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Research article

Ulam-Hyers stabilities of fractional functional differential equations

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Abstract: From the first results on Ulam-Hyers stability, what has been noted is the exponential growth of the researchers dedicated to investigating Ulam-Hyers stability of fractional differential equation solutions whether they are functional, evolution, impulsive, among others. However, some issues and problems still need to be addressed. An intensifying problem is the small amount of work on Ulam-Hyers stability of solutions of fractional functional differential equations through more general fractional operators. In this sense, in this paper, we present a study on the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the solution of the fractional functional differential equation using the Banach fixed point theorem.

Keywords: ψ-Hilfer fractional derivative; Ulam-Hyers stability; Ulam-Hyers-Rassias stability; fractional functional differential equations; Banach fixed point theorem
Mathematics Subject Classification: 26A33, 34A08, 34K37, 34K20

1. Introduction

From an exchange of questions and answers between Ulam and Hyers, the research on the stability of solutions of functional differential equations was started several years ago [12, 44]. More precisely, Ulam raised the following question: Let H_1 and H_2 be a group and a metric group endowed with the metric $d(\cdot,\cdot)$, respectively. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if the function $f : H_1 \to H_2$ satisfies the following inequality $d(f(x,y), f(x)f(y)) < \delta$, $\forall x, y \in H_1$, then there exists a homeomorphism $F : H_1 \to H_2$ with $d(f(x), F(x)) < \varepsilon$, $\forall x \in H_1$? And so Hyers, presents his answer, in the case where H_1 and H_2 are Banach spaces [44]. Since then, a significant number of researchers have devoted themselves to developing their research which address stability and many important works have been published not only on functional differential equations, but also other types of equations [15, 16, 17, 18, 32].

On the other hand, with the expansion of the fractional calculus and the number of researchers investigating more and more problems involving the stability of solutions of fractional functional differential equations, specially in Banach spaces, this field started to gain more attention [2, 3, 4, 39]. In addition, not only stability has been the subject of study, but investigating the existence and uniqueness, as well as the controllability of solutions of fractional differential equations, has called, and still call, a lot of attention over the years [5, 11, 14, 20, 21, 22, 23, 26, 30].

In 2012 Zhao et al. [49], investigated the existence of positive solutions of the fractional functional differential equation introduced by means of the Caputo fractional derivative and using the Krasnoselskii fixed point theorem. In this paper, the results obtained on the existence of positive solutions for the fractional functional differential equation improves and generalize the existing results. There are numerous works on the existence and uniqueness of fractional functional differential equations, both locally and globally in the Hilbert, Banach and Fréchet spaces. For better reading we suggest the works [7, 9, 25, 47].

In the middle of 2017, Abbas et al. [4], investigated the existence of Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the random solution of the fractional functional differential equation of the Hilfer and Hilfer-Hadamard type by means of fixed point theorems. Abbas et al. [2], also investigated the Ulam stability of functional partial differential equations through Picard's operator theory and provided some examples. Further work on stability of fractional functional differential equations and even functional integral equation can be found in the following works [1, 8, 13, 45]. The stability study is broad and there are other types of stability in which we will not discuss in this paper, but in the paper of Stamova and Stamov [40], they perform a system stability analysis of fractional functional differential equations using the Lyapunov method and the principle of comparison.

In addition, it is worth mentioning the work done in 2019 by Liu et al., on Ulam–Hyers–Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations using Picard's method and Gronwall inequality [27]. On the other hand, the theory of fractional differential equations with almost sectorial operators has been investigated over the years and results on the existence, uniqueness, stability and attractivity of mild solutions are the subject of study by many researchers [6, 50, 51]. This shows that research in the field of fractional differential equations over the years has been construed and some important and were interesting results are obtained [24, 29, 35, 36]. Another significant result in this field of study, is the investigation of approximate controllability of mild solutions in Banach space using the Banach principle technique [28, 48].

Since the theory about the Ulam-Hyers stability of functional differential equations is in wide growth, and the number of papers on this matter is, in our opinion, still small, one of the objectives for the realization of this paper is to provide an investigation of the fractional differential equation Eq. (1.1), in order to be a good research material in this matter.

Consider the delay fractional differential equation of the form

$$ {}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} y(t) = \mathcal{F}(t, y(t), y(t-a)) \tag{1.1} $$

where ${}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative with $0 < \upsilon \leq 1$, $0 \leq \zeta \leq 1$, $\mathcal{F} : \mathbb{R}^3 \to \mathbb{R}$ is a bounded and continuous function, $a > 0$ is a real constant and $t > a$.

Motivated by the works [31, 41, 42], in this paper, we have as main purpose to investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the fractional functional differential equation Eq. (1.1) by means of Banach fixed point theorem.

This paper is organized as follows: in Section 2, we present as preliminaries the continuous functions and the weighted function space, in order to introduce the ψ- Riemann-Liouville fractional integral and the ψ-Hilfer fractional derivative. In this sense, we present the concepts of Ulam-Hyers and Ulam-Hyers-Rassias stabilities, as well as Banach fixed point theorem, which is fundamental for obtaining the main results. In Section 3, we present the first result of this paper, the Ulam-Hyers-Rassias stability by means of Banach fixed point theorem. In Section 4, again by means of Banach fixed point theorem, we present the second result of this paper, the Ulam-Hyers stability. Concluding remarks close the paper.

2. Preliminaries

In this section we present some important concepts that will be useful to write our mains results. First, we present the definitions of the ψ-Riemann-Liouville fractional integral and the ψ-Hilfer fractional derivative. In this sense, we present the Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stabilities concepts for the ψ-Hilfer fractional derivative. We conclude the section with the Banach fixed point theorem, an important result to obtain the stability of the fractional functional differential equation.

Let $[a,b]$ $(0 < a < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$ and $C[a,b]$, $AC^n[a,b]$, $C^n[a,b]$ be the spaces of continuous functions, n-times absolutely continuous functions, n-times continuously differentiable functions on $[a,b]$, respectively.

The space of the continuous functions f on $[a,b]$ with the usual norm is defined by [33]

$$\|f\|_{C[a,b]} = \max_{t\in[a,b]} |f(t)|.$$

On the order hand, we have n-times absolutely continuous functions given by

$$AC^n[a,b] = \left\{f : [a,b] \to \mathbb{R};\; f^{(n-1)} \in AC([a,b])\right\}.$$

The weighted space $C_{\gamma,\psi}[a,b]$ of functions f on $(a,b]$ is defined by [33]

$$C_{\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R};\; (\psi(t) - \psi(a))^{\gamma} f(t) \in C[a,b]\},\; 0 \le \gamma < 1$$

with the norm

$$\|f\|_{C_{\gamma;\psi}[a,b]} = \|(\psi(t) - \psi(a))^{\gamma} f(t)\|_{C[a,b]} = \max_{t\in[a,b]} |(\psi(t) - \psi(a))^{\gamma} f(t)|.$$

The weighted space $C^n_{\gamma;\psi}[a,b]$ of function f on $(a,b]$ is defined by [33]

$$C^n_{\gamma;\psi}[a,b] = \left\{f : (a,b] \to \mathbb{R};\; f(t) \in C^{n-1}[a,b];\; f^{(n)}(t) \in C_{\gamma;\psi}[a,b]\right\},\; 0 \le \gamma < 1$$

with the norm

$$\|f\|_{C^n_{\gamma;\psi}[a,b]} = \sum_{k=0}^{n-1} \left\|f^{(k)}\right\|_{C[a,b]} + \left\|f^{(n)}\right\|_{C_{\gamma;\psi}[a,b]}.$$

For $n = 0$, we have, $C^0_{\gamma,\psi}[a,b] = C_{\gamma,\psi}[a,b]$.

The weighted space $C^{\upsilon,\zeta}_{\gamma,\psi}[a,b]$ is defined by

$$C^{\upsilon,\zeta}_{\gamma;\psi}[a,b]=\left\{f\in C_{\gamma;\psi}[a,b];\ {}^{\mathbf{H}}\mathrm{D}^{\upsilon,\zeta;\psi}_{a+}f\in C_{\gamma;\psi}[a,b]\right\},\ \gamma=\upsilon+\zeta(1-\upsilon).$$

Let $\upsilon>0$, $[a,b]$ a interval $(\infty\leq a<b\leq\infty)$ and $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t)$ on $[a,b]$. The Riemann-Liouville fractional integral with respect to another function ψ on $[a,b]$ is defined by [33, 34]

$$I^{\upsilon;\psi}_{t_0+}y(t):=\frac{1}{\Gamma(\upsilon)}\int_{t_0}^{t}\mathcal{N}^{\upsilon}_{\psi}(t,s)y(s)\,ds \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function with $0<\upsilon\leq 1$ and $\mathcal{N}^{\upsilon}_{\psi}(t,s):=\psi'(s)(\psi(t)-\psi(s))^{\upsilon-1}$. The ψ-Riemann-Liouville fractional integral on the left is defined in an analogous way.

On the other hand, let $n-1<\upsilon\leq n$ with $n\in\mathbb{N}$, $J=[a,b]$ be an interval such that $-\infty\leq a<b\leq+\infty$ and let $f,\psi\in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t)\neq 0$, for all $t\in J$. The ψ-Hilfer fractional derivative is given by [33, 34]

$${}^{\mathbf{H}}\mathrm{D}^{\upsilon,\zeta;\psi}_{t_0+}y(t)=I^{\zeta(n-\upsilon);\psi}_{t_0+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}I^{(1-\zeta)(n-\upsilon);\psi}_{t_0+}y(t).$$

The ψ-Hilfer fractional derivative on the left is defined in an analogous way.

Let X be a nonempty set. A function $d:X\times X\rightarrow[0,\infty]$ is called generalized metric on X if, and only if, d satisfies [43]:

1. $d(x,y)=0$ if $x=y$;
2. $d(x,y)=d(y,x)$, for all $x,y\in X$;
3. $d(x,z)\leq d(x,y)+d(y,z)$, for all $x,y,z\in X$.

For the study of Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stabilities, we will adapt such definitions [31, 41, 43].

Definition 1. *Let $\varepsilon\geq 0$, $\Phi\in C_{1-\gamma;\psi}[t_0-a,t_0]$ and $t_0,T\in\mathbb{R}$ with $T>t_0$. Assume that for any continuous function $f:[t_0-a,T]\rightarrow\mathbb{R}$ satisfying*

$$\begin{cases}\left|{}^{\mathbf{H}}\mathrm{D}^{\upsilon,\zeta;\psi}_{t_0+}f(t)\mathcal{F}(t,f(t),f(t-a))\right| & < \ \varepsilon,\ t\in[t_0,T]\\ |f(t)-\Phi(t)| & < \ \varepsilon,\ t\in[t_0-a,t_0],\end{cases}$$

exists a continuous function $f_0:[t_0-a,T]\rightarrow\mathbb{R}$ satisfying:

$$\begin{cases}{}^{\mathbf{H}}\mathrm{D}^{\upsilon,\zeta;\psi}_{t_0+}f_0(t) & = \ \mathcal{F}(t,f_0(t),f_0(t-a)),\ t\in[t_0,T]\\ f_0(t) & = \ \Phi(t),\ t\in[t_0-a,t_0]\end{cases}$$

and

$$|f(t)-f_0(t)|\leq\mathcal{K}(\varepsilon),\ t\in[t_0-a,T]$$

where $\mathcal{K}(\varepsilon)$ in dependant of ε only. Then, we say that the solution of Eq. (1.1) is Ulam-Hyers stable.

Definition 2. *Let* $\varepsilon \geq 0$*,* $\Phi \in C_{1-\gamma;\psi}[t_0 - a, t_0]$ *and* $t_0, T \in \mathbb{R}$ *with* $T > t_0$*. Assume that for any continuous function* $f : [t_0 - a, T] \to \mathbb{R}$ *satisfying*

$$\begin{cases} \left|{}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} f(t) - \mathcal{F}(t, f(t), f(t-a))\right| & < \varphi, \ t \in [t_0, T] \\ |f(t) - \Phi(t)| & < \varphi, \ t \in [t_0 - a, t_0]; \end{cases}$$

exists a continuous function $f_0 : [t_0 - a, T] \to \mathbb{R}$ *satisfying*

$$\begin{cases} {}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} f_0(t) & = \mathcal{F}(t, f_0(t), f_0(t-a)), \ t \in [t_0, T] \\ f_0(t) & = \Phi(t), \ t \in [t_0 - a, t_0] \end{cases}$$

and

$$|f(t) - f_0(t)| \leq \Phi_1, \ t \in [t_0 - a, T]$$

where Φ_1 *is a function not depending on* f *and* f_0 *explicitly. Then, we say that the solution of* Eq. (1.1) is the Ulam-Hyers-Rassias stable.

Definition 3. Eq.(1.1) is said to be generalized Ulam-Hyers-Rassias stable with respect to ϕ if there exists $c_\phi > 0$ such that for each solution $y \in C^1_{1-\gamma;\psi}([t_0 - a, T], \mathbb{R})$ to

$$\left|{}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} y(t) - \mathcal{F}(t, y(t), y(t-a))\right| \leq \phi(t), \ t \in [t_0 - a, T]$$

there exists a solution $x \in C^1_{1-\gamma;\psi}([t_0 - a, T], \mathbb{R})$ of Eq.(1.1) with

$$|y(t) - x(t)| \leq c_\phi \phi(t), \ t \in [t_0 - a, T].$$

The following is the result of the Banach fixed point theorem, however its proof will be omitted.

Theorem 1. [10] Let (X, d) be a generalized complete metric space. Assume that $\Omega : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d\left(\Omega^{k+1}x, \Omega^k x\right) < \infty$ for some $x \in X$, then the following are true:

1. The sequence $\{\Omega^n x\}$ converges to a fixed x^* of Ω;
2. x^* is the unique fixed point of Ω in

$$X^* = \left\{y \in X : d\left(\Omega^k x, y\right) < \infty\right\}.$$

3. If $y \in X^*$, then

$$d(y, X^*) \leq \frac{1}{1-L} d(\Omega y, y).$$

3. Ulam-Hyers-Rassias stability

By means of the Banach fixed point theorem, in this section we present the first result of this paper, the Ulam-Hyers-Rassias stability for the delay fractional differential equation, Eq. (1.1).

Theorem 2. *Consider the interval* $I = [t_0 - a, T]$ *and suppose that* $\mathcal{F} : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function with the following Lipschitz condition:*

$$|\mathcal{F}(t, x, y) - \mathcal{F}(t, z, w)| \leq \mathcal{L}_1 |x - z| + \mathcal{L}_2 |y - w|$$

for all $(t, x, y), (t, z, w) \in I \times \mathbb{R} \times \mathbb{R}$.

Let $\phi : I \to (0, \infty)$ *be a continuous function. Assume that* $\Phi \in C_{1-\gamma;\psi}[t_0 - a, t_0]$*,* $\mathcal{K}, \mathcal{L}_1$ *and* $\mathcal{L}_2$ *are positive constants with*

$$0 < \mathcal{K}(\mathcal{L}_1 + \mathcal{L}_2) < 1$$

and

$$\left|\frac{1}{\Gamma(\upsilon)} \int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,u)\phi(u)\,du\right| \leq \mathcal{K}\phi(t),$$

for all $t \in I = [t_0 - a, T]$.

Then, if a continuous function $y : I \to \mathbb{R}$ *and* $\varphi : I \to (0, \infty)$ *satisfies*

$$\begin{cases} \left|{}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} y(t) - \mathcal{F}(t, y(t), y(t-a))\right| & < \varphi(t), \ t \in [t_0, T] \\ |y(t) - \Phi(t)| & < \varphi(t), \ t \in [t_0 - a, t_0] \end{cases}$$

then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ *such that*

$$\begin{cases} {}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} y_0(t) & = \mathcal{F}(t, y_0(t), y_0(t-a)), \ t \in [t_0, \ T] \\ y_0(t) & = \Phi(t), \ t \in [t_0 - a, t_0] \end{cases} \tag{3.1}$$

and

$$|y(t) - y_0(t)| \leq \frac{1}{1 - \mathcal{K}(\mathcal{L}_1 + \mathcal{L}_2)} \mathcal{K}\phi(t), \text{ for all } t \in I. \tag{3.2}$$

Proof. For the proof of this theorem, first consider the set S given by

$$S = \left\{\varphi : I \to \mathbb{R} : \varphi \in C_{1-\gamma;\psi}, \ \varphi(t) = \Phi(t), \text{ if } t \in [t_0 - a, t_0]\right\}$$

and the following generalized metric over S

$$d(\varphi, \mu) = \inf\{M \in [0, \infty) : |\varphi(t) - \mu(t)| \leq M\phi(t), \ \forall t \in I\}. \tag{3.3}$$

Note that, (S, d) is a generalized complete metric space. Now, we introduce the following function $\Omega : S \to S$ given by

$$\begin{cases} (\Omega\varphi)(t) & = \Phi(t), \ t \in [t_0 - a, t_0] \\ (\Omega\varphi)(t) & = \Phi(t_0)\Psi^{\gamma}(t, t_0) + \dfrac{1}{\Gamma(\upsilon)} \displaystyle\int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,s)\mathcal{F}(u, \varphi(u), \varphi(u-a))\,du, \ t \in [t_0, T], \end{cases} \tag{3.4}$$

where $\Psi^{\gamma}(t, t_0) := \dfrac{(\psi(t) - \psi(t_0))^{1-\gamma}}{\Gamma(\gamma)}$, with $\gamma = \upsilon + \zeta(1 - \upsilon)$. Note that, for $\varphi \in S$, the function $\Omega\varphi$ is continuous. In this way, we can write $\Omega\varphi \in S$. Let $\varphi, \mu \in S$ and by Eq.(3.4), we have

$$\begin{aligned} & |(\Omega\varphi)(t) - (\Omega\mu)(t)| \\ \leq & \left|\frac{1}{\Gamma(\upsilon)} \int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,u)\left(\mathcal{F}(u, \varphi(u), \varphi(u-a)) - \mathcal{F}(u, \mu(u), \mu(u-a))\right)du\right| \\ \leq & \frac{1}{\Gamma(\upsilon)} \int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,u)\left(\mathcal{L}_1|\varphi(u) - \mu(u)| - \mathcal{L}_2|\varphi(u-a) - \mu(u-a)|\right)du \end{aligned}$$

$$\begin{aligned}
&\leq \frac{M\mathcal{L}_1}{\Gamma(\upsilon)}\int_{t_0}^{t}\mathcal{N}_{\psi}^{\upsilon}(t,u)\phi(u)\,du+\frac{M\mathcal{L}_2}{\Gamma(\upsilon)}\int_{t_0}^{t}\mathcal{N}_{\psi}^{\upsilon}(t,u)\phi(u)\,du\\
&\leq \frac{M(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(\upsilon)}\left|\int_{t_0}^{t}\mathcal{N}_{\psi}^{\upsilon}(t,u)\phi(u)\,du\right|\\
&\leq M\mathcal{K}(\mathcal{L}_1+\mathcal{L}_2)\phi(t),\ t\in[t_0,T]
\end{aligned}$$

and

$$|(\Omega\varphi)(t)-(\Omega\mu)(t)|=\Phi(t)-\Phi(t)=0,\ t\in[t_0-a,t]$$

which implies that $d(\Omega\varphi-\Omega\mu)\leq\mathcal{K}(\mathcal{L}_1+\mathcal{L}_2)\,d(\varphi,\mu)$. Since $0<\mathcal{K}(\mathcal{L}_1+\mathcal{L}_2)<1$, then Ω is strictly contractive on S.

Let $\xi\in S$ arbitrary and $\min_{t\in I}\phi(t)>0$. As $\mathcal{F}(t,\xi(t),\xi(t-a))$ and $\xi(t)$ are bounded on I, then there exists a constant $0<M<\infty$ such that

$$\begin{aligned}
|(\Omega\xi)(t)-\xi(t)| &= \left|\Psi^{\gamma}(t,t_0)\Phi(t_0)+\frac{1}{\Gamma(\upsilon)}\int_{t_0}^{t}\mathcal{N}_{\psi}^{\upsilon}(t,u)\mathcal{F}(u,\xi(u),\xi(u-a))\,du-\xi(t)\right|\\
&\leq M\varphi(t).
\end{aligned}\tag{3.5}$$

Thus, by means of Eq. (3.5), it follows that $d(\Omega\xi,\xi)<\infty$. By means of the Theorem 1 (1), there exists a continuous function $y_0 : I\to\mathbb{R}$ such that $\Omega^n\xi\to y_0$ in (S,d) and $\Omega y_0=y_0$, then y_0 satisfies

$$\begin{cases}
{}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi}y_0(t) = \mathcal{F}(t,y_0(t),y_0(t-a)),\ t\in[t_0,T]\\
y_0(t) = \Phi(t),\ t\in[t_0-a,t_0].
\end{cases}$$

Now consider for any $g\in S$, such that g and ξ are bounded on I, then exist a constant $0<M_g<\infty$ such that

$$|\xi(t)-g(t)|\leq M_g\varphi(t)$$

for $t\in I$.

Thus, we can write $\forall g\in S$, $d(\xi,g)<\infty$ with $S=\{g\in S; d(\xi,g)<\infty\}$. Furthermore, it is clear that

$$-\phi(t)\leq {}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi}y(t)-\mathcal{F}(t,y(t),y(t-a))\leq\phi(t),\ \forall t\in[t_0,T].\tag{3.6}$$

Applying the fractional integral $I_{t_0}^{\upsilon;\psi}(\cdot)$ on both sides of Eq.(3.6), we have

$$\begin{aligned}
&\left|y(t)-\Psi^{\gamma}(t,t_0)\Phi(t_0)-\frac{1}{\Gamma(\upsilon)}\int_{t_0}^{t}\mathcal{N}_{\psi}^{\upsilon}(t,u)\mathcal{F}(u,y(u),y(u-a))\,du\right|\\
&\leq\left|\frac{1}{\Gamma(\upsilon)}\int_{t_0}^{t}\mathcal{N}_{\psi}^{\upsilon}(t,u)\phi(u)\,du\right|\leq\mathcal{K}\phi(t),\ t\in[t_0,T].
\end{aligned}$$

This form, by definition Ω, finishes

$$|y(t)-(\Omega y)(t)|\leq\mathcal{K}\phi(t),\ t\in I.$$

Consequently, it implies that $d(y,\Omega y)\leq\mathcal{K}$. By means of Theorem 1 (3) and the last estimative, we have

$$d(y,y_0)\leq\frac{1}{1-\mathcal{K}(\mathcal{L}_1+\mathcal{L}_2)}d(\Omega y,y)\leq\frac{\mathcal{K}\phi(t)}{1-\mathcal{K}(\mathcal{L}_1+\mathcal{L}_2)},\ \forall t\in I.$$

Thus, by Theorem 1 (2), we conclude that there exists y_0, the unique continuous function with the property Eq. (3.1). □

Remark 1. *One of the advantages of working with Ulam-Hyers and Ulam-Hyers-Rassias stabilities, or any other type of stability with the global fractional differential operator so-called ψ-Hilfer, is that the results obtained in this way, are also valid for their respective individual cases.*

4. Ulam-Hyers stability

In this section, we investigate the second main result of the paper, the Ulam-Hyers stability, again making use of the Banach fixed point theorem.

Theorem 3. *Suppose that $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the following Lipschitz condition*

$$|\mathcal{F}(t,x,y) - \mathcal{F}(t,z,w)| \leq \mathcal{L}_1 |x-z| + \mathcal{L}_2 |y-w|,$$

where $(t,x,y), (t,z,w) \in I \times \mathbb{R} \times \mathbb{R}$ and $0 < \dfrac{(\psi(T))^{\upsilon}(\mathcal{L}_1 + \mathcal{L}_2)}{\Gamma(\upsilon+1)} < 1$.

Let $\Phi \in C_{1-\gamma;\psi}[t_0 - a, t]$ and $\varepsilon \geq 0$. If a continuous function $y : I \to \mathbb{R}$ satisfies

$$\begin{cases} \left| {}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} y(t) - \mathcal{F}(t, y(t), y(t-a)) \right| & < \varepsilon, \ t \in [t_0, T] \\ |y(t) - \Phi(t)| & < \varepsilon, \ t \in [t_0 - a, t_0] \end{cases}$$

then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$\begin{cases} {}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} y_0(t) & = \mathcal{F}(t, y_0(t), y_0(t-a)), \ t \in [t_0, T] \\ y_0(t) & = \Phi(t), \ t \in [t_0 - a, t_0] \end{cases} \tag{4.1}$$

and

$$|y(t) - y_0(t)| \leq \frac{\varepsilon(\psi(T))^{\upsilon}}{\Gamma(\upsilon+1) - (\psi(T))^{\upsilon}(\mathcal{L}_1 + \mathcal{L}_2)}, \ \forall t \in I. \tag{4.2}$$

Proof. For the proof, we consider the following generalized metric over S, given by

$$d_1(\varphi, \mu) = \inf\{M \in [0, \infty] : |\varphi(t) - \mu(t)| \leq M, \ \forall t \in I\}.$$

Note that, (S, d_1) is a generalized complete metric space.

For any $\varphi, \mu \in S$ and $M_{\varphi,\mu} \in \{M \in [0,\infty] : |\varphi(t) - \mu(t)| \leq M, \ \forall t \in I\}$, using Eq. (3.4), we obtain

$$\begin{aligned} |(\Omega\varphi)(t) - (\Omega\mu)(t)| &= \left| \begin{array}{l} \dfrac{1}{\Gamma(\upsilon)} \displaystyle\int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,u) \mathcal{F}(u, \phi(u), \phi(u-a))\, du \\ -\dfrac{1}{\Gamma(\upsilon)} \displaystyle\int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,u) \mathcal{F}(u, \mu(u), \mu(u-a))\, du \end{array} \right| \\ &\leq \frac{1}{\Gamma(\upsilon)} \int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,u) \left(\mathcal{L}_1 |\phi(u) - \mu(u)| + \mathcal{L}_2 |\phi(u-a) - \mu(u-a)|\right) du \\ &\leq \frac{\mathcal{L}_1 M_{\varphi,\mu}}{\Gamma(\upsilon)} \int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,u) du + \frac{\mathcal{L}_2 M_{\varphi,\mu}}{\Gamma(\upsilon)} \int_{t_0}^{t} \mathcal{N}_{\psi}^{\upsilon}(t,u) du \\ &\leq \frac{(\mathcal{L}_1 + \mathcal{L}_2) M_{\varphi,\mu}}{\Gamma(\upsilon+1)} (\psi(T))^{\upsilon} \end{aligned}$$

and

$$|(\Omega\varphi)(t) - (\Omega\mu)(t)| = \Phi(t) - \Phi(t) = 0, \ \forall t \in [t_0 - a, t_0]$$

which imply that $d_1(\Omega\varphi,\Omega\mu)\leq\dfrac{(L_1+L_2)(\psi(T))^{\upsilon}}{\Gamma(\upsilon+1)}d(\varphi,\mu)$.

Since $0<\dfrac{(\psi(T))^{\upsilon}(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(\upsilon+1)}<1$, then Ω is strictly contractive on S.

Now, for an arbitrary $\delta\in S$ and using the fact that $\mathcal{F}(t,\delta(t),\delta(t-a))$ and $\delta(t)$, are bounded on S, we can show $d_1(\Omega\delta,\delta)<\infty$. Hence, from Theorem 1 (1), there exists a continuous function $y_0:I\to\mathbb{R}$ such that $\Omega^n\xi\to y_0$ in (S,d_1) and $\Omega y_0=y_0$, then y_0 satisfies

$$\begin{cases} {}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi}y_0(t) &= \mathcal{F}(t,y_0(t),y_0(t-a)),\ t\in[t_0,T]\\ y_0(t) &= \Phi(t),\ t\in[t_0-a,t_0]. \end{cases}$$

Using the fact that g and δ are bounded on I, then $d_1(\delta,g)<\infty$, $\forall g\in S$, with $S=\{d_1(\delta,y)<\infty\}$.

Then, using Theorem 1 (2), y_0 is the unique continuous function with the property Eq.(4.1). Note that,

$$-\varepsilon\leq {}^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi}y(t)-\mathcal{F}(t,y(t),y(t-a))\leq\varepsilon \tag{4.3}$$

for all $t\in[t_0,T]$.

Applying the fractional integral $I_{t_0}^{\upsilon;\psi}(\cdot)$, on both sides of Eq.(4.3), we get

$$\left|y(t)-\Psi^{\gamma}(t,t_0)-\frac{1}{\Gamma(\upsilon)}\int_{t_0}^{t}\mathcal{N}_{\psi}^{\upsilon}(t,u)\mathcal{F}(u,y(u),y(u-a))\,du\right|\varepsilon I_{t_0}^{\upsilon;\psi}(1)\leq\frac{\varepsilon(\psi(T))^{\upsilon}}{\Gamma(\upsilon+1)} \tag{4.4}$$

for each $t\in I$.

By means of Theorem 1 (3) and Eq. (4.4), we get

$$\begin{aligned} d_1(y,y_0) &\leq \frac{\varepsilon(\psi(T))^{\upsilon}}{\Gamma(\upsilon+1)\left(1-\dfrac{(\psi(T))^{\upsilon}(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(\upsilon+1)}\right)}\\ &= \frac{\varepsilon(\psi(T))^{\upsilon}}{\Gamma(\upsilon+1)-(\psi(T))^{\upsilon}(\mathcal{L}_1+\mathcal{L}_2)}, \end{aligned}$$

which concludes the proof. □

Corollary 1. *Suppose the conditions of the* Theorem 2. If a continuous function $y:I\to\mathbb{R}$ satisfies

$$\begin{cases} \left|\mathcal{D}_{t_0+}^{\upsilon}y(t)-\mathcal{F}(t,y(t),y(t-a))\right| &< \varepsilon,\ t\in[t_0,T]\\ |y(t)-\Phi(t)| &< \varepsilon,\ t\in[t_0-a,t_0] \end{cases} \tag{4.5}$$

then there exists a unique continuous function $y_0:I\to\mathbb{R}$ such that

$$\begin{cases} \mathcal{D}_{t_0+}^{\upsilon}y_0(t) &= \mathcal{F}(t,y_0(t),y_0(t-a)),\ t\in[t_0,T]\\ y_0(t) &= \Phi(t),\ t\in[t_0-a,t_0] \end{cases} \tag{4.6}$$

and

$$|y(t)-y_0(t)|\leq\frac{\varepsilon(\ln T)^{\upsilon}}{\Gamma(\upsilon+1)-(\ln T)^{\upsilon}(\mathcal{L}_1+\mathcal{L}_2)},\ \forall t\in I, \tag{4.7}$$

where $\mathcal{D}_{t_0+}(\cdot)$ is the Hadamard fractional derivative.

Proof. The proof is a direct consequence of the Theorem 2. □

Corollary 2. *Suppose the conditions of the* Theorem 2. If a continuous function $y : I \to \mathbb{R}$ satisfies

$$\begin{cases} \left|y'(t) - \mathcal{F}(t, y(t), y(t-a))\right| < \varepsilon, \ t \in [t_0, T] \\ \left|y(t) - \Phi(t)\right| < \varepsilon, \ t \in [t_0 - a, t_0] \end{cases} \tag{4.8}$$

then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$\begin{cases} y_0'(t) = \mathcal{F}(t, y_0(t), y_0(t-a)), \ t \in [t_0, T] \\ y_0(t) = \Phi(t), \ t \in [t_0 - a, t_0] \end{cases} \tag{4.9}$$

and

$$\left|y(t) - y_0(t)\right| \leq \frac{\varepsilon T}{1 - T(\mathcal{L}_1 + \mathcal{L}_2)}, \ \forall t \in I. \tag{4.10}$$

Proof. The proof is a direct consequence of the Theorem 2. □

Remark 2. *The following fractional differential equation*

$$^{\mathbf{H}}\mathrm{D}_{t_0+}^{\upsilon,\zeta;\psi} y(t) = \mathcal{F}(t, y(t)) \tag{4.11}$$

is a special case of Eq. (1.1). Consequently, the results proposed here are also valid for Eq. (4.11).

Applying the limit $\upsilon \to 1$ on both sides of the Eq. (4.11), we obtain the following first order differential equation [19]

$$y'(t) = \mathcal{F}(t, y(t)),$$

which, in turn, the results proposed here, are also valid.

5. Concluding remarks

The study of Ulam-Hyers-type stability of solutions of the fractional functional differential equations has been the object of much study and investigated by many researchers [1, 5, 8, 20, 21, 22, 26, 30, 45]. Although it is yet a field of mathematics that is in expansion, over the years countless works have been published and others are yet to come. In this sense, the paper presented a discussion on the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the fractional functional differential equation Eq. (1.1) through the Banach fixed point theorem, which contributes to the growth of this area.

From this contribution, the natural question that arises is whether by means of the ψ-Hilfer fractional derivative it is also possible to obtain the stabilities investigated here in the function space $L_{p,\upsilon}(I, \mathbb{R})$? And using another fixed point theorem? Another possibility of study is to investigate other types of stabilities such as δ-Ulam-Hyers-Rassias, semi-Ulam-Hyers-Rassias and Mittag-Leffler-Ulam using the same fractional differentiable operator [37, 38, 46]. Studies in this direction are being prepared and will be published in the near future.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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