeta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(x-a)^{\tau};p\right). \tag{7.4}$$

$$\left(D_{a+}^{\zeta}\left\{(z-a)^{\vartheta}\ J_{\vartheta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(z-a)^{\tau};p\right)\right\}\right)(x)$$
$$= (x-a)^{\vartheta-\zeta}\ J_{\vartheta-\zeta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(x-a)^{\tau};p\right). \tag{7.5}$$

$$\left(D_{a+}^{\zeta,\sigma}\left\{(z-a)^{\vartheta}\ J_{\vartheta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(z-a)^{\tau};p\right)\right\}\right)(x)$$
$$= (x-a)^{\vartheta-\zeta}\ J_{\vartheta-\zeta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(x-a)^{\tau};p\right). \tag{7.6}$$

Proof. By virtue of the formulas Eq (7.1) and Eq (1.3), the term by term fractional integration and use of the relation [4]

$$\left(I_{a+}^{\zeta}(z-a)^{\vartheta-1}\right)(x) = \frac{\Gamma(\vartheta)}{\Gamma(\zeta+\vartheta)}(x-a)^{\zeta+\vartheta-1} \qquad (\vartheta,\zeta\in\mathbb{C},\ \Re(\zeta)>0,\Re(\vartheta)>0) \tag{7.7}$$

yield for $x>a$.

$$\left(I_{a+}^{\zeta}\left\{(z-a)^{\vartheta}\ J_{\vartheta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(z-a)^{\tau};p\right)\right\}\right)(x)$$
$$= \left(I_{a+}^{\zeta}\left\{\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\omega}(\varsigma+nq,s-\varsigma;p)}{B(\varsigma,s-\varsigma)}\frac{(s)_{nq}}{\Gamma(\tau n+\vartheta+1)}\frac{(-\lambda)^{n}(z-a)^{\tau n+\vartheta}}{n!}\right\}\right)(x),$$
$$= (x-a)^{\zeta+\vartheta}\ J_{\vartheta+\zeta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(x-a)^{\tau};p\right). \tag{7.8}$$

Consequent, by Eq (7.5) and Eq (1.3), we find that

$$\left(D_{a+}^{\zeta}\left\{(z-a)^{\vartheta}\ J_{\vartheta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(z-a)^{\tau};p\right)\right\}\right)(x)$$

$$= \left(\frac{d}{dx}\right)^{\ell}\left(I_{a+}^{\ell-\zeta}\left\{(z-a)^{\vartheta}\ J_{\vartheta,q}^{\tau,\varsigma,s;\omega}\left(\lambda(z-a)^{\tau};p\right)\right\}\right)(x)$$

$$= \left(\frac{d}{dx}\right)^{\ell} \left((x-a)^{\vartheta+\ell-\zeta-1} \ J^{\tau,\varsigma,s;\omega}_{\vartheta+\ell-\zeta,q} \left(\lambda\left(x-a\right)^{\tau};p\right)\right)(x), \tag{7.9}$$

Applying Eq (4.4), we are led to the desired result Eq (7.5). Lastly, by Eq (7.3) and Eq (1.3), we becomes

$$\left(D^{\zeta,\sigma}_{a+}\left\{(z-a)^{\vartheta} \ J^{\tau,\varsigma,s;\omega}_{\vartheta,q}\left(\lambda\left(z-a\right)^{\tau};p\right)\right\}\right)(x)$$

$$= \left(D^{\zeta,\sigma}_{a+}\left\{\sum_{n=0}^{\infty}\frac{\mathcal{B}_{\omega}\left(\varsigma+nq,s-\varsigma;p\right)}{B\left(\varsigma,s-\varsigma\right)}\frac{(s)_{nq}}{\Gamma\left(\tau n+\vartheta+1\right)}\frac{(-\lambda)^{n}\left(z-a\right)^{\tau n+\vartheta}}{n!}\right\}\right)(x)$$

$$= \sum_{n=0}^{\infty}\frac{\mathcal{B}_{\omega}\left(\varsigma+nq,s-\varsigma;p\right)}{B\left(\varsigma,s-\varsigma\right)}\frac{(s)_{nq}}{\Gamma\left(\tau n+\vartheta+1\right)}\frac{(-\lambda)^{n}}{n!}\left(D^{\zeta,\sigma}_{a+}\left\{(z-a)^{\tau n+\vartheta}\right\}\right)(x), \tag{7.10}$$

Using the familiar relation of Srivatava and Tomovski [11]:

$$\left(D^{\zeta,\sigma}_{a+}\left\{(z-a)^{\vartheta-1}\right\}\right)(x) = \frac{\Gamma(\vartheta)}{\Gamma(\vartheta-\zeta)}\ (x-a)^{\vartheta-\zeta-1} \tag{7.11}$$

$$\left(x>a;\ 0<\zeta<1;\ 0\le\sigma\le1,\ \Re(\vartheta)>0\right)$$

In Eq (7.10), we are led to the result Eq (7.6). □

8. Conclusions

In the present paper, The properties, integral transform and fractional calculus of the newly defined extended generalized Bessel-Maitland type function are investigated here and find their connection with other functions scattered in the literature of special function. Various special cases of the derived results in the paper can be evaluate by taking suitable values of parameters involved. For example if we set $\omega = 0, \beta = \beta - 1$ and $z = -z$ in (1.3), we immediately obtain the result due to Mittal et al [18]. For various other special cases we refer [19,21] and we left results for the interested readers.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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