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Research article

Global attractor and exponential attractor for a Parabolic system of Cahn-Hilliard with a proliferation term

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Abstract: In this article, we are interested in the study of Parabolic system of Cahn-Hilliard with a proliferation term and Dirichet boundary conditions. In particular, we prove the existence and the uniqueness of the solution, the existence of the global attractor and the existence of an exponential attractor.

Keywords: Cahn-Hilliard system; proliferation term; dissipativity; global attractor; exponential attractors
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1. Introduction

The generalization of the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(u) = 0, \tag{1.1}$$

where g is the proliferation term, has been proposed in [3, 5] as a model for the growth of cancerous tumors and other biological entities.

Generally, g can be the linear function $g(s) = \alpha s$, $\alpha > 0$ in which case (1.1) is known as the Cahn-Hilliard-Oono equation and accounts for long-ranged interactions in the phase equations and in the phase separation process (see [8]). The other possibility is the quadratic function $g(s) = \alpha s(s-1)$, $\alpha > 0$ (nonlinear). In that case (1.1) has applications in biology and, precisely, models wound healing and tumour growth (see [12]). To be more precise, such a model deals with cells which move, proliferate and interact via diffusion and cell-cell adhesion. Here, u is the order parameter, f is the local free energy and α is the proliferation rate.

These equation has been studied; we refer to, e.g., [3, 5, 9], in which authors have proved the existence and the uniqueness of the solution and the existence of the finite-dimensional attractors.

Precisely, we are going to study the model for the growth of cancerous tumors and other biological entities, model obtained by combinaison of (1.1) with the following temperature equation

$$\frac{\partial\theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t}. \tag{1.2}$$

The same kind of model without proliferation term are knwon as the conserved phase field model and has been studied (see for instance D. Brochet, X. Chen and D. Hilhost G [1], L. Cherfils and A. Miranville [2], Gilardi [6], C. Giorgi, M. Grasseli, and V. Pata A [7], Miranville [10], A. J. Ntsokongo and N. Batangouna [11]).

This work is structured as follows: firstly, we have the setting of the problem followed by a priori estimates which allow us to construct the dissipative semigroup associated to the problem, and finally we prove the existence of the exponential attractors and, thus, of finite-dimensional global attractors.

2. Setting of the problem

We consider the following parabolic system of Cahn-Hilliard with a proliferation term

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(u) = \theta \text{ in } \mathbb{R}_+^* \times \Omega, \tag{2.1}$$

$$\frac{\partial\theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} \text{ in } \mathbb{R}_+^* \times \Omega, \tag{2.2}$$

$$\theta = \Delta u = u = 0 \text{ on } \mathbb{R}_+^* \times \Gamma, \tag{2.3}$$

$$\theta(0, x) = \theta_0(x); \;\; u(0, x) = u_0(x), \;\; \forall x \in \Omega, \tag{2.4}$$

in a bounded and regular domain $\Omega \subset \mathbb{R}^n$($n = 1, 2$ or 3) with boundary Γ.
f and g verifies the following properties:

$$f \in C^2(\mathbb{R}),\; f(0) = 0,\; g \in C^1(\mathbb{R}) \tag{2.5}$$

$$f^{'}(s) \geqslant -c_0,\; s \in \mathbb{R},\; c_0 \geqslant 0 \tag{2.6}$$

$$f(s)s \geqslant c_1 F(s) - c_2 \geqslant -c_3,\; s \in \mathbb{R},\; c_1 > 0,\; c_2, c_3 \geqslant 0, \tag{2.7}$$

where $F(s) = \displaystyle\int_0^s f(\xi)d\xi$ and for all $u \in L^2(\Omega)$, such that $\displaystyle\int_\Omega F(u)dx < +\infty$, we have

$$\|g(u)\|\|u\| \leqslant \epsilon \int_\Omega F(u)dx + c_\epsilon,\; \forall\, \epsilon > 0, \tag{2.8}$$

$$\|g^{'}(u)\|^2 \leqslant c_6 \int_\Omega F(u)dx + c_7,\; c_6 \geqslant 0, \tag{2.9}$$

Notation. We denote by $(.,.)$ the usual L^2-scalar product, with associated norm $\|.\|$, and we set $\|.\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}.\|$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet boundary conditions. More generally, $\|.\|_X$ denotes the norm in the Banach space X.

Remark 2.1. *The properties (2.8) and (2.9) essentially mean that g is subordinated to f. In particular, they hold for the usual choices* $f(s) = s^3 - s$ *and* $g(s) = \alpha s$ *or* $g(s) = \alpha s(s-1)$, $\alpha > 0$. *More generally,* *(2.8) and (2.9) hold when f and g have polynomial growths of order $2p+1$, $p \geqslant 1$ and $q \geqslant 1$ such that $q \leqslant p+1$. Indeed, let us assume, for simplicity, that $f(s) = \sum_{i=1}^{2p+1} a_i s^i$, $a_{2p+1} > 0$ and $g(s) = \sum_{i=1}^{q} b_i s^i$, $b_q > 0$. Then,*

$$\|g(u)\|\|u\| \leqslant c(\|u\|^q_{L^{2p}(\Omega)} + 1)\|u\| \leqslant c(\|u\|^{q+1}_{L^{2p}(\Omega)} + 1) \leqslant \epsilon \int_\Omega F(u)dx + c_\epsilon \ \forall \epsilon > 0,$$

since $2q \leqslant 2p+2$ and $q+1 < 2p+2$. Furthermore ,

$$\|g^{'}(u)\|^2 \leqslant c\int_\Omega u^{2q-2}dx + c^{'} \leqslant c\int_\Omega u^{2q+2}dx + c^{'}, \ \textit{hence } (2.9).$$

In this article, we assume $f(s) = s^3 - s$ and $g(s) = \alpha s(s-1)$, $\alpha > 0$.

Finally, the same letter c, $c^{'}$ and $c^{''}$ denotes constants which may vary from line to line, or even in a same line. Similary, the same letter Q denotes monotone increasing functions which may vary from line to line, or even in a same line.

3. A priori estimates

In what follows, the Poincaré, Hölder and Young inequalities are extensively used, without further referring to them.

We multiply (2.1) by $(-\Delta)^{-1}u$ and integrate over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{-1} + \|\nabla u\|^2 + (f(u),u) + (g(u),(-\Delta)^{-1}u) = (\theta,(-\Delta)^{-1}u), \tag{3.1}$$

owing to (2.7) and (2.8), we obtain

$$\frac{d}{dt}\|u\|^2_{-1} + c\left(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)dx\right)dx \leqslant \frac{c}{2}\int_\Omega F(u)dx + c^{'}\|\theta\|^2 + c^{''}, \ c > 0,$$

hence

$$\frac{d}{dt}\|u\|^2_{-1} + c\left(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)dx\right) \leqslant c^{'}\|\theta\|^2 + c^{''}, \ c > 0. \tag{3.2}$$

We now multiply (2.2) by θ and integrate over Ω, we find

$$\frac{d}{dt}\|\theta\|^2 + 2\|\nabla\theta\|^2 = -2\left(\frac{\partial u}{\partial t},\theta\right). \tag{3.3}$$

We then multiply (2.1) by $\frac{\partial u}{\partial t}$ and integrate over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 = \left(\Delta f(u),\frac{\partial u}{\partial t}\right) - \left(g(u),\frac{\partial u}{\partial t}\right) + \left(\theta,\frac{\partial u}{\partial t}\right). \tag{3.4}$$

Here,

$$\left|\left(g(u), \frac{\partial u}{\partial t}\right)\right| \leqslant \|g(u)\| \left\|\frac{\partial u}{\partial t}\right\| \leqslant \|g(u)\|^2 + \frac{1}{4}\left\|\frac{\partial u}{\partial t}\right\|^2.$$

Furthermore,

$$\left(\Delta f(u), \frac{\partial u}{\partial t}\right) \leqslant c\|f(u)\|^2_{H^2(\Omega)} + \frac{1}{4}\left\|\frac{\partial u}{\partial t}\right\|^2,$$

which yields

$$\frac{d}{dt}\|\Delta u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \leqslant c(\|f(u)\|^2_{H^2(\Omega)} + \|g(u)\|^2) + 2\left(\theta, \frac{\partial u}{\partial t}\right). \tag{3.5}$$

We recal that $H^2(\Omega)$ is continuously embedded in $C(\overline{\Omega})$ and owing to (2.5),

$$\|f(u)\|^2_{H^2(\Omega)} + \|g(u)\|^2 \leqslant Q(\|u\|_{H^2(\Omega)}), \tag{3.6}$$

and inserting (3.6) into (3.5), we obtain

$$\frac{d}{dt}\|\Delta u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \leqslant Q(\|u\|_{H^2(\Omega)}) + 2\left(\theta, \frac{\partial u}{\partial t}\right). \tag{3.7}$$

Summing finally (3.3) and (3.7), we find

$$\frac{d}{dt}(\|\Delta u\|^2 + \|\theta\|^2) + \left\|\frac{\partial u}{\partial t}\right\|^2 + 2\|\nabla\theta\|^2 \leqslant Q(\|\Delta u\|^2).$$

In particular, we deduce

$$\frac{d}{dt}(\|\Delta u\|^2 + \|\theta\|^2) \leqslant Q(\|\Delta u\|^2). \tag{3.8}$$

We set

$$y = \|\Delta u\|^2 + \|\theta\|^2,$$

we deduce from (3.8) an inequation of the form

$$y^{'} \leqslant Q(y). \tag{3.9}$$

Let z be the solution to the ordinary differential equation

$$z^{'} = Q(z), \ z(0) = y(0) = \|\Delta u_0\|^2 + \|\theta_0\|^2.$$

It follows from the comparison principle, that there exists a time $T_0 = T_0(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) > 0$ belonging to, say $(0, \frac{1}{2})$ such that

$$y(t) \leqslant z(t) \quad \forall t \leqslant T_0,$$

hence

$$\|u\|^2_{H^2(\Omega)} + \|\theta\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|), \quad \forall t \leqslant T_0. \tag{3.10}$$

We multiply (2.1) by $(-\Delta)^{-1}\dfrac{\partial u}{\partial t}$ and integrate over Ω, we have

$$\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)dx\right) + \left\|\frac{\partial u}{\partial t}\right\|^2_{-1} \leqslant c\|g(u)\|^2 + 2\left(\theta, (-\Delta)^{-1}\frac{\partial u}{\partial t}\right). \tag{3.11}$$

We then multiply (2.2) by $(-\Delta)^{-1}\theta$ and integrate over Ω, we obtain

$$\frac{d}{dt}\|\theta\|^2_{-1} + 2\|\theta\|^2 = -2\left((-\Delta)^{-1}\frac{\partial u}{\partial t}, \theta\right). \tag{3.12}$$

Summing (3.11) and (3.12), owing to (3.6)-(3.10), we find

$$\frac{dE_1}{dt} + c\left(\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + \|\theta\|^2\right) \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|),\ c > 0, \tag{3.13}$$

where

$$E_1 = \|\nabla u\|^2 + 2\int_\Omega F(u)dx + \|\theta\|^2_{-1}.$$

Summing ψ_1(3.2) and (3.13) , where $\psi_1 > 0$ is small enough, we obtain

$$\frac{dE_2}{dt} + c\left(E_2 + \left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + \|\theta\|^2\right) \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c',\ c > 0, \tag{3.14}$$

where

$$E_2 = E_1 + \psi_1\|u\|^2_{-1},$$

satisfies

$$c\left(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)dx + \|\theta\|^2_{-1}\right) + c' \leqslant E_2 \leqslant c''\left(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)dx + \|\theta\|^2_{-1}\right) + c''',$$
$$c, c'' > 0.$$

In particular, we deduce from (3.14) and Gronwall's lemma the dissipative estimate

$$E_2(t) + \int_0^t e^{-c(t-s)}\left\|\frac{\partial u(s)}{\partial t}\right\|^2_{-1} ds \leqslant ce^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c'',\ c > 0, t \geqslant 0, \tag{3.15}$$

where we have used the continuous embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ to deduce that

$$|\int_\Omega F(u_0)dx| \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|).$$

Furthermore,

$$\int_t^{t+1}\left(\left\|\frac{\partial u(s)}{\partial t}\right\|^2_{-1} + \|\theta(s)\|^2\right)ds \leqslant ce^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c'',\ c > 0, t \geqslant 0. \tag{3.16}$$

Finally, more generally, for every $r > 0$

$$\int_t^{t+r}\left(\left\|\frac{\partial u(s)}{\partial t}\right\|_{-1}^2 + \|\theta(s)\|^2\right)ds \leqslant ce^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c^{''}(r),\ c > 0, t \geqslant 0. \tag{3.17}$$

We now differentiate (2.1) with respect to time and have, noting that $\frac{\partial\theta}{\partial t} = -\frac{\partial u}{\partial t} + \Delta\theta$,

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + f^{'}(u)\frac{\partial u}{\partial t} + (-\Delta)^{-1}(g^{'}(u)\frac{\partial u}{\partial t}) = -(-\Delta)^{-1}\frac{\partial u}{\partial t} - \theta. \tag{3.18}$$

We multiply (3.18) by $t\frac{\partial u}{\partial t}$ and integrate over Ω, owing (2.6), we find for $t \leqslant T_0$

$$\frac{t}{2}\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\right) + t\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \leqslant c_0 t\left\|\frac{\partial u}{\partial t}\right\|^2 + t\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + t\left|\left(\theta, \frac{\partial u}{\partial t}\right)\right| + t\left|\left((-\Delta)^{-1}(g^{'}(u)\frac{\partial u}{\partial t}), \frac{\partial u}{\partial t}\right)\right|. \tag{3.19}$$

We know that $H^2(\Omega) \subset L^\infty(\Omega)$ with continuous injection and $(-\Delta)^{-1}\frac{\partial u}{\partial t} \in H^2(\Omega)$. Then

$$\begin{aligned}
\left|\left((-\Delta)^{-1}(g^{'}(u)\frac{\partial u}{\partial t}), \frac{\partial u}{\partial t}\right)\right| &\leqslant \int_\Omega \left|g^{'}(u)\right|\left|\frac{\partial u}{\partial t}\right|\left|(-\Delta)^{-1}\frac{\partial u}{\partial t}\right|dx \\
&\leqslant \|g^{'}(u)\|\left\|\frac{\partial u}{\partial t}\right\|\left\|(-\Delta)^{-1}\frac{\partial u}{\partial t}\right\|_{L^\infty(\Omega)} \\
&\leqslant \|g^{'}(u)\|\left\|\frac{\partial u}{\partial t}\right\|\left\|(-\Delta)^{-1}\frac{\partial u}{\partial t}\right\|_{H^2(\Omega)} \\
&\leqslant c\|g^{'}(u)\|\left\|\frac{\partial u}{\partial t}\right\|\left\|(-\Delta)(-\Delta)^{-1}\frac{\partial u}{\partial t}\right\| \\
&\leqslant c\|g^{'}(u)\|\left\|\frac{\partial u}{\partial t}\right\|^2 \\
&\leqslant c^{'}\|g^{'}(u)\|^2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \frac{1}{4}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \\
&\leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|)\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \frac{1}{4}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2.
\end{aligned} \tag{3.20}$$

Inserting (3.20) into (3.19), we have

$$\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\right) + t\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|)\left(t\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\right) + 2c_0 t\left\|\frac{\partial u}{\partial t}\right\|^2 + 2ct\|\theta\|^2 + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2.$$

Noting that,

$$\left\|\frac{\partial u}{\partial t}\right\|^2 \leqslant c\left\|\frac{\partial u}{\partial t}\right\|_{-1}\left\|\nabla\frac{\partial u}{\partial t}\right\|,$$

hence

$$\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\right) + \frac{t}{2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|)\left(t\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\right) + c^{'}t\|\theta\|^2 + c\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2.$$

(3.21)

Using the estimates (3.17), (3.21) and Gronwall's lemma, we find

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2 \leqslant \frac{1}{t}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|), \ \forall t \in (0, T_0]. \tag{3.22}$$

Multiplying then (3.18) by $\frac{\partial u}{\partial t}$ and integrate over Ω, we obtain, proceeding as above,

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \frac{1}{2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \ \leqslant \ c\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \|\theta\|^2\right). \tag{3.23}$$

It thus follows from (3.17), (3.23) and Gronwall's lemma

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2 \ \leqslant \ e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|)\left\|\frac{\partial u(T_0)}{\partial t}\right\|_{-1}^2, \ c \geqslant 0, \ t \geqslant T_0, \tag{3.24}$$

and it finally follows from (3.22) that

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2 \ \leqslant \ e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|), \ c \geqslant 0, t \geqslant T_0. \tag{3.25}$$

We now rewrite, for $t \geqslant T_0$ fixed, (2.1) in the form

$$-\Delta u + f(u) + (-\Delta)^{-1}g(u) = h_u(t), \ u = 0 \text{ on } \Gamma, \tag{3.26}$$

where

$$h_u(t) = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + (-\Delta)^{-1}\theta. \tag{3.27}$$

We multiply (3.27) by $h_u(t)$ and integrate over Ω, we obtain

$$\|h_u(t)\|^2 \ \leqslant \ c'\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \|\theta\|^2\right),$$

owing to (3.15)-(3.25), we have

$$\|h_u(t)\|^2 \ \leqslant \ e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|), \ c > 0, \ t \geqslant T_0. \tag{3.28}$$

We multiply (3.26) by u and integrate over Ω, we find

$$\|\nabla u\|^2 + (f(u), u) \ \leqslant \ \|h_u(t)\|\|u\| + c\|g(u)\|\|u\|,$$

which yields, owing to (2.7) and (2.8)

$$\|\nabla u\|^2 + c_1\int_\Omega F(u)dx \leqslant c\|h_u(t)\|^2 + c', \ c_1 > 0. \tag{3.29}$$

Then, multiplying (3.26) by $-\Delta u$ and integrate over Ω, we obtain

$$\|\Delta u\|^2 + (f'(u)\nabla u, \nabla u) \leqslant \|h_u(t)\|\|\Delta u\| + \|g(u)\|\|u\|,$$

which yields, owing to (2.6) and (2.8)

$$\|\Delta u\|^2 \leqslant \|h_u(t)\|^2 + c'\|\nabla u\|^2 + c\int_\Omega F(u)dx + c''. \tag{3.30}$$

Summing (3.29) and ψ_2(3.30), where $\psi_2 > 0$ is small enough, we find

$$\|\nabla u\|^2 + c(\|u\|^2_{H^2(\Omega)} + \int_\Omega F(u)dx) \quad \leqslant \quad c'\|h_u(t)\|^2 + c'',\ c > 0. \tag{3.31}$$

We thus deduce from (3.28) and (3.31) that

$$\|u\|^2_{H^2(\Omega)} \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c'',\ c \geqslant 0,\ t \geqslant T_0. \tag{3.32}$$

The estimate (3.3) implies

$$\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 \quad \leqslant \quad c\left\|\frac{\partial u}{\partial t}\right\|^2. \tag{3.33}$$

It follows from (3.10), (3.23) and (3.25) that

$$\int_{T_0}^t \left\|\frac{\partial u(s)}{\partial t}\right\|^2 ds \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|),\ c > 0,\ t \geqslant T_0. \tag{3.34}$$

We then deduce from (3.33) and (3.34) that

$$\|\theta\|^2 + \int_{T_0}^t \|\nabla\theta(s)\|^2 ds \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|),\ c > 0,\ t \geqslant T_0,$$

which gives

$$\|\theta\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|),\ c > 0,\ t \geqslant T_0. \tag{3.35}$$

Combining (3.32) and (3.35), we obtain

$$\|u\|^2_{H^2(\Omega)} + \|\theta\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c'',\ c \geqslant 0,\ t \geqslant T_0. \tag{3.36}$$

Finally, we deduce from (3.10) and (3.36), that

$$\|u\|^2_{H^2(\Omega)} + \|\theta\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c'',\ c \geqslant 0,\ t \geqslant 0. \tag{3.37}$$

We now note, it follows from (3.16) that

$$\int_0^1 \|\theta\|^2 dt \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c''. \tag{3.38}$$

Furthermore, multiplying (2.1) by u and integrate over Ω, we find, thanks to (2.6)

$$\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 \leqslant c(\|\nabla u\|^2 + \|\theta\|^2) + c'\|g(u)\|^2.$$

Owing to (3.6) and (3.10), we have

$$\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|). \tag{3.39}$$

We thus deduce from (3.39) that

$$\int_0^1 \|u\|^2_{H^2(\Omega)} dt \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|). \tag{3.40}$$

Therefore, the estimates (3.38) and (3.40) allow to affirm, there exists $T \in (0, 1)$ such that

$$\|u(T)\|^2_{H^2(\Omega)} + \|\theta(T)\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c''. \tag{3.41}$$

Actually, repeating the above estimates, starting from $t = T$ instead of t=0, we see that (3.41) holds for $T = 1$, i.e. we have the smoothing property

$$\|u(1)\|^2_{H^2(\Omega)} + \|\theta(1)\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c''. \tag{3.42}$$

In particular, having this smoothing property, it is not difficult to prove that we have, owing to (3.15), (3.37) and (3.42), the dissipative estimate

$$\|u(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2 \leqslant e^{-ct} Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|) + c',\ c > 0,\ t \geqslant 0, \tag{3.43}$$

We multiply (2.2) by $-\Delta\frac{\partial\theta}{\partial t}$ and integrate over Ω, we obtain

$$\frac{d}{dt}\|\Delta\theta\|^2 + \left\|\nabla\frac{\partial\theta}{\partial t}\right\|^2 \leqslant \left\|\nabla\frac{\partial u}{\partial t}\right\|^2. \tag{3.44}$$

Thanks to (3.17) and (3.23), we have

$$\int_t^{t+r} \left\|\nabla\frac{\partial u}{\partial t}\right\|^2 ds \leqslant e^{-ct} Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|, r) + c''(r),\ c > 0, t \geqslant t_0 + r.$$

Setting $y = \|\Delta\theta\|^2$, $g = 0$ and $h = \left\|\nabla\frac{\partial u}{\partial t}\right\|^2$, we deduce from (3.44) that

$$y' \leqslant gy + h, t \geqslant t_0,$$

where, owing to the above estimates, y, g and h satisfy the assumptions of the uniform Gronwall's lemma (for $t \geqslant t_0$), which yields that, for $t \geqslant t_0 + r$,

$$\int_t^{t+r} \|\Delta\theta\|^2 ds \leqslant e^{-ct} Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|, r) + c''(r),\ c > 0, t \geqslant r,$$

hence

$$\|\theta\|^2_{H^2(\Omega)} \leqslant e^{-ct} Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|, r) + c''(r),\ c > 0, t \geqslant r. \tag{3.45}$$

Combining (3.43) and (3.45), we obtain the dissipative estimate

$$\|u(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^2(\Omega)} \leqslant e^{-ct} Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}, r) + c''(r),\ c > 0, t \geqslant r. \tag{3.46}$$

From where the

Theorem 3.1. *We assume that* $(u_0, \theta_0) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(\Omega)$. *Then, the system (2.1)-(2.4) possesses at least solution* (u, θ) *such that* $u \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega))$, $\theta \in L^\infty(0, T; L^2(\Omega))$ *and* $\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$, $\forall T > 0$.

The proof of existence is based on the estimates (3.15), (3.43) and a standard Galerkin scheme.

4. The dissipative semigroup

We have the

Theorem 4.1. *We assume that* $(u_0, \theta_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$. *Then, the system (2.1)-(2.4) possesses a unique solution* (u, θ) *such that* $u \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, $\theta \in L^\infty(0, T; L^2(\Omega))$ *and* $\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$, $\forall T > 0$.

Proof. Let now (u_1, θ_1) and (u_2, θ_2) be two solutions to (2.1)-(2.4) with initial data $(u_{1,0}\ ,\ \theta_{1,0})$ and $(u_{2,0}\ ,\ \theta_{2,0}) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, respectively. We set $(u\ ,\ \theta) = (u_1\ ,\ \theta_1) - (u_2\ ,\ \theta_2)$ and $(u_0\ ,\ \theta_0) = (u_{1,0}\ ,\ \theta_{1,0}) - (u_{2,0}\ ,\ \theta_{2,0})$. Then $(u\ ,\ \theta)$ verifies the following problem

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u_1) - f(u_2)) + g(u_1) - g(u_2) = \theta \ \text{ in } \ [0, T] \times \Omega, \tag{4.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} \ \text{ in } \ [0, T] \times \Omega, \tag{4.2}$$

$$\theta = \Delta u = u = 0 \ \text{ on } \ [0, T] \times \Gamma, \tag{4.3}$$

$$\theta(0, x) = \theta_0(x); \ \ u(0, x) = u_0(x), \ \forall x \in \Omega. \tag{4.4}$$

We multiply (4.1) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and (4.2) by $(-\Delta)^{-1}\theta$ integrate over Ω, summing the two resulting equations, we have

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2 + \|\theta\|_{-1}^2) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \|\theta\|^2 \ \leqslant \ \left|\left(f(u_1) - f(u_2), \frac{\partial u}{\partial t})\right)\right| + \left|\left(g(u_1) - g(u_2), (-\Delta)^{-1}\frac{\partial u}{\partial t}\right)\right|. \tag{4.5}$$

We have thanks to lagrange theorem, the following estimate

$$\begin{aligned}
&\left|\left(g(u_1) - g(u_2), (-\Delta)^{-1}\frac{\partial u}{\partial t}\right)\right| \\
&\leqslant \int_\Omega |u| \left|(-\Delta)^{-1}\frac{\partial u}{\partial t}\right| \int_0^1 |g'(su_1 + (1 - s)u_2)| ds dx \\
&\leqslant \alpha \int_\Omega |u| \left|(-\Delta)^{-1}\frac{\partial u}{\partial t}\right| \int_0^1 |2(su_1 + (1 - s)u_2) - 1| ds dx \\
&\leqslant c \int_\Omega (2(|u_1| + |u_2|) + 1)|u| \left|(-\Delta)^{-1}\frac{\partial u}{\partial t}\right| ds dx \\
&\leqslant c(\|u_1\|_{L^4(\Omega)} + \|u_2\|_{L^4(\Omega)} + 1)\|u\|_{L^4(\Omega)} \left\|\frac{\partial u}{\partial t}\right\|_{-1}.
\end{aligned}$$

Noting that $H^1(\Omega) \subset L^4(\Omega)$ with continuous injection and while using (3.43), we have

$$\left|\left(g(u_1) - g(u_2), (-\Delta)^{-1}\frac{\partial u}{\partial t}\right)\right| \leqslant Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)}, \|\theta_{1,0}\|, \|\theta_{2,0}\|)\|\nabla u\|^2 + \frac{1}{4}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2. \tag{4.6}$$

Besides

$$\left|\left(f(u_1) - f(u_2), \frac{\partial u}{\partial t}\right)\right| \leqslant \|\nabla(f(u_1) - f(u_2))\| \left\|\frac{\partial u}{\partial t}\right\|_{-1},$$

and owing to (3.43),

$$
\begin{aligned}
&\|\nabla(f(u_1)-f(u_2))\|\\
&=\left\|\nabla\left(\int_0^1 f^{'}(su_1+(1-s)u_2)dsu\right)\right\|\\
&\leqslant\left\|\int_0^1 f^{'}(su_1+(1-s)u_2)ds\right\|\|\nabla u\|+\left\|\int_0^1 f^{''}(su_1+(1-s)u_2)ds\right\|(\||u|\nabla u_1|\|+\||u|\nabla u_2|\|)\\
&\leqslant Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\theta_{1,0}\|,\|\theta_{2,0}\|)(\|\nabla u\|+\||u|\nabla u_1|\|+\||u|\nabla u_2|\|)\\
&\leqslant Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\theta_{1,0}\|,\|\theta_{2,0}\|)\|\nabla u\|. \qquad (4.7)
\end{aligned}
$$

We insert the estimates (4.6) and (4.7) into (4.5), we find

$$
\frac{d}{dt}(\|\nabla u\|^2+\|\theta\|_{-1}^2)+\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\|\theta\|^2 \leqslant Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\theta_{1,0}\|,\|\theta_{2,0}\|)\|\nabla u\|^2. \qquad (4.8)
$$

Where Q is monotone increasing with respect to both arguments. We deduce from
(4.8) and Gronwall's lemma that

$$
\|u(t)\|_{H^1(\Omega)}^2+\|\theta(t)\|_{-1}^2 \leqslant e^{ct}Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\theta_{1,0}\|,\|\theta_{2,0}\|)(\|u_0\|_{H^1(\Omega)}^2+\|\theta_0\|_{-1}^2),
$$

hence the uniqueness, as well as the continuous depending with respect to the initial data. □

We set $\Psi=(H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega))$. It follows from Theorem 4.1, that we have the continuous (with respect to the $H^1(\Omega)\times H^{-1}(\Omega)$ − norm) of the following semigroup

$$
\begin{aligned}
S(t):\Psi &\longrightarrow \Psi,\\
(u_0,\theta_0) &\longrightarrow (u(t),\theta(t)),
\end{aligned}
$$

(i.e, $S(0)=I,\ S(t)\circ S(s)=S(t+s),\ t,s\geqslant 0$). We then deduce from (3.43) the following theorem.

Theorem 4.2. *The semigroup $S(t)$ is dissipative in Ψ, i.e., there exists a bounded set $\mathcal{B}_0\in\Psi$(called absorbing set) such that, for every bounded $B\in\Psi$, there exists $t_0=t_0(B)\geqslant 0$ such that $t\geqslant t_0$ implies $S(t)B\subset\mathcal{B}_0$.*

Remark 4.1. *It is easy to see that we can assume, without loss of generality, that $\mathcal{B}_0$ is positively invariant by $S(t)$, i.e., $S(t)\mathcal{B}_0\subset\mathcal{B}_0$, $\forall t\geqslant 0$. Furthermore, it follows from (3.46) that $S(t)$ is dissipative in $(H^2(\Omega))^2$ and it follows from (3.45) that we can take $\mathcal{B}_0$ in $(H^2(\Omega))^2$.*

Corollary 4.1. *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ who is bounded in $(H^2(\Omega))^2$ and compact in Ψ.*

The existence of the global attractor being established, one question is to know whether this attractor has a finite dimension in terms of the fractal or Hausdorff dimension. This is the aim of the final section.

5. Existence of exponential attractors

The aim of this section is to prove the existence of exponential attractors for the semigroup $S(t), t \geqslant 0$, associated to the problem (2.1)-(2.4). To do so, we need the semigroup that has to be Lipschitz continuous, satisfying the smoothing property and checking a Hölder continuous with respect to time. This is enough to conclude on the existence of exponential attractors .

Lemma 5.1. *Let (u_1, θ_1) and (u_2, θ_2) be two solutions to (2.1)-(2.4) with initial data $(u_{1,0}\ ,\ \theta_{1,0})$ and $(u_{2,0}\ ,\ \theta_{2,0})$, respectively, belonging to $\mathcal{B}_0$. Then, the corresponding solutions of the problem (2.1)-(2.4) satisfy the following estimate*

$$\|u_1(t) - u_2(t)\|^2_{H^2} + \|\theta_1(t) - \theta_2(t)\|^2 \leqslant ce^{c^{'}t}(\|u_{1,0} - u_{2,0}\|^2_{H^1} + \|\theta_{1,0} - \theta_{2,0}\|^2_{-1}),\ t \geqslant 1, \tag{5.1}$$

where the constants only depend on $\mathcal{B}_0$.

Proof. We set $(u\ ,\ \theta) = (u_1\ ,\ \theta_1) - (u_2\ ,\ \theta_2)$ and $(u_0\ ,\ \theta_0) = (u_{1,0}\ ,\ \theta_{1,0}) - (u_{2,0}\ ,\ \theta_{2,0})$, then $(u\ ,\ \theta)$ satisfies

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u_1) - f(u_2)) + g(u_1) - g(u_2) = \theta \text{ in } [0, T] \times \Omega, \tag{5.2}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} \text{ in } [0, T] \times \Omega, \tag{5.3}$$

$$\theta = \Delta u = u = 0 \text{ on } [0, T] \times \Gamma, \tag{5.4}$$

$$\theta(0, x) = \theta_0(x);\ \ u(0, x) = u_0(x) \text{ in } \Omega. \tag{5.5}$$

We first deduce from (4.8) that

$$\|\nabla u(t)\|^2 + \|\theta(t)\|^2_{-1} \leqslant ce^{c^{'}t}(\|u_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2_{-1}),\ c^{'} > 0,\ t \geqslant 0, \tag{5.6}$$

and

$$\int_0^t \left(\left\|\frac{\partial u(s)}{\partial t}\right\|^2_{-1} + \|\theta(s)\|^2\right)ds \leqslant ce^{c^{'}t}(\|u_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2_{-1}),\ c^{'} > 0,\ t \geqslant 0, \tag{5.7}$$

where the constants only depend on $\mathcal{B}_0$.
We differentiate (5.2) with respect to time and have, owing to (5.3),

$$\begin{aligned}&(-\Delta)^{-1}\frac{\partial \varphi}{\partial t} - \Delta\varphi + f^{'}(u_1)\varphi + (f^{'}(u_1) - f^{'}(u_2))\frac{\partial u_2}{\partial t}\\ &+(-\Delta)^{-1}\left(g^{'}(u_1)\varphi + (g^{'}(u_1) - g^{'}(u_2))\frac{\partial u_2}{\partial t}\right) = -\theta - (-\Delta)^{-1}\varphi,\end{aligned} \tag{5.8}$$

where $\varphi = \dfrac{\partial u}{\partial t}$.
We multiply (5.8) by $(t - T_0)\varphi$ and integrate over Ω, where T_0 is same as in one of previous section, owing to (2.6), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left((t - T_0)\left\|\varphi\right\|^2_{-1}\right) + (t - T_0)\|\nabla\varphi\|^2\\ &\leqslant \frac{1}{2}\|\varphi\|^2_{-1} + c_0(t - T_0)\|\varphi\|^2 + (t - T_0)\|\varphi\|^2_{-1} + (t - T_0)\left|\left((f^{'}(u_1) - f^{'}(u_2))\frac{\partial u_2}{\partial t}, \varphi\right)\right|\end{aligned}$$

$$+(t-T_0)\left|(\theta,\varphi)\right|+(t-T_0)\left|((-\Delta)^{-1}(g^{'}(u_1)\varphi),\varphi)\right|$$
$$+(t-T_0)\left|\left((-\Delta)^{-1}(g^{'}(u_1)-g^{'}(u_2))\frac{\partial u_2}{\partial t},\varphi\right)\right|. \quad (5.9)$$

Noting that $u_1, u_2 \in H^2(\Omega)$, then

$$\begin{aligned}\left|((f^{'}(u_1)-f^{'}(u_2))\frac{\partial u_2}{\partial t},\varphi)\right| &\leqslant \int_\Omega \left|f^{'}(u_1)-f^{'}(u_2)\right|\left|\varphi\right|\left|\frac{\partial u_2}{\partial t}\right|dx\\ &\leqslant \int_\Omega \left|3u_1^2-3u_2^2\right|\left|\varphi\right|\left|\frac{\partial u_2}{\partial t}\right|dx\\ &\leqslant c(\|u_1\|_{L^\infty(\Omega)}+\|u_2\|_{L^\infty(\Omega)})\int_\Omega |u|\left|\varphi\right|\left|\frac{\partial u_2}{\partial t}\right|dx\\ &\leqslant c\int_\Omega |u|\left|\varphi\right|\left|\frac{\partial u_2}{\partial t}\right|dx\\ &\leqslant c\|u\|_{L^4(\Omega)}\left\|\varphi\right\|_{L^4(\Omega)}\left\|\frac{\partial u_2}{\partial t}\right\|\\ &\leqslant c\|\nabla u\|\left\|\nabla\varphi\right\|\left\|\frac{\partial u_2}{\partial t}\right\|, \end{aligned} \quad (5.10)$$

proceeding as in (3.20), we find

$$\left|\left(((-\Delta)^{-1}(g'(u_1))\varphi),\varphi\right)\right| \leqslant c\left\|\nabla\varphi\right\|\left\|\varphi\right\|_{-1}, \quad (5.11)$$

and

$$\begin{aligned}\left|\left((-\Delta)^{-1}((g^{'}(u_1)-g^{'}(u_2))\frac{\partial u_2}{\partial t}),\varphi\right)\right| &\leqslant 2\alpha\int_\Omega \left|(-\Delta)^{-1}\varphi\right||u|\left|\frac{\partial u_2}{\partial t}\right|dx\\ &\leqslant c\|u\|_{L^4(\Omega)}\left\|(-\Delta)^{-1}\varphi\right\|_{L^4(\Omega)}\left\|\frac{\partial u_2}{\partial t}\right\|\\ &\leqslant c\|\nabla u\|\|(-\Delta)^{-1}\varphi\|_{H^1(\Omega)}\left\|\frac{\partial u_2}{\partial t}\right\|\\ &\leqslant c\|\nabla u\|\|\varphi\|_{-1}\left\|\frac{\partial u_2}{\partial t}\right\|\\ &\leqslant c\|\nabla u\|\|\varphi\|\left\|\frac{\partial u_2}{\partial t}\right\|\\ &\leqslant c\|\nabla u\|\|\nabla\varphi\|\left\|\frac{\partial u_2}{\partial t}\right\|, \end{aligned} \quad (5.12)$$

where the constants only depend on $\mathcal{B}_0$.
By substituting (5.10), (5.11) and (5.12) into (5.9), we have, owing to the interpolation inequality,

$$\frac{d}{dt}\left((t-T_0)\left\|\varphi\right\|_{-1}^2\right)+(t-T_0)\left\|\nabla\varphi\right\|^2$$
$$\leqslant c(t-T_0)\left(\left\|\varphi\right\|_{-1}^2+\|\theta\|^2\right)+c(t-T_0)\|\nabla u\|^2\left\|\frac{\partial u_2}{\partial t}\right\|^2+\left\|\varphi\right\|_{-1}^2. \quad (5.13)$$

We now multiply (5.3) by $-(t-T_0)\theta$ and integrate over Ω, we obtain

$$\frac{d}{dt}\left((t-T_0)\|\theta\|^2\right)+(t-T_0)\|\nabla\theta\|^2 \leqslant c(t-T_0)\left\|\varphi\right\|^2+\|\theta\|^2. \tag{5.14}$$

Therefore, noting that it follows from (3.13), (3.15), (3.23) and (3.25) (for $(u,\theta)=(u_2,\theta_2)$) that

$$\int_{T_0}^{t}\left\|\frac{\partial u_2}{\partial t}\right\|^2 ds \leqslant ce^{c^{'}t}, t\geqslant T_0,$$

where the constants only depend on $\mathcal{B}_0$.

Combining (5.13) and (5.14), we find, owing to Gronwall's lemma over (T_0,t); note that $T_0<1$,

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2+\|\theta(t)\|^2 \leqslant ce^{c^{'}t}(\|u_0\|_{H^1(\Omega)}^2+\|\theta_0\|_{-1}^2),\ t\geqslant 1, \tag{5.15}$$

where the constants only depend on $\mathcal{B}_0$.
We rewrite (5.2) in the form

$$-\Delta u=\tilde{h}_u(t), u=0 \text{ sur } \partial\Omega, \tag{5.16}$$

for $t\geqslant 1$ fixed, where

$$\tilde{h}_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t}-(f(u_1)-f(u_2))-(-\Delta)^{-1}(g(u_1)-g(u_2))+(-\Delta)^{-1}\theta. \tag{5.17}$$

We multiply (5.17) by $\tilde{h}_u(t)$ and integrate over Ω, we obtain

$$\|\tilde{h}_u(t)\|^2 \leqslant c\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\|\theta\|^2\right)+c\|\nabla u\|^2.$$

It then follows from (5.6) and (5.15) that

$$\|\tilde{h}_u(t)\|^2 \leqslant ce^{c^{'}t}(\|u_0\|_{H^1(\Omega)}^2+\|\theta_0\|_{-1}^2),\ t\geqslant 1, \tag{5.18}$$

where the constants only depend on $\mathcal{B}_0$.
We multiply (5.16) by $-\Delta u$ and integrate over Ω, we find

$$\|\Delta u\|^2\leqslant\|\tilde{h}_u(t)\|^2,$$

hence, owing to (5.18), we have

$$\|u\|_{H^2(\Omega)}^2\leqslant ce^{c^{'}t}(\|u_0\|_{H^1(\Omega)}^2+\|\theta_0\|_{-1}^2),\ t\geqslant 1, \tag{5.19}$$

where the constants only depend on $\mathcal{B}_0$.
We finally deduce from (5.15) and (5.19), the estimate (5.1) which concludes the proof. □

Lemma 5.2. *Let (u_1,θ_1) and (u_2,θ_2) be two solutions to (2.1)-(2.4) with initial data $(u_{1,0}\ ,\ \theta_{1,0})$ and $(u_{2,0}\ ,\ \theta_{2,0})$, respectively, belonging to $\mathcal{B}_0$. Then, the semigroup $\{S(t)\}_{t\geqslant 0}$ is Lipschitz continuity with respect to space, i.e, there exists the constant $c>0$ such that*

$$\|u_1(t)-u_2(t)\|^2_{H^1(\Omega)}+\|\theta_1(t)-\theta_2(t)\|^2_{-1} \ \leqslant\ ce^{c^{'}t}(\|u_{1,0}-u_{2,0}\|^2_{H^1(\Omega)}+\|\theta_{1,0}-\theta_{2,0}\|^2_{-1}),\ c^{'}>0,\ t\geqslant 0, \tag{5.20}$$

where the constants only depend on $\mathcal{B}_0$.

Proof. The proof of the lemma 5.2 is a direct consequence of the estimate (5.6). □

It just remains to prove the Hölder continuity with respect to time .

Lemma 5.3. *Let (u,θ) be the solution of (5.2)-(5.5) with intial data (u_0,θ_0) in $\mathcal{B}_0$. Then, the semigroup $\{S(t)\}_{t\geqslant 0}$ is Hölder continuous with respect to time ,i.e, there exists the constant $c>0$ such that* $\forall t_1,\ t_2\in[0,T]$

$$\|S(t_1)(u_0,\theta_0)-S(t_2)(u_0,\theta_0)\|_\Psi \ \leqslant\ c|t_1-t_2|^{\frac{1}{2}}, \tag{5.21}$$

where the constants only depends on $\mathcal{B}_0$ and T .

Proof.

$$\begin{aligned}\|S(t_1)(u_0,\theta_0)-S(t_2)(u_0,\theta_0)\|_\Psi &= \|(u(t_1)-u(t_2),\theta(t_1)-\theta(t_2))\|_\Psi\\ &\leqslant \|u(t_1)-u(t_2)\|_{H^1(\Omega)}+\|\theta(t_1)-\theta(t_2)\|_{-1}\\ &\leqslant c(\|\nabla(u(t_1)-u(t_2))\|+\|\theta(t_1)-\theta(t_2)\|_{-1})\\ &\leqslant c\left(\left\|\int_{t_1}^{t_2}\nabla\frac{\partial u}{\partial t}ds\right\|+\left\|\int_{t_1}^{t_2}\frac{\partial\theta}{\partial t}ds\right\|_{-1}\right)\\ &\leqslant c|t_1-t_2|^{\frac{1}{2}}\left|\int_{t_1}^{t_2}\left(\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\left\|\frac{\partial\theta}{\partial t}\right\|^2_{-1}\right)ds\right|^{\frac{1}{2}}.\end{aligned} \tag{5.22}$$

Noting that, thanks to (3.16), (3.23) and (3.25), we have

$$\left|\int_{t_1}^{t_2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 ds\right|\leqslant c, \tag{5.23}$$

where the constant c depends only on $\mathcal{B}_0$ and $T\geqslant T_0$ such that $t_1,\ t_2\in[0,T]$.
Furthermore, multiplying (2.2) by $(-\Delta)^{-1}\dfrac{\partial\theta}{\partial t}$ and integrate over Ω, we obtain

$$\frac{d}{dt}\|\theta\|^2+\left\|\frac{\partial\theta}{\partial t}\right\|^2_{-1}\leqslant c\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}, \tag{5.24}$$

and it follows from (5.23) and (5.24) that

$$\left|\int_{t_1}^{t_2}\left\|\frac{\partial\theta}{\partial t}\right\|^2_{-1} ds\right|\leqslant c, \tag{5.25}$$

where c only depends on $\mathcal{B}_0$ and T such that $t_1,\ t_2\in[0,T]$.
Finally, we obtain thanks to (5.23) and (5.25), the estimate (5.21). Thus, the lemma is proved. □

We finally deduce from Lemma 5.1, Lemma 5.2 and Lemma 5.3 the following result (see,e.g, [9, 10]).

Theorem 5.1. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M} \subset \mathcal{B}_0$, i.e,*

(i) *$\mathcal{M}$ is compact in $H^1(\Omega) \times H^{-1}(\Omega)$;*

(ii) *$\mathcal{M}$ is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geqslant 0$;*

(iii) *$\mathcal{M}$ has finite fractal dimension in $H^1(\Omega) \times H^{-1}(\Omega)$;*

(iv) *$\mathcal{M}$ attracts exponentially fast the bounded subsets of Ψ*

$$\forall B \in \Psi \text{ bounded},\ dist_{H^1(\Omega)\times H^{-1}(\Omega)}(S(t)B, \mathcal{M}) \leqslant Q(\|B\|_\Psi)e^{-ct}, c > 0,\ t \geqslant 0,$$

where the constant c is independent of B and $dist_{H^1(\Omega)\times H^{-1}(\Omega)}$ denotes the Hausdorff semidistance between sets defined by

$$dist_{H^1(\Omega)\times H^{-1}(\Omega)}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_{H^1(\Omega)\times H^{-1}(\Omega)}.$$

Remark 5.1. *Setting $\tilde{\mathcal{M}} = S(1)\mathcal{M}$, we can prove that $\tilde{\mathcal{M}}$ is an exponential attractor for $S(t)$, but now in the topology of Ψ.*
Since $\mathcal{M}$ (or $\tilde{\mathcal{M}}$) is a compact attracting set, we deduce from Theorem 5.1 and standard results (see,e.g, [4, 10]) the

Corollary 5.1. *The semigroup $S(t)$ possesses the finite-dimensional global attractor $\mathcal{A} \subset \mathcal{B}_0$.*

Remark 5.2. *We note that the global attractor $\mathcal{A}$ is the smallest (for inlusion) compact set of the phase space which is invariant by the flow (i.e. $S(t)\mathcal{A} = \mathcal{A}, \forall t \geqslant 0$) and attractors all bounded sets of initial data as time goes to infinity; thus, it appears as a suitable object in view of the study of the asymptotic behaviour of the system. Furthermore, the finite dimensionality means, roughy speaking, that, even though the initial phase space is infinite dimensional, the reduced dynamics is, in some proper sense, finite dimensional and can be described by a finite number of parameters.*

Remark 5.3. *Compared to the global attractor, an exponentiel attractor is expected to be more robust under perturbations. Indeed, the rate of attraction of trajectories to the global attractor may be slow and it is very difficult, if not impossible, to estimate this rate of attraction with respect to the physical parameters of the problem in general. As a consequence, global attractors may change drastically under small perturbations.*

6. Conclusion

This manuscript explains in a clear way, the context of dynamic system with a proliferation term, when the relative solution exists. The existence of exponential attractor, associated to the problem (2.1)-(2.4) that we have proved, allow to assert that the existing solution of the problem (2.1)-(2.4) that we have shown in this work, belongs to the finite-dimensional subset called global attractor, from a certain time.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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