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Research article

Several explicit and recursive formulas for generalized Motzkin numbers

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Abstract: In the paper, the authors find two explicit formulas and recover a recursive formula for generalized Motzkin numbers. Consequently, the authors deduce two explicit formulas and a recursive formula for the Motzkin numbers, the Catalan numbers, and the restricted hexagonal numbers respectively.

Keywords: explicit formula; recursive formula; generalized Motzkin number; Motzkin number; restricted hexagonal number; Catalan number; generating function
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1. Introduction

The Motzkin numbers M_n enumerate various combinatorial objects. In 1977, Donaghey and Shapiro [3] gave fourteen different manifestations of the Motzkin numbers M_n. In particular, the Motzkin numbers M_n give the numbers of paths from $(0,0)$ to $(n,0)$ which never dip below the x-axis $y=0$ and are made up only of the steps $(1,0)$, $(1,1)$, and $(1,-1)$.

The first seven Motzkin numbers M_n for $0 \le n \le 6$ are $1,1,2,4,9,21,51$. All the Motzkin numbers M_n can be generated by

$$M(x)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}=\frac{1}{1-x+\sqrt{1-2x-3x^2}}=\sum_{k=0}^{\infty}M_kx^k.$$

In 2007, Mansour *et al* [12] introduced the (u,l,d)-Motzkin numbers $m_n^{(u,l,d)}$ and obtained [12, Theorem 2.1] that $m_n^{(u,l,d)} = m_n^{(1,l,ud)}$,

$$M_{u,l,d}(x) = \frac{1 - lx - \sqrt{(1 - lx)^2 - 4udx^2}}{2udx^2} = \sum_{n=0}^{\infty} m_n^{(u,l,d)} x^n, \tag{1.1}$$

and

$$m_n^{(u,l,d)} = l^n \sum_{j=0}^{n/2} \frac{1}{j+1}\binom{2j}{j}\binom{n}{2j}\left(\frac{ud}{l^2}\right)^j. \tag{1.2}$$

From (1.1) and (1.2), it is easy to see that $m_n^{(u,l,d)} = m_n^{(d,l,u)}$.

In 2014, Sun [42] generalized the Motzkin numbers M_n to

$$M_n(a,b) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k a^{n-2k} b^k \tag{1.3}$$

for $a, b \in \mathbb{N}$ in terms of the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{1.4}$$

and established the generating function

$$M_{a,b}(x) = \frac{1 - ax - \sqrt{(1 - ax)^2 - 4bx^2}}{2bx^2} = \frac{1}{1 - ax + \sqrt{(1 - ax)^2 - 4bx^2}} = \sum_{k=0}^{\infty} M_k(a,b) x^k, \tag{1.5}$$

where $\lfloor \lambda \rfloor$ denotes the floor function defined by the largest integer less than or equal to $\lambda \in \mathbb{R}$. Wang and Zhang pointed out in [43] that

$$M_n(1,1) = M_n, \quad M_n(2,1) = C_{n+1}, \quad \text{and} \quad M_n(3,1) = H_n, \tag{1.6}$$

where H_n denote the restricted hexagonal numbers described by Harary and Read [4].

For more information on many results, applications, and generalizations of the Motzkin numbers M_n, please refer to the papers [3, 9, 10, 42, 43] and closely related references therein. For more information on many results, applications, and generalizations of the Catalan numbers C_n, please refer to the monograph [5], the newly published papers [11, 17, 19, 26, 27, 31, 36–38, 40, 41], the survey articles [25, 29], and closely related references therein.

Comparing (1.1) with (1.5) reveals that $M_k(a,b)$ and $m_k^{(u,l,d)}$ are equivalent to each other and satisfy

$$M_k(a,b) = m_n^{(1,a,b)} = m_k^{(b,a,1)} \quad \text{and} \quad m_k^{(u,l,d)} = M_k(l, ud). \tag{1.7}$$

Therefore, it suffices to consider generalized Motzkin numbers $M_k(a,b)$, rather than the (u,l,d)-Motzkin numbers $m_n^{(u,l,d)}$, in this paper.

By the second relation in (1.7), one can reformulated the formula (1.2) as

$$M_n(a,b) = a^n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{j+1}\binom{2j}{j}\binom{n}{2j}\left(\frac{b}{a^2}\right)^j. \tag{1.8}$$

Substituting (1.4) into (1.3) recovers (1.8) once again.

In 2015, Wang and Zhang [43, Theorem 1] combinatorially obtained, among other things, the recursive formula

$$M_{n+2}(a,b) = aM_{n+1}(a,b) + b\sum_{\ell=0}^{n} M_\ell(a,b)M_{n-\ell}(a,b), \quad n \ge 0. \tag{1.9}$$

It is not difficult to see that the function $(1-ax)^2 - 4bx^2 = (a^2-4b)x^2 - 2ax + 1$ is nonnegative if and only if

1. either $b = 0$ and $x \in \mathbb{R}$,
2. or $a^2 - 4b = 0$, $a \ne 0$, and $x \le \frac{1}{2a}$,
3. or $a^2 - 4b > 0$, $b < 0$, and $x \in \mathbb{R}$,
4. or $a^2 - 4b > 0$, $b > 0$, and $x \ge \frac{1}{a-2\sqrt{b}}$ or $x \le \frac{1}{a+2\sqrt{b}}$.

Consequently, the generating function $M_{a,b}(x)$ defined by (1.5) in [42] is defined for either $b \le 0$ or $a \ge 2\sqrt{b} > 0$.

In this paper, we will find two explicit formulas, which are different from (1.8), and recover the recursive formula (1.9) for generalized Motzkin numbers $M_n(a,b)$. Consequently, we will derive two explicit formula and a recursive formula for the Motzkin numbers M_n, the Catalan numbers C_n, and the restricted hexagonal numbers H_n respectively.

We can state our main results as the following three theorems.

Theorem 1. *For $n \ge 0$, we can compute generalized Motzkin numbers $M_n(a,b)$ by*

$$M_n(a,b) = \frac{1}{2b}\left(\frac{4b-a^2}{2a}\right)^{n+2}\sum_{\ell=0}^{n+2}\left(\frac{2a^2}{4b-a^2}\right)^{\ell}\frac{(2\ell-3)!!}{\ell!}\binom{\ell}{n-\ell+2}, \tag{1.10}$$

where $\binom{p}{q} = 0$ for $q > p \ge 0$ and the double factorial of negative odd integers $-(2n+1)$ is

$$[-(2n+1)]!! = \frac{(-1)^n}{(2n-1)!!} = (-1)^n\frac{2^n n!}{(2n)!}, \quad n = 0, 1, \dots .$$

Consequently, we can compute the Motzkin numbers M_n and the restricted hexagonal numbers H_n respectively by

$$M_n = \frac{9}{8}\left(\frac{3}{2}\right)^n\sum_{\ell=0}^{n+2}\left(\frac{2}{3}\right)^{\ell}\frac{(2\ell-3)!!}{\ell!}\binom{\ell}{n-\ell+2} \tag{1.11}$$

and

$$H_n = (-1)^n\frac{25}{72}\left(\frac{5}{6}\right)^n\sum_{\ell=0}^{n+2}(-1)^\ell\left(\frac{18}{5}\right)^{\ell}\frac{(2\ell-3)!!}{\ell!}\binom{\ell}{n-\ell+2}. \tag{1.12}$$

Theorem 2. *For $n \ge 0$, we can compute generalized Motzkin numbers $M_n(a,b)$ by*

$$M_n(a,b) = -\frac{(a-2\sqrt{b})^{n+2}}{2b}\sum_{\ell=0}^{n+2}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!}\left(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\right)^{\ell}. \tag{1.13}$$

Consequently, we can compute the Motzkin numbers M_n and the restricted hexagonal numbers H_n respectively by

$$M_n = \frac{(-1)^{n+1}}{2}\sum_{\ell=0}^{n+2}(-1)^\ell 3^\ell \frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!}$$

and

$$H_n = -\frac{1}{2}\sum_{\ell=0}^{n+2} 5^\ell \frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!}.$$

Theorem 3. *For $n \ge 0$, generalized Motzkin numbers $M_n(a,b)$ satisfy*

$$M_0(a,b) = 1, \quad M_1(a,b) = a, \tag{1.14}$$

and the recursive formula (1.9)*. Consequently, for $n \ge 0$, the Motzkin numbers M_n, the Catalan numbers C_n, and the restricted hexagonal numbers H_n meet the recursive formulas*

$$M_{n+2} = M_{n+1} + \sum_{\ell=0}^{n} M_\ell M_{n-\ell}, \tag{1.15}$$

$$C_{n+2} = 2C_{n+1} + \sum_{\ell=0}^{n} C_\ell C_{n-\ell}, \tag{1.16}$$

and

$$H_{n+2} = 3H_{n+1} + \sum_{\ell=0}^{n} H_\ell H_{n-\ell} \tag{1.17}$$

respectively.

2. Lemmas

In order to prove the explicit formula (1.10), we need the following lemmas.

Lemma 1 ([1, p. 40, Exercise 5)], [16, Section 2.2, p. 849], [22, p. 94], [34, Lemma 3], and [44, Lemma 2.1])**.** *Let $u(x)$ and $v(x) \ne 0$ be two differentiable functions. Let $U_{(n+1)\times 1}(x)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(x)$ be an $(n+1)\times n$ matrix whose elements*

$$v_{i,j}(x) = \begin{cases} \binom{i-1}{j-1} v^{(i-j)}(x), & i-j \ge 0 \\ 0, & i-j < 0 \end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{(n+1)\times(n+1)}(x)|$ denote the determinant of the $(n+1)\times(n+1)$ matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{pmatrix} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{pmatrix}.$$

Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n \frac{\left|W_{(n+1)\times(n+1)}(x)\right|}{v^{n+1}(x)}.$$

Lemma 2 ([2, p. 134, Theorem A and p. 139, Theorem C]). *The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind*

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1} \ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n \ge k \ge 0$ *by*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[f \circ h(t)] = \sum_{k=0}^{n} f^{(k)}(h(t))\mathrm{B}_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)) \tag{2.1}$$

for $n \ge 0$.

Lemma 3 ([2, p. 135]). *The Bell polynomials of the second kind* $\mathrm{B}_{n,k}$ *satisfy*

$$\mathrm{B}_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{2.2}$$

for $n \ge k \ge 0$.

Lemma 4. *For* $n \ge k \ge 0$*, we have*

$$\mathrm{B}_{n,k}(x, 1, 0, \dots, 0) = \frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.3}$$

More generally, for $n \ge k \ge 0$ *and* $\lambda, \alpha \in \mathbb{C}$*, we have*

$$\mathrm{B}_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-q\lambda) \tag{2.4}$$

or, equivalently,

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\alpha\ell\rangle_n. \tag{2.5}$$

Proof. The formula (2.3) can be found in [24, Theorem 5.1], [35, p. 7, (19)], [39, Section 3], and [44, Lemma 2.5]. The explicit formula (2.4) was first established in [30, Remark 1] and then was applied in [18, Section 2], [20, First proof of Theorem 2], [21, Lemma 2.2], [24, Remark 6.1], [28, Lemma 4], and [32, Lemma 2.6]. The formula (2.5) and the equivalence were presented in [33, Theorems 2.1 and 4.1]. □

Remark 1. In recent years, there have been some literature such as [6–8, 13–15, 23, 24, 30, 35, 45–48] devoting to deep investigation and extensive applications of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$. Specially, in the papers [13, 14], the generalized Dyck paths (namely, various type of Motzkin paths) and the Bell polynomials were connected closely.

3. Proofs of Theorems 1 and 3

We are now in a position to prove our main results.

Proof of Theorem 1. By virtue of (2.1), (2.2), and (2.3), we obtain for $k \ge 0$ that

$$\begin{aligned}
\Big[\sqrt{(1-ax)^2-4bx^2}\,\Big]^{(k+2)} &= \sum_{\ell=0}^{k+2}\Big\langle\frac{1}{2}\Big\rangle_{\ell}[(1-ax)^2-4bx^2]^{1/2-\ell} \\
&\quad\times \mathrm{B}_{k+2,\ell}(-2[a+(4b-a^2)x], 2(a^2-4b), 0, \dots, 0) \\
&\quad\to \sum_{\ell=0}^{k+2}\Big\langle\frac{1}{2}\Big\rangle_{\ell}\mathrm{B}_{k+2,\ell}(-2a, 2(a^2-4b), 0, \dots, 0) \\
&= \sum_{\ell=0}^{k+2}\Big\langle\frac{1}{2}\Big\rangle_{\ell}[2(a^2-4b)]^{\ell}\mathrm{B}_{k+2,\ell}\Big(\frac{a}{4b-a^2}, 1, 0, \dots, 0\Big) \\
&= \sum_{\ell=0}^{k+2}\Big\langle\frac{1}{2}\Big\rangle_{\ell}[2(a^2-4b)]^{\ell}\frac{(k-\ell+2)!}{2^{k-\ell+2}}\binom{k+2}{\ell}\binom{\ell}{k-\ell+2}\Big(\frac{a}{4b-a^2}\Big)^{2\ell-k-2}
\end{aligned} \tag{3.1}$$

as $x \to 0$, where

$$\langle x\rangle_n = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

denotes the falling factorial of $x \in \mathbb{R}$.

Letting $u(x) = 1 - ax - \sqrt{(1-ax)^2-4bx^2}$ and $v(x) = x^2$ in Lemma 1 gives

$$\frac{\mathrm{d}^n M_{a,b}(x)}{\mathrm{d}x^n} = \frac{1}{2b}\frac{(-1)^n}{x^{2(n+1)}}\begin{vmatrix}
u(x) & \binom{0}{0}x^2 & 0 & \cdots & 0 & 0 & 0 \\
u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & \cdots & 0 & 0 & 0 \\
u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \cdots & 0 & 0 & 0 \\
u^{(3)}(x) & 0 & 2\binom{3}{1} & \cdots & 0 & 0 & 0 \\
u^{(4)}(x) & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\
u^{(n-2)}(x) & 0 & 0 & \cdots & 2\binom{n-2}{n-3}x & \binom{n-2}{n-2}x^2 & 0 \\
u^{(n-1)}(x) & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & 2\binom{n-1}{n-2}x & \binom{n-1}{n-1}x^2 \\
u^{(n)}(x) & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & 2\binom{n}{n-1}x
\end{vmatrix}$$

$$= \frac{1}{2b}\frac{(-1)^n}{x^{2(n+1)}}\left[(-1)^n u^{(n)}(x)\begin{vmatrix}
\binom{0}{0}x^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
2\binom{1}{0}x & \binom{1}{1}x^2 & 0 & \cdots & 0 & 0 & 0 \\
2\binom{2}{0} & 2\binom{2}{1}x & \binom{2}{2}x^2 & \cdots & 0 & 0 & 0 \\
0 & 2\binom{3}{1} & 2\binom{3}{2}x & \cdots & 0 & 0 & 0 \\
0 & 0 & 2\binom{4}{2} & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 2\binom{n-2}{n-3}x & \binom{n-2}{n-2}x^2 & 0 \\
0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & 2\binom{n-1}{n-2}x & \binom{n-1}{n-1}x^2
\end{vmatrix}\right.$$

$$+2\binom{n}{n-1}x\begin{vmatrix} u(x) & \binom{0}{0}x^2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & 0 & 0 & \cdots & 0 & 0 \\ u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \binom{2}{2}x^2 & 0 & \cdots & 0 & 0 \\ u^{(3)}(x) & 0 & 2\binom{3}{1} & 2\binom{3}{2}x & \binom{3}{3}x^2 & \cdots & 0 & 0 \\ u^{(4)}(x) & 0 & 0 & 2\binom{4}{2} & 2\binom{4}{3}x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ u^{(n-2)}(x) & 0 & 0 & 0 & 0 & \cdots & 2\binom{n-2}{n-3}x & \binom{n-2}{n-2}x^2 \\ u^{(n-1)}(x) & 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & 2\binom{n-1}{n-2}x \end{vmatrix}$$

$$-2\binom{n}{n-2}\binom{n-1}{n-1}x^2\begin{vmatrix} u(x) & \binom{0}{0}x^2 & 0 & 0 & \cdots & 0 & 0 \\ u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & 0 & \cdots & 0 & 0 \\ u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \binom{2}{2}x^2 & \cdots & 0 & 0 \\ u^{(3)}(x) & 0 & 2\binom{3}{1} & 2\binom{3}{2}x & \cdots & 0 & 0 \\ u^{(4)}(x) & 0 & 0 & 2\binom{4}{2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ u^{(n-3)}(x) & 0 & 0 & 0 & \cdots & 2\binom{n-3}{n-4}x & \binom{n-3}{n-3}x^2 \\ u^{(n-2)}(x) & 0 & 0 & 0 & \cdots & 2\binom{n-2}{n-4} & 2\binom{n-2}{n-3}x \end{vmatrix}\Bigg]$$

$$= \frac{1}{2b}\frac{u^{(n)}(x)}{x^2} - \frac{2n}{x}\frac{1}{2b}\frac{(-1)^{n-1}}{x^{2n}}\begin{vmatrix} u(x) & \binom{0}{0}x^2 & 0 & \cdots & 0 & 0 \\ u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & \cdots & 0 & 0 \\ u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \cdots & 0 & 0 \\ u^{(3)}(x) & 0 & 2\binom{3}{1} & \cdots & 0 & 0 \\ u^{(4)}(x) & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ u^{(n-2)}(x) & 0 & 0 & \cdots & 2\binom{n-2}{n-3}x & \binom{n-2}{n-2}x^2 \\ u^{(n-1)}(x) & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & 2\binom{n-1}{n-2}x \end{vmatrix}$$

$$-n(n-1)\frac{1}{x^2}\frac{1}{2b}\frac{(-1)^{n-2}}{x^{2(n-1)}}\begin{vmatrix} u(x) & \binom{0}{0}x^2 & 0 & 0 & \cdots & 0 & 0 \\ u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & 0 & \cdots & 0 & 0 \\ u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \binom{2}{2}x^2 & \cdots & 0 & 0 \\ u^{(3)}(x) & 0 & 2\binom{3}{1} & 2\binom{3}{2}x & \cdots & 0 & 0 \\ u^{(4)}(x) & 0 & 0 & 2\binom{4}{2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ u^{(n-3)}(x) & 0 & 0 & 0 & \cdots & 2\binom{n-3}{n-4}x & \binom{n-3}{n-3}x^2 \\ u^{(n-2)}(x) & 0 & 0 & 0 & \cdots & 2\binom{n-2}{n-4} & 2\binom{n-2}{n-3}x \end{vmatrix}$$

$$= \frac{1}{2b}\frac{u^{(n)}(x)}{x^2} - \frac{2n}{x}\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}} - \frac{n(n-1)}{x^2}\frac{\mathrm{d}^{n-2}M_{a,b}(x)}{\mathrm{d}x^{n-2}}$$

$$= \frac{1}{x^2}\left[\frac{u^{(n)}(x)}{2b} - 2nx\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}} - n(n-1)\frac{\mathrm{d}^{n-2}M_{a,b}(x)}{\mathrm{d}x^{n-2}}\right].$$

Therefore, by L'Hôspital's rule, we have

$$\begin{aligned}\lim_{x\to0}\frac{\mathrm{d}^n M_{a,b}(x)}{\mathrm{d}x^n}&=\lim_{x\to0}\left\{\frac{1}{x^2}\left[\frac{u^{(n)}(x)}{2b}-2nx\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}}-n(n-1)\frac{\mathrm{d}^{n-2}M_{a,b}(x)}{\mathrm{d}x^{n-2}}\right]\right\}\\&=\lim_{x\to0}\left\{\frac{1}{2x}\left[\frac{u^{(n+1)}(x)}{2b}-2nx\frac{\mathrm{d}^{n}M_{a,b}(x)}{\mathrm{d}x^{n}}-n(n+1)\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}}\right]\right\}\\&=\frac{1}{2}\lim_{x\to0}\left[\frac{u^{(n+2)}(x)}{2b}-2nx\frac{\mathrm{d}^{n+1}M_{a,b}(x)}{\mathrm{d}x^{n+1}}-n(n+3)\frac{\mathrm{d}^{n}M_{a,b}(x)}{\mathrm{d}x^{n}}\right]\\&=\frac{1}{2}\left[\lim_{x\to0}\frac{u^{(n+2)}(x)}{2b}-n(n+3)\lim_{x\to0}\frac{\mathrm{d}^{n}M_{a,b}(x)}{\mathrm{d}x^{n}}\right]\end{aligned}$$

which is equivalent to

$$\lim_{x\to0}\frac{\mathrm{d}^n M_{a,b}(x)}{\mathrm{d}x^n}=\frac{1}{(n+1)(n+2)}\lim_{x\to0}\frac{u^{(n+2)}(x)}{2b}=\frac{1}{2b(n+1)(n+2)}\lim_{x\to0}u^{(n+2)}(x).$$

Considering

$$\lim_{x\to0}\frac{\mathrm{d}^n M_{a,b}(x)}{\mathrm{d}x^n}=n!M_n(a,b),$$

making use of (3.1), and simplifying lead to the explicit formula (1.10).

Letting $(a,b)=(1,1)$ and $(a,b)=(3,1)$ respectively in (1.10) and considering the three relations in (1.6) derive (1.11) and (1.12) immediately. The proof of Theorem 1 is complete. □

Proof of Theorem 2. From (1.5), it is derived that

$$\sqrt{(1-ax)^2-4bx^2}=1-ax-2b\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}.$$

This implies that

$$M_k(a,b)=-\frac{1}{2b}\frac{1}{(k+2)!}\lim_{x\to0}\left[\sqrt{(1-ax)^2-4bx^2}\right]^{(k+2)},\quad k\ge0. \tag{3.2}$$

It is easy to see that

1. when $a^2-4b>0$ and $x\le\min\left\{\frac{1}{a+2\sqrt{b}},\frac{1}{a-2\sqrt{b}}\right\}=\frac{1}{a+2\sqrt{b}}$, we have

$$\begin{aligned}\left[\sqrt{(1-ax)^2-4bx^2}\right]^{(k+2)}&=\left[\sqrt{(a^2-4b)\left(x-\frac{1}{a+2\sqrt{b}}\right)\left(x-\frac{1}{a-2\sqrt{b}}\right)}\right]^{(k+2)}\\&=\sqrt{a^2-4b}\left(\sqrt{\frac{1}{a+2\sqrt{b}}-x}\sqrt{\frac{1}{a-2\sqrt{b}}-x}\right)^{(k+2)}\\&=\sqrt{a^2-4b}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\left(\sqrt{\frac{1}{a+2\sqrt{b}}-x}\right)^{(\ell)}\left(\sqrt{\frac{1}{a-2\sqrt{b}}-x}\right)^{(k-\ell+2)}\\&=(-1)^k\sqrt{a^2-4b}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\left\langle\frac{1}{2}\right\rangle_\ell\left(\frac{1}{a+2\sqrt{b}}-x\right)^{1/2-\ell}\left\langle\frac{1}{2}\right\rangle_{k-\ell+2}\left(\frac{1}{a-2\sqrt{b}}-x\right)^{\ell-k-3/2}\end{aligned}$$

$$\to (-1)^k \sqrt{a^2-4b}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\left\langle\frac{1}{2}\right\rangle_\ell\left(\frac{1}{a+2\sqrt{b}}\right)^{1/2-\ell}\left\langle\frac{1}{2}\right\rangle_{k-\ell+2}\left(\frac{1}{a-2\sqrt{b}}\right)^{\ell-k-3/2}$$

$$= (k+2)!(a-2\sqrt{b})^{k+2}\sum_{\ell=0}^{k+2}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(k-\ell+2)-3]!!}{[2(k-\ell+2)]!!}\left(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\right)^{\ell}$$

as $x \to 0$;

2. when $a^2-4b<0$ and

$$\frac{1}{a+2\sqrt{b}} = \max\left\{\frac{1}{a+2\sqrt{b}}, \frac{1}{a-2\sqrt{b}}\right\} > x > \min\left\{\frac{1}{a+2\sqrt{b}}, \frac{1}{a-2\sqrt{b}}\right\} = \frac{1}{a-2\sqrt{b}},$$

we have

$$\left[\sqrt{(1-ax)^2-4bx^2}\right]^{(k+2)} = \left[\sqrt{(4b-a^2)\left(\frac{1}{a+2\sqrt{b}}-x\right)\left(x-\frac{1}{a-2\sqrt{b}}\right)}\right]^{(k+2)}$$

$$= \sqrt{4b-a^2}\left(\sqrt{\frac{1}{a+2\sqrt{b}}-x}\sqrt{x-\frac{1}{a-2\sqrt{b}}}\right)^{(k+2)}$$

$$= \sqrt{4b-a^2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\left(\sqrt{\frac{1}{a+2\sqrt{b}}-x}\right)^{(\ell)}\left(\sqrt{x-\frac{1}{a-2\sqrt{b}}}\right)^{(k-\ell+2)}$$

$$= \sqrt{4b-a^2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}(-1)^\ell\left\langle\frac{1}{2}\right\rangle_\ell\left(\frac{1}{a+2\sqrt{b}}-x\right)^{1/2-\ell}\left\langle\frac{1}{2}\right\rangle_{k-\ell+2}\left(x-\frac{1}{a-2\sqrt{b}}\right)^{1/2-(k-\ell+2)}$$

$$\to \sqrt{4b-a^2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}(-1)^\ell\left\langle\frac{1}{2}\right\rangle_\ell\left(\frac{1}{a+2\sqrt{b}}\right)^{1/2-\ell}\left\langle\frac{1}{2}\right\rangle_{k-\ell+2}\left(\frac{1}{2\sqrt{b}-a}\right)^{1/2-(k-\ell+2)}$$

$$= (2\sqrt{b}-a)^{k+1}\sqrt{4b-a^2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\frac{(2\ell-3)!!}{2^\ell}\left(\frac{a+2\sqrt{b}}{2\sqrt{b}-a}\right)^{\ell-1/2}(-1)^{k-\ell}\frac{[2(k-\ell+2)-3]!!}{2^{k-\ell+2}}$$

$$= (2\sqrt{b}-a)^{k+2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\frac{(2\ell-3)!!}{2^\ell}\left(\frac{a+2\sqrt{b}}{2\sqrt{b}-a}\right)^{\ell}(-1)^{k-\ell}\frac{[2(k-\ell+2)-3]!!}{2^{k-\ell+2}}$$

$$= (a-2\sqrt{b})^{k+2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\frac{(2\ell-3)!!}{2^\ell}\frac{[2(k-\ell+2)-3]!!}{2^{k-\ell+2}}\left(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\right)^{\ell}$$

$$= (k+2)!(a-2\sqrt{b})^{k+2}\sum_{\ell=0}^{k+2}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(k-\ell+2)-3]!!}{[2(k-\ell+2)]!!}\left(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\right)^{\ell}$$

as $x \to 0$.

By virtue of (3.2), we obtain the formula (1.13) readily.

Letting $(a,b)=(1,1)$ and $(a,b)=(3,1)$ respectively in (1.13) and making use of the first and third relations in (1.6) lead to (1.11) and (1.12) immediately. The proof of Theorem 2 is complete. □

Proof of Theorem 3. From (1.5), it is derived that

$$\sqrt{(1-ax)^2-4bx^2} = 1-ax-2b\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}.$$

Squaring on both sides of the above equation gives

$$
\begin{aligned}
(1-ax)^2-4bx^2 &= 1-2ax+(a^2-4b)x^2 = \left[1-ax-2b\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}\right]^2 \\
&= 1+a^2x^2+4b^2\left[\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}\right]^2-2ax-4b\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}+4abx\sum_{k=0}^{\infty}M_k(a,b)x^{k+2} \\
&= 1-2ax+a^2x^2+4b^2x^4\sum_{k=0}^{\infty}\left[\sum_{\ell=0}^{k}M_\ell(a,b)M_{k-\ell}\right]x^k \\
&\quad -4b\sum_{k=2}^{\infty}M_{k-2}(a,b)x^k+4ab\sum_{k=3}^{\infty}M_{k-3}(a,b)x^k \\
&= 1-2ax+a^2x^2-4b\sum_{k=2}^{\infty}M_{k-2}(a,b)x^k+4ab\sum_{k=3}^{\infty}M_{k-3}(a,b)x^k \\
&\quad +4b^2\sum_{k=4}^{\infty}\left[\sum_{\ell=0}^{k-4}M_\ell(a,b)M_{k-\ell-4}(a,b)\right]x^k \\
&= 1-2ax+a^2x^2-4b[M_0(a,b)x^2+M_1(a,b)x^3]+4abM_0(a,b)x^3 \\
&\quad -4b\sum_{k=4}^{\infty}M_{k-2}(a,b)x^k+4ab\sum_{k=4}^{\infty}M_{k-3}(a,b)x^k+4b^2\sum_{k=4}^{\infty}\left[\sum_{\ell=0}^{k-4}M_\ell(a,b)M_{k-\ell-4}(a,b)\right]x^k \\
&= 1-2ax+[a^2-4bM_0(a,b)]x^2+4b[aM_0(a,b)-M_1(a,b)]x^3 \\
&\quad -4b\sum_{k=4}^{\infty}\left[M_{k-2}(a,b)-aM_{k-3}(a,b)-b\sum_{\ell=0}^{k-4}M_\ell(a,b)M_{k-\ell-4}(a,b)\right]x^k
\end{aligned}
$$

which means that

$$
a^2-4b=a^2-4bM_0(a,b), \quad 4b[aM_0(a,b)-M_1(a,b)]=0,
$$

and

$$
M_{k-2}(a,b)-aM_{k-3}(a,b)-b\sum_{\ell=0}^{k-4}M_\ell(a,b)M_{k-\ell-4}(a,b)=0, \quad k\ge 4.
$$

Consequently, the identities in (1.14) and the recursive formula (1.9) follow.

Taking $(a,b)=(1,1)$, $(a,b)=(2,1)$, and $(a,b)=(3,1)$ respectively in (1.9) and considering the three relations in (1.6) lead to (1.15), (1.16), and (1.17) immediately. The proof of Theorem 3 is complete. □

4. Two more remarks

Remark 2. From the proof of Theorem 1, we can conclude that

$$
x^2\frac{\mathrm{d}^nM_{a,b}(x)}{\mathrm{d}x^n}+2nx\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}}+n(n-1)\frac{\mathrm{d}^{n-2}M_{a,b}(x)}{\mathrm{d}x^{n-2}}=\frac{u^{(n)}(x)}{2b}, \quad n\ge 2.
$$

This implies that the generating function $M_{a,b}(x)$ expressed in (1.5) is an explicit solution of the linear ordinary differential equations

$$x^2 f^{(n)}(x) + 2nxf^{(n-1)}(x) + n(n-1)f^{(n-2)}(x) = F_{n;a,b}(x)$$

for all $n \ge 2$, where, by (2.2) and (2.3) or (2.4),

$$F_{n;a,b}(x) = \frac{n!(4b-a^2)^n}{2^{n+1}b} \frac{\sqrt{(1-ax)^2-4bx^2}}{[a+(4b-a^2)x]^n} \sum_{\ell=1}^{n} \frac{2^\ell(2\ell-3)!!}{\ell!(4b-a^2)^\ell} \binom{\ell}{n-\ell} \frac{[a+(4b-a^2)x]^{2\ell}}{[(1-ax)^2-4bx^2]^\ell}.$$

Remark 3. This paper is a continuation of the article [49] and a revised version of the preprint [28].

Conflict of interest

The authors declare that they have no conflict of interest in this paper.

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