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Research article

S-function associated with fractional derivative and double Dirichlet average

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Abstract: The object of this article is to investigate the double Dirichlet averages of S-functions. Representations of such relations are obtained in terms of fractional derivative. Some interesting special cases are also stated.

Keywords: double Dirichlet average; S-function; fractional derivative; Gamma and Beta function
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1. Introduction and Preliminaries

The so-called Dirichlet average of a function is an integral average of the function with respect to the Dirichlet measure. The concept of Dirichlet average was introduced by Carlson in [2, 3, 5]. It is studied, among others, by Zu Castell [6], Daiya [7], Daiya and Ram [8], Massopust and Forster [21], Neuman [22], Neuman and Van Fleet [23], Saxena et al. [27], and others. A detailed and comprehensive account of various types Dirichlet averages has been given by Carlson in his monograph [4].

In this paper we will investigate the Dirichlet averages of the S-function defined and studied by Saxena and Daiya [25]. Throughout our present paper, we denote by $\mathbb{R}, \mathbb{N}$ and $\mathbb{C}$ the sets of real, natural and complex numbers, respectively.

1.1. The S-function

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1, \cdots, a_p; b_1, \cdots, b_q; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots \left(a_p\right)_n (\gamma)_{n\tau,k}}{(b_1)_n \cdots \left(b_q\right)_n \Gamma_k(n\alpha + \beta)} \frac{x^n}{n!}, \tag{1.1}$$

where, $k \in \mathbb{R}$, $\alpha,\beta,\gamma,\tau \in \mathbb{C}$, $\Re(\alpha) > 0$, a_i $(i = 1,2,\cdots,p)$, b_j $(j = 1,2,\cdots,q)$, $\Re(\alpha) > k\Re(\tau)$ and $p < q + 1$. The Pochhammer symbol $(\lambda)_\mu$ $(\lambda,\mu \in \mathbb{C})$ with $(1)_n = n!$ for $n \in \mathbb{N}$ defined in terms of Gamma function as (see, [25, p. 199])

$$(\lambda)_\mu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 \quad (\mu = 0;\ \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+\mu-1) \ \ (\mu \in \mathbb{N};\ \lambda \in \mathbb{C}) \end{cases}.$$

The k-Pochhammer symbol was introduced by Diaz and Pariguan [9], defined as

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \tag{1.2}$$

$$(x)_{(n+r)q,k} = (x)_{rq,k}(x+qrk)_{nq,k}, \tag{1.3}$$

where $x \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}$.
Let $\gamma \in \mathbb{C}$ and $k, s \in \mathbb{R}$, then the following identity holds true

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1}\Gamma_k\left(\frac{k\gamma}{s}\right), \tag{1.4}$$

and in particular

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right). \tag{1.5}$$

Let $\gamma \in \mathbb{C}$, $k, s \in \mathbb{R}$ and $n \in \mathbb{N}$, then we have following identity:

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq}\left(\frac{k\gamma}{s}\right)_{nq,k}, \tag{1.6}$$

and in particular

$$(\gamma)_{nq,k} = k^{nq}\left(\frac{\gamma}{k}\right)_{nq}. \tag{1.7}$$

1.1.1. Special cases

(i). when $p = q = 0$ in (1.1) it reduces to generalized k-Mittag-Leffler function, defined by Saxena et al. [26].

$$\underset{(0,0)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}(-;-;x) = \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{x^n}{n!} = E^{\gamma,\tau}_{k,\alpha,\beta}(x). \tag{1.8}$$

(ii). For $\tau = q$, (1.1) yields

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,q,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;x\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{nq,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)}\frac{x^n}{n!}, \tag{1.9}$$

where $\Re(\alpha) > kp$.
(iii). If we set $\tau = 1$ in (1.1), then we have

$$\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,1,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;x\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)}\frac{x^n}{n!}$$

$$= \overset{(\alpha,\beta,\gamma,k)}{\underset{(p,q)}{S}} \left(a_1, \cdots, a_p; b_1, \cdots, b_q; x\right), \tag{1.10}$$

where $\Re(\alpha) > kp$.

We will need some more notations in the further exposition. In the sequel, symbol E_{n-1} will denote the Euclidean simplex, defined by

$$E_{n-1} = \left\{(u_1, \cdots, u_{n-1}); u_j \geq 0,\ j = 1, 2, \cdots, n;\ u_1, \cdots, u_{n-1} \leq 1\right\}. \tag{1.11}$$

1.2. Dirichlet average

Carlson [3,4] introduced the concept of connecting elementary functions with higher transcendental functions using averaging technique. The Dirichlet average is a certain kind of integral average with respect to Dirichlet measure, which in Statistics called as beta distribution of several variables.

Let Ω be a convex set in $\mathbb{C}$ and let $z = (z_1, \cdots, z_n) \in \Omega^n$ for $n \geq 2$, and let f be a measurable function on Ω. Then we have

$$F(b, z) = \int_{E_{n-1}} f(u \circ z)\, \mathrm{d}\mu_b(u), \tag{1.12}$$

and

$$u \circ z = \sum_{i=1}^{n-1} u_i z_i + (1 - u_1 - \cdots u_{n-1}) z_n, \tag{1.13}$$

where, $\Gamma(.)$ being the gamma function. In particular, for $n = 1$, $F(b, z) = f(z)$.
Here, $\mathrm{d}\mu_b$ is the Dirichlet measure. Let $b \in C^n$, $n \geq 2$ and $E = E_{n-1}$ be the standard simplex in R^{n-1}, the complex measure μ_b, then Dirichlet measure defined on E, by

$$\mathrm{d}\mu_b(u) = \frac{1}{B(b)} \int_E u_1^{b_1-1} \cdots u_{n-1}^{b_{n-1}-1} (1 - u_1 - \cdots - u_{n-1})^{b_n-1}\, \mathrm{d}u_1 \cdots \mathrm{d}u_{n-1}, \tag{1.14}$$

with the multivariable Beta function

$$B(b) = \frac{\Gamma(b_1) \cdots \Gamma(b_k)}{\Gamma(b_1 + \cdots + b_k)} \quad \left(\Re\left(b_j\right) > 0,\ j = 1, 2, \cdots, k\right). \tag{1.15}$$

For $n = 2$, we have

$$\mathrm{d}\mu_{\beta,\beta'}(u) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} u^{\beta-1} (1 - u)^{\beta'-1}. \tag{1.16}$$

Carlson [3] investigated the average (1.12) for $f(z) = z^k$, $k \in \mathbb{R}$, given as

$$R_k(b, z) = \int_{E_{n-1}} (u \circ z)^k\, \mathrm{d}\mu_b(u). \tag{1.17}$$

If $n = 2$, then we have (see, [3,4])

$$R_k(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 [ux + (1 - u) y]^k\, u^{\beta-1} (1 - u)^{\beta'-1}\, \mathrm{d}u, \tag{1.18}$$

where $\beta, \beta' \in \mathbb{C}; \min[\Re(\beta), \Re(\beta')] > 0;\ x, y \in \mathbb{R}$.
Gupta and Agrawal [11] have shown that the double Dirlchlet average is equivalent to fractional derivative of two variables.

Let z be $akXx$ matrix with complex elements Z_{ij}. Further $u = (u_1, \cdots, u_k)$ and $v = (v_1, \cdots, v_x)$ be an ordered K-tuple and x-tuple of real non-negative weights $\sum u_i = 1$ and $\sum v_j = 1$, respectively.
Here

$$u \circ z \circ v = \sum_{i=1}^{k} \sum_{j=1}^{k} u_i z_{ij} v_j \tag{1.19}$$

If $Z_{i\,j}$ is regarded as a point of the complex plane all these convex combinations are points in the convex hull of $(z_{11} - z_{kx})$ denote by $H(z)$.
Let $b = (b_1, \cdots, b_k)$ be an ordered K-tuple of complex numbers with positive real part $\Re(b) > 0$ and similarly for $\beta = (\beta_1, \cdots, \beta_x)$ then define $\mathrm{d}\mu_b(u)$ and $\mathrm{d}\mu_\beta(v)$.
If $\Re(b) > 0$, $\Re(\beta) > 0, H(z) \subset D$ and f be the holomorphic on a domain D in the complex plane, then we have

$$F(b, z, \beta) = \int \int f(u \circ z \circ v)\, \mathrm{d}\mu_b(u)\, \mathrm{d}\mu_\beta(v). \tag{1.20}$$

Double average for $(k = x = 2)$ of $(u \cdot z \cdot v)^t$ is the $\mathcal{R}$ function is defined by Gupta and Agrawal [11], given by

$$\mathcal{R}(\mu, \mu', z, \rho, \sigma) = \int_0^1 \int_0^1 (u \circ z \circ v)^t\, \mathrm{d}m_{(\mu,\mu')}(\mu)\ \mathrm{d}m_{(\rho,\sigma)}(v), \tag{1.21}$$

where $\Re(\mu) > 0$, $\Re(\mu') > 0$, $\Re(\rho) > 0$, $\Re(\sigma) > 0$, and

$$\begin{aligned} u \circ z \circ v &= \sum_{i=1}^{2} \sum_{j=1}^{2} \left(u_i \circ z_{ij} \circ v_j\right) = \sum_{i=1}^{2} \left[u_i \left(z_{i1} v_1 + z_{i2} v_2\right)\right] \\ &= \left[u_i z_{11} v_i + u_2 z_{21} v_1 + u_1 z_{12} v_2 + u_2 z_{22} v_2\right]. \end{aligned} \tag{1.22}$$

Let $z_{11} = a, z_{12} = b, z_{21} = c, z_{22} = d$ and
$\begin{cases} u_1 = u & u_2 = 1 - u \\ v_1 = v & v_2 = 1 - v \end{cases}$ thus $z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Then we have

$$\begin{aligned} u \circ z \circ v &= uva + ub(1 - v) + (1 - u)cv + (1 - u)d(1 - v) \\ &= \left[uv(a - b - c + d) + u(b - d) + v(c - d) + d\right], \end{aligned} \tag{1.23}$$

and

$$\mathrm{d}m_{\mu,\mu'}(u) = \frac{\Gamma(\mu + \mu')}{\Gamma(\mu)\Gamma(\mu')} u^{\mu-1}(1 - u)^{\mu'-1}\ \mathrm{d}u, \tag{1.24}$$

$$\mathrm{d}m_{\rho,\sigma}(v) = \frac{\Gamma(\rho + \sigma)}{\Gamma(\rho)\Gamma(\sigma)}\ v^{\rho-1}(1 - v)^{\sigma-1}\ \mathrm{d}v. \tag{1.25}$$

Thus

$$\begin{aligned} &\mathcal{R}_t(\mu, \mu', z, \rho, \sigma) \\ &= \frac{\Gamma(\mu + \mu')}{\Gamma(\mu)\Gamma(\mu')} \frac{\Gamma(\rho + \sigma)}{\Gamma(\rho)\Gamma(\sigma)} \int_0^1 \int_0^1 \left[uv(a - b - c + d) + u(b - d) + v(c - d) + d\right]^t \end{aligned}$$

$$\times\, u^{\mu-1}\,(1-u)^{\mu'-1}\; v^{\rho-1}\,(1-v)^{\sigma-1}\; \mathrm{d}u\,\mathrm{d}v. \tag{1.26}$$

For further details of S-function, Dirichlet and Double Dirichlet average reader can refer to the work by Daiya [7], Daiya and Ram [8] and Saxena et al. [27].

1.3. Fractional derivative

The Fractional Calculus is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer (fractional) order. The concept of fractional calculus was introduced by mathematicians to solve problems that could not be handled by a local derivative: the fractional order alpha that appears in the concept of fractional derivative can be used to represent some physical parameters. Riemann and Liouville introduced the fractional derivative as a derivative of the convolution of a given function and the power law function. In recent years numerous works have been dedicated to the fractional calculus of variations. Most of them deal with Riemann-Liouville fractional derivatives (see, [10, 24]), a few with Caputo or Riesz derivatives [1,28]. For more details, reader can refer recent work [12–14,16–18].

For $\Re(\alpha) > 0$, the Riemann-Liouville fractional integral (left-hand sided variants of operators) of order $\alpha \in \mathbb{R}\,(\alpha > 0)$ is given by (see, [10,15,19])

$$I^{\alpha}_{0+}\,(F\,(x)) = \frac{1}{\Gamma\,(\alpha)} \int_0^x \frac{F\,(t)}{(x-t)^{1-\alpha}}\,\mathrm{d}t = x^{\alpha} \int_0^1 \frac{(1-\rho)^{\alpha-1}}{\Gamma\,(\alpha)} F\,(x\rho)\;\mathrm{d}\rho. \tag{1.27}$$

The Riemann-Liouville fractional differintegral operator D^{α}_{x} of order $\alpha(\alpha \in \mathbb{C})$ is defined as (see, for details, [10, Chapter 13] and [20, Page 2470, Eqn.(1)])

$$D^{\alpha}_{x}\,(F\,(x)) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1}\, F(t)\,\mathrm{d}t & (\Re(\alpha) < 0) \\ \frac{d^n}{dx^n}\left\{D^{\alpha-n}_{x}\,(f(x))\right\} & (n-1 \leq \Re(\alpha) < n;\; n \in \mathbb{N}) \end{cases}, \tag{1.28}$$

provided that the defining integral in (1.28) exists, and

$$D^{\alpha}_{x-x_0}\,(F\,(x)) = \frac{1}{\Gamma\,(-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha-1}\, F(t)\,\mathrm{d}t \qquad (\Re(\alpha) < 0)\,, \tag{1.29}$$

where $F(x)$ is of the form $x^p f(x)$ and $f(x)$ is analytic at $x = 0$.
It readily follows from (1.28) that

$$D^{\alpha}_{x}\left\{x^{\lambda}\right\} = \frac{\Gamma\,(\lambda+1)}{\Gamma\,(\lambda-\alpha+1)}\, x^{\lambda-\alpha} \qquad (\alpha > 0,\; \Re(\lambda) > -1)\,. \tag{1.30}$$

The Beta function is given by

$$B\,(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,\mathrm{d}x, \tag{1.31}$$

we also have the relationship between Gamma and Beta function, as follows

$$B\,(m,n) = \frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)} \quad (m,n > 0)\,. \tag{1.32}$$

2. Main results

Theorem 2.1. *Let $k \in \mathbb{R}$; $\alpha,\beta,\gamma \in \mathbb{C}$; $\Re(\alpha) > 0$ and $\tau \in \mathbb{C}$, then double Dirichlet average is established of the function* $\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;\ (u\circ z\circ v)\right)$ *for* $(k=x=2)$*, is given by*

$$J_n(\mu_1,\mu_2;z;\rho_1\rho_2) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\ (x-y)^{1-\rho_1-\rho_2} \times\ D_{x-y}^{-\rho_2}\left(t^{\rho_1-1}\right)\left\{\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right)\right\}(x-y). \tag{2.1}$$

Proof. By using equation (1.21) and (1.23), we have

$$\begin{aligned}&J_n(\mu_1,\mu_2;z;\rho_1,\rho_2)\\ &=\int_0^1\int_0^1 \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;(u\circ z\circ v)\right)\mathrm{d}m_{\mu_1\mu_2}(u)\ \mathrm{d}m_{\rho_1\rho_2}(v)\\ &=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}\int_0^1\int_0^1[u\circ z\circ v]^n\,\mathrm{d}m_{\mu_1\mu_2}(u)\ \mathrm{d}m_{\rho_1\rho_2}(v),\end{aligned}$$

where $\Re(\mu_1)>0$ $\Re(\mu_2)>0$, $\Re(\rho_1)>0$ and $\Re(\rho_2)>0$.

$$\begin{aligned}J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) &= \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} - \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}\\ &\times\int_0^1\int_0^1[uv(a-b-c+d)+u(b-d)+v(c-d)+d]^n\\ &\times\ u^{\mu_1-1}(1-u)^{\mu_2-1}\,v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

To obtain the fractional derivative, we assume $a=c=x$, $b=d=y$, then we have

$$\begin{aligned}J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) &= \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}\\ &\times\int_0^1\int_0^1[y+v(x-y)]^n\,u^{\mu_1-1}(1-u)^{\mu_2-1}\,v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Next, by using definition of Beta function (1.31) and relation (1.32), we arrive at

$$\begin{aligned}J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) &= \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}\\ &\times\int_0^1[y+v(x-y)]^n\,v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{d}v.\end{aligned} \tag{2.2}$$

By putting $v(x-y)=t$, we have

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}$$

$$\times \int_0^{x-y} (y+t)^n \left(\frac{t}{x-y}\right)^{\rho_1-1} \left(1-\frac{t}{x-y}\right)^{\rho_2-1} \frac{\mathrm{d}t}{x-y}$$

$$= \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} (x-y)^{1-\rho_1-\rho_2} \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots \left(a_p\right)_n (\gamma)_{n\tau,k}}{(b_1)_n \cdots \left(b_q\right)_n \Gamma_k(n\alpha+\beta)\, n!}$$
$$\times \int_0^{x-y} (y+t)^n\, t^{\rho_1-1} (x-y-t)^{\rho_2-1}\, \mathrm{d}t$$

$$= \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} (x-y)^{1-\rho_1-\rho_2}$$
$$\times \int_0^{x-y} \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} \left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right) t^{\rho_1-1} (x-y-t)^{\rho_2-1}\, \mathrm{d}t.$$

By using definition of fractional derivative (1.28), we get

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2}$$
$$\times\ D_{x-y}^{-\rho_2}\left(t^{\rho_1-1}\right)\left\{\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right)\right\}(x-y).$$

This completes the proof of Theorem 2.1. □

Corollary 2.1. *Put $\tau = q$ in* (2.1)*, then it reduces in the following from:*

$$J_n(\mu_1,\mu_2;z;\rho_1\rho_2) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2}$$
$$\times\ D_{x-y}^{-\rho_2}\left(t^{\rho_1-1}\right)\left\{\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,q,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right)\right\}(x-y). \tag{2.3}$$

Corollary 2.2. *Put $\tau = 1$ in Theorem 2.1, then* (2.1) *reduces in the following from:*

$$J_n(\mu_1,\mu_2;z;\rho_1\rho_2) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2}$$
$$\times\ D_{x-y}^{-\rho_2}\left(t^{\rho_1-1}\right)\left\{\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,1,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right)\right\}(x-y). \tag{2.4}$$

Theorem 2.2. *Let $k \in \mathbb{R}; \alpha,\beta,\gamma \in \mathbb{C};\ \Re(\alpha) > 0$ and $\tau \in \mathbb{C}$, then we have*

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n\,\Gamma(\rho_1+\rho_2)}{(\mu_1+\mu_2)_n\,\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2}$$
$$\times\ D_{x-y}^{-\rho_2}\left(t^{\rho_1-1}\right)\left\{\underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right)\right\}(x-y). \tag{2.5}$$

Proof. By using equation (1.21), (1.23), (1.24) and (1.25), we have

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}$$
$$\times\int_0^1\int_0^1[uv(a-b-c+d)+u(b-d)+v(c-d)+d]^n$$
$$\times\ u^{\mu_1-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{d}u\,\mathrm{d}v.$$

By setting $a=x$; $b=y$ and $c=d=0$, then we have

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}$$
$$\times\int_0^1\int_0^1[uv(x-y)+uy]^n\,u^{\mu_1-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{d}u\,\mathrm{d}v,$$

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}$$
$$\times\int_0^1\int_0^1[vx+(1-v)y]^n\,u^{\mu_1+n-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{d}u\,\mathrm{d}v.$$

By using definition of Beta function (1.31) and equation (1.32), we have

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\frac{\Gamma(\mu_1+n)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2+n)}$$
$$\times\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}\int_0^1[vx+(1-v)y]^n\,v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{d}v,$$

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n\Gamma(\rho_1+\rho_2)}{(\mu_1+\mu_2)_n\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}$$
$$\times\int_0^1[v(x-y)+y]^n\,v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{d}v.$$

Next, we put $v(x-y)=t$, then we arrive at

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n\Gamma(\rho_1+\rho_2)}{(\mu_1+\mu_2)_n\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots\left(a_p\right)_n(\gamma)_{n\tau,k}}{(b_1)_n\cdots\left(b_q\right)_n\Gamma_k(n\alpha+\beta)\,n!}$$
$$\times\int_0^{x-y}[t+y]^n\left(\frac{t}{x-y}\right)^{\rho_1-1}\left(1-\frac{t}{x-y}\right)^{\rho_2-1}\frac{\mathrm{d}v}{x-y}.$$

Now, by using definition of fractional derivatives, we have

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n \Gamma(\rho_1+\rho_2)}{(\mu_1+\mu_2)_n \Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2}$$
$$\times\ D_{x-y}^{-\rho_2}\left(t^{\rho_1-1}\right)\left\{ \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,\tau,k)}{S}} \left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right)\right\}(x-y).$$

This completes the proof of Theorem 2.2. □

Corollary 2.3. *If we put* $\tau = q$ *in* (2.5), *then it reduces in the following from:*

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n \Gamma(\rho_1+\rho_2)}{(\mu_1+\mu_2)_n \Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2}$$
$$\times\ D_{x-y}^{-\rho_2}\left(t^{\rho_1-1}\right)\left\{ \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,q,k)}{S}} \left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right)\right\}(x-y). \tag{2.6}$$

Corollary 2.4. *If we put* $\tau = 1$ *in theorem 2.2, then* (2.5) *reduces in the following from:*

$$J_n(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n \Gamma(\rho_1+\rho_2)}{(\mu_1+\mu_2)_n \Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2}$$
$$\times\ D_{x-y}^{-\rho_2}\left(t^{\rho_1-1}\right)\left\{ \underset{(p,q)}{\overset{(\alpha,\beta,\gamma,1,k)}{S}} \left(a_1,\cdots,a_p;b_1,\cdots,b_q;(y+t)\right)\right\}(x-y). \tag{2.7}$$

3. Concluding remarks

In this paper, we study the double Dirichlet averages of S-functions [25]. Representations of such relations are obtained in terms of Riemann-Liouville fractional differintegral. The present work shows that every analytic function can be measured as double Dirichlet average by using fractional differintegral operator. Also, the relation between double Dirichlet average of any analytic function and fractional differintegral can be converted into single Dirichlet average of those functions by using fractional differintegrals of the functions. The obtained results can be used for further study in double Dirichlet average of any analytic function.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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