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Research article

Schrödinger-Poisson system without growth and the Ambrosetti-Rabinowitz conditions

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Abstract: We consider the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = \lambda f(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, \ \lim\limits_{|x|\to+\infty} \phi = 0, & \text{in } \mathbb{R}^3. \end{cases}$$

Unlike most other papers on this problem, the Schrödinger-Poisson system without any growth and Ambrosetti-Rabinowitz condition is considered in this paper. Firstly, by Jeanjean's monotonicity trick and the mountain pass theorem, we prove that the problem possesses a positive solution for large value of λ . Secondly, we establish the multiplicity of solutions via the symmetric mountain pass theorem.

Keywords: Schrödinger-Poisson system; supercritical growth; variational methods; L^∞-estimate
Mathematics Subject Classification: 35B45, 35J20, 35J50

1. Introduction

In this paper, we are interested in finding nontrivial solutions to the following one-parameter family of Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = \lambda f(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, \ \lim\limits_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

In recent years, system (1.1) has been studied widely due to the fact that it arises in several physical phenomena (see [3–5, 9, 18]). From the viewpoint of quantum mechanics, this system describes a charged wave interacting with its own electrostatic field in the case that magnetic effects could be ignored. The terms u and ϕ describe the wave functions associated to the particle and electric potential.

The term ϕu is nonlocal and concerns the interactions with electric field. The nonlinearity models the interaction between the particles and external nonlinear perturbations.

There has been a lot of contributions about the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \ \lim_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

which was first introduced in [4]. The case $V(x) = 1$ and $f(x,u) = |u|^{p-2}u$, $2 < p < 6$, has been studied in [7], where Ruiz gave existence and nonexistence results. The existence of a ground state solution of (1.2) with $f(x,u) = |u|^{p-2}u$ and $3 < p < 6$ was proved by Azzollini [1]. For the general nonlinearity f and the potential $V(x)$, in [6, 12, 13, 15, 20, 21], the authors studied the existence and multiplicity of nontrivial solutions for the Schrödinger-Poisson system with superlinear and subcritical growth condition. The following global Ambrosetti-Rabinowitz type condition plays a crucial role in the above mentioned papers:

$$0 < F(u) := \int_0^u f(s)ds \le \frac{1}{\gamma}uf(u), \tag{$A-R$}$$

where $\gamma > 4$. Since the nonlocal term $\int_{\mathbb{R}^3} \phi_u u^2$ in the energy functional of (1.2) is homogeneous of degree 4, if $\gamma > 4$ from (A-R) then Ambrosetti-Rabinowitz condition guarantees boundedness of Palais-Smale sequences as well as existence of a mountain pass geometry.

It is very natural for us to pose the question: Can we replace (A-R) with a weaker condition? When $V(x)$ is periodic or asymptotically periodic and $f(u)$ does not satisfy the Ambrosetti-Rabinowitz condition, Alves, Souto and Soares [2] established the existence of positive ground state solutions by using the mountain pass theorem. In [14] Mao et al. studied the following Schrödinger-Poisson system of the form

$$\begin{cases} -\Delta u + V(x)u + \varepsilon\phi u = \lambda f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \ \lim_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3, \\ u > 0, & \text{in } \mathbb{R}^3, \end{cases}$$

where f satisfies $0 < 4F(s) \le sf(s)$, for $s > 0$ is small. Under the conditions that ε is small and λ is large, the authors proved the existence of a positive solution. Differently from the above-mentioned results, the purpose of this paper is to present some existence and multiplicity results of solutions of problem (1.1) under the nonlinearity $f(t)$ which possesses only conditions in a neighborhood of the origin. More importantly, we consider the case that f satisfies $0 < \gamma F(t) \le tf(t)$ where $\gamma \in (3,4]$, for $t > 0$ is small. To the best of our knowledge, there are less results in the literatures on the case $\gamma \in (3,4)$.

Firstly, we study problem (1.1) under the following conditions:

(V_1): $V \in C(\mathbb{R}^3,\mathbb{R})$, $0 < V_L \le V(x)$ for all $x \in \mathbb{R}^3$ and $V(x)$ is coercive, *i.e.*, $\lim_{|x|\to\infty} V(x) = \infty$;

(V_2): $V \in C^1(\mathbb{R}^3,\mathbb{R})$ and $2V(x) + \nabla V(x)\cdot x \ge 0$ for *a.e.* $x \in \mathbb{R}^3$ and $\nabla V(x)\cdot x \in L^r(\mathbb{R}^3)$ for some $r \in [\frac{3}{2},\infty]$.

(f_0): $f(t) = 0$ for $t \le 0$;

(f_1): there exists $\alpha \in (4,2^*)$ such that $\limsup_{t\to0^+} \frac{f(t)t}{t^\alpha} < +\infty$, where $2^* = 6$;

(f_2): there exists $\beta \in (4,2^*)$ such that $\liminf_{t\to0^+} \frac{F(t)}{t^\beta} > 0$;

(f_3): there exists $3 < \gamma \le 4$ such that $0 < \gamma F(t) \le tf(t)$, for $t > 0$ is small ;

(f_1'): there exists $\alpha \in (4, 2^*)$ such that $\limsup\limits_{t\to 0} \frac{f(t)t}{|t|^\alpha} < +\infty$;
(f_2'): there exists $\beta \in (4, 2^*)$ such that $\liminf\limits_{t\to 0} \frac{F(t)}{|t|^\beta} > 0$, where $F(t) = \int_0^t f(s)ds$;
(f_3'): there exists $\widetilde{\gamma} > 4$ such that $0 < \widetilde{\gamma}F(t) \le tf(t)$, for $|t|$ small and $t \ne 0$;
(f_4): $f(-t) = -f(t)$, for $|t|$ small.

Next, we give our main results.

Theorem 1.1. *Assume (V_1), (V_2) and $(f_0) - (f_3)$ hold. Then the problem (1.1) has a positive solution $u \in X$ (X is defined in Sect.2) for all sufficiently large λ.*

Remark 1.1. (V_2) is used to obtain a special bounded Palais-Smale sequence with Jeanjean's monotonicity trick. That $\nabla V(x) \cdot x \in L^r(\mathbb{R}^3)$ for some $r \in [\frac{3}{2}, \infty]$ plays an important role in deriving the Pohozaev identity for the weak solutions of (2.2).

To prove Theorem 1.1, we are faced with several difficulties. On one hand, due to the nonlinearity f without any growth condition at infinity, the natural variational functional associated to (1.1) may be not well defined. Inspired by work of Costa [8] and Huang [10], we modify $f(t)$ to a new well-defined nonlinearity. Furthermore by Moser iteration, we shall show that for large λ, the solutions of the modified problem are the solutions of the original problem.

On the other hand, different from [8] and [14], since we don't assume the global Ambrosetti-Rabinowitz condition about $f(t)$, the boundedness of Palais-Smale sequence seems hard to verify. We use an argument developed by Jeanjean [11] to overcome this difficulty. Then Pohozaev type identity [19] and the condition (V_2) are used to construct a special bounded Palais-Smale sequence for the modified functional J_λ (will be defined in Section 2) .

Theorem 1.2. *Assume (V_1), $(f_1') - (f_3')$ and (f_4) hold. Then for any given positive integer $k \ge 1$ the problem (1.1) has k pairs of solutions $\pm u_i \in X (i = 1, 2, ..., k)$ for all sufficiently large λ.*

The key to prove Theorem 1.2 is a priori estimate of the weak solution for the modified problem. Firstly, we modify f to a new nonlinearity h which is odd and satisfies the Ambrosetti-Rabinowitz condition (see Lemma 2.1). By symmetric mountain pass theorem, the modified problem has a sequence of weak solutions. Secondly, it will be shown that the solutions converge to zero in L^∞-norm as $\lambda \to \infty$. Therefore, for λ large, they are solutions of the original problem.

Remark 1.2. It is evident that the following function satisfies hypotheses $(f_1') - (f_3')$ and (f_4):

$$f(t) = C_1|t|^{\alpha-2}t + C_2|t|^{q-2}t,$$

where $4 < \alpha < 2^* < q < \infty$ and $C_1,\ C_2$ are positive constants.

This paper is organized as follows. In Section 2, we describe the related mathematical tools and give the proof of Theorem 1.1. Theorem 1.2 is proved in Section 3.

In what follows, C and C_i will denote positive generic constants.

2. Proof of Theorem 1.1

As usual, the norm of $L^s(\mathbb{R}^N)$ $(s \ge 1)$ is denoted by $|\cdot|_s$. Define

$$X=\left\{u\in D^{1,2}(\mathbb{R}^3):\ \int_{\mathbb{R}^3}V(x)u^2dx<\infty\right\}$$

endowed with the following norm

$$\|u\|=\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)dx\right)^{1/2}.$$

By (V_1), it well known that $X\hookrightarrow L^p(\mathbb{R}^3)$ continuously for $p\in[2,6]$, compactly for $p\in[2,6)$.

By the conditions (f_0), (f_1) and (f_2), there exist positive constants $\delta\in(0,\frac{1}{2})$, C_3 and C_4 such that

$$F(t)\le C_3t^\alpha \text{ and } F(t)\ge C_4t^\beta \text{ for } 0\le t\le 2\delta. \tag{2.1}$$

For the fixed $\delta>0$, we now consider $d(t)\in C^1(\mathbb{R},\mathbb{R})$ is a cut-off function satisfying

$$d(t)=\begin{cases}1, & \text{if } t\le\delta,\\ 0, & \text{if } t\ge 2\delta,\end{cases}$$

$|d'(t)|\le\frac{2}{\delta}$ and $0\le d(t)\le 1$ for $t\in[\delta,2\delta]$. Define $G(t)=d(t)F(t)+(1-d(t))F_\infty(t)$, where

$$F_\infty(t)=\begin{cases}C_3|t|^\alpha, & \text{if } t>0,\\ 0, & \text{if } t\le 0.\end{cases}$$

Set $g(t)=G'(t)$. We observe that the conditions (f_0)–(f_3) imply some properties of $g(t)$.

Lemma 2.1. *(1)* $g\in C(\mathbb{R},\mathbb{R})$, $g(t)=0$, *for all* $t\le 0$ *and* $g(t)=o(1)$ *as* $t\to 0^+$;
(2) $\lim_{t\to+\infty}\frac{g(t)}{t^3}=+\infty$;
(3) there exists $C_5>0$ *such that* $g(t)\le C_5t^{\alpha-1}$, *for all* $t\ge 0$;
(4) for any $T>0$, *there exists a constant* $C(T)>0$ *such that* $G(t)\ge C(T)t^\beta$ *for all* $t\in[0,T]$;
(5) for all $t>0$, *we have* $0<\gamma G(t)\le tg(t)$.

By [17], for every $u\in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u\in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u=u^2$$

and

$$\int_{\mathbb{R}^3}\nabla\phi_u\cdot\nabla v dx=\int_{\mathbb{R}^3}u^2vdx, \text{ for all } v\in D^{1,2}(\mathbb{R}^3).$$

It has the following properties:

Lemma 2.2. *For any* $u\in X\subset H^1(\mathbb{R}^3)$, *we have*
(1) $\phi_u\ge 0$;
(2) $\phi_{tu}=t^2\phi_u$;
(3) $\|\phi_u\|^2_{D^{1,2}}=\int_{\mathbb{R}^3}\phi_uu^2dx\le C_6|u|^4_{12/5}\le C_7\|u\|^4$, *where* C_6, C_7 *are constants.*

We now consider the modified equation of (1.1) given by

$$\begin{cases}-\Delta u+V(x)u+\phi u=\lambda g(u), & \text{in } \mathbb{R}^3,\\ -\Delta\phi=u^2,\ \lim_{|x|\to+\infty}\phi(x)=0, & \text{in } \mathbb{R}^3.\end{cases} \tag{2.2}$$

By definition of G and Lemma 2.1, for $u \in X$, the functional associated to (2.2) given by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2dx - \lambda\int_{\mathbb{R}^3}G(u)dx \tag{2.3}$$

is well-defined.

Noticing that we can not ensure that the modified nonlinearity g satisfies the classical Ambrosetti-Rabinowitz condition, the boundedness of Palais-Smale sequence seems hard to prove. The following abstract result [11] is used to construct a special Palais-Smale sequence.

Proposition 1. *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and let $\mathfrak{J} \subset \mathbb{R}^+$ be an interval. $\{\Phi_\mu\}_{\mu\in\mathfrak{J}}$ are C^1-functionals on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \textit{ for all } \mu \in \mathfrak{J},$$

where $B(u) \geq 0$ for all $u \in X$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to +\infty$. Suppose that there exist two points $u_1, u_2 \in X$ such that

$$c_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\mu(\gamma(t)) > \max\{\Phi_\mu(u_1), \Phi_\mu(u_2)\}, \textit{ for all } \mu \in \mathfrak{J},$$

where $\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = u_1, \ \gamma(1) = u_2\}$. Then, for almost every $\mu \in \mathfrak{J}$, there exists a sequence $\{u_n(\mu)\} \subset X$ such that
(1) $\{u_n(\mu)\}$ is bounded in X,
(2) $\Phi_\mu(u_n(\mu)) \to c_\mu$,
(3) $\Phi'_\mu(u_n(\mu)) \to 0$, in X^*, where X^* is dual space of X.
Furthermore, the map $\mu \to c_\mu$ is continuous from the left.

Consider a family of functionals

$$\begin{aligned} J_{\mu,\lambda}(u) : = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)dx \\ &+ \frac{1}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2dx - \mu\int_{\mathbb{R}^3}\lambda G(u)dx, \quad u \in X. \end{aligned} \tag{2.4}$$

Denote $A(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2dx$, $B(u) = \int_{\mathbb{R}^3}\lambda G(u)dx$ and $\mathfrak{J} = [\frac{1}{2}, 1]$. Then $J_{\mu,\lambda}(u) = A(u) - \mu B(u)$. The next lemma ensures that $J_{\mu,\lambda}$ satisfies all assumptions of Proposition 1.

Lemma 2.3. *Assume that (V_1) and $(f_0) - (f_2)$ hold. For all $u \in X$, then*
(1) $B(u) \geq 0$;
(2) $A(u) \to \infty$ as $\|u\| \to \infty$;
(3) there exists $u_0 \in X$, independent of μ, such that $J_{\mu,\lambda}(u_0) < 0$ for all $\mu \in [\frac{1}{2}, 1]$;
(4) for all $\mu \in [\frac{1}{2}, 1]$, it holds

$$c_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J_{\mu,\lambda}(\gamma(t)) > \max\{J_{\mu,\lambda}(\gamma(0)), J_{\mu,\lambda}(\gamma(1))\},$$

where $\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \ \gamma(1) = u_0\}$.

Proof. From Lemma 2.1-(1) and (V_1), (1) and (2) are proved directly. To prove (3), let us fix some nonnegative radially symmetric function $e(x) \in C_0^\infty(\mathbb{R}^3) \setminus \{0\}$. Then, for $t > 0$, we have

$$\begin{aligned} J_{1/2,\lambda}(te) &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla e|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)e^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_e(x)e^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} \lambda G(te)dx \\ &\leq \frac{t^4}{2}\left(\frac{1}{t^2}\|e\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} \phi_e(x)e^2 dx - \int_{\mathbb{R}^3} \frac{\lambda G(te)}{t^4} dx\right). \end{aligned} \tag{2.5}$$

By Lemma 2.1-(2), it is easy to see that $J_{1/2,\lambda}(te) < 0$ for t large.

It remains to prove (4). By Lemma 2.1-(3) and the Sobolev embedding theorem, we have

$$\begin{aligned} J_{\mu,\lambda}(u) &\geq \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx - \mu \int_{\mathbb{R}^3} \lambda G(u)dx \\ &\geq \frac{1}{2}\|u\|^2 - C_8\|u\|^\alpha. \end{aligned}$$

From this, we get $c_\mu > 0$ and complete the proof. □

Remark 2.1. By Lemma 2.3 and Proposition 1, then for almost every $\mu \in [\frac{1}{2}, 1]$, there exists a sequence $\{u_n\} \subset X$ satisfying

$$\{u_n\} \text{ is bounded in } X,\ J_{\mu,\lambda}(u_n) \to c_\mu \text{ and } J'_{\mu,\lambda}(u_n) \to 0 \text{ in } X^*. \tag{2.6}$$

Lemma 2.4. *The sequence $\{u_n\}$ given in (2.6), up to subsequence, converges to a positive critical point u_μ of $J_{\mu,\lambda}$ with $J_{\mu,\lambda}(u_\mu) = c_\mu$.*

Proof. Since $\{u_n\}$ is bounded in X, we have

$$u_n \rightharpoonup u_\mu \text{ in } X,\ \ u_n \to u_\mu \text{ in } L^\alpha(\mathbb{R}^3),\ \ u_n \to u_\mu \text{ a.e. in } \mathbb{R}^N,$$

for some $u_\mu \in X$. For all $\varphi \in C_0^\infty(\mathbb{R}^3)$, using Lebesgue's Theorem, we have that

$$\langle J'_{\mu,\lambda}(u_n) - J'_{\mu,\lambda}(u_\mu), \varphi\rangle \to 0,$$

where we used $u_n \rightharpoonup u_\mu$ in X, Lemma 2.1-(3) and $u_n \to u_\mu$ in $L^\alpha(\mathbb{R}^3)$. Thus recalling that $J'_{\mu,\lambda}(u_n) \to 0$ we indeed have $J'_{\mu,\lambda}(u_\mu) = 0$.

Next we prove $u_n \to u_\mu$ in X. Using Lemma 2.1-(3) and the fact $u_n \to u_\mu$ in $L^\alpha(\mathbb{R}^3)$, we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} (g(u_n) - g(u_\mu))(u_n - u_\mu)dx = 0.$$

Hence

$$\begin{aligned} o_n(1) =&\langle J'_{\mu,\lambda}(u_n) - J'_{\mu,\lambda}(u_\mu), u_n - u_\mu\rangle \\ =& \int_{\mathbb{R}^3} \left(\nabla u_n - \nabla u_\mu\right)^2 dx + \int_{\mathbb{R}^3} V(x)(u_n - u_\mu)^2 dx \\ &+ \int_{\mathbb{R}^3} \left(\phi_{u_n} u_n - \phi_{u_\mu} u_\mu\right)(u_n - u_\mu)dx \\ &- \mu \int_{\mathbb{R}^3} \lambda(g(u_n) - g(u_\mu))(u_n - u_\mu)dx \\ =&\|u_n - u_\mu\|^2 + o_n(1), \end{aligned} \tag{2.7}$$

where we used the elementary inequalities

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\left(\phi_{u_n}u_n-\phi_{u_\mu}u_\mu\right)(u_n-u_\mu)dx\right|\\
&\le\left|\int_{\mathbb{R}^3}\phi_{u_n}(u_n-u_\mu)^2dx\right|+\left|\int_{\mathbb{R}^3}\left(\phi_{u_n}-\phi_{u_\mu}\right)u_\mu(u_n-u_\mu)dx\right| \qquad (2.8)\\
&\le|\phi_{u_n}|_6|u_n-u_\mu|_{12/5}^2+|\phi_{u_n}-\phi_{u_\mu}|_6|u_n-u_\mu|_{12/5}|u|_{12/5}.
\end{aligned}
$$

Therefore, $u_n \to u_\mu$ in X. The positivity of u_μ follows by a standard argument (see [19]). □

To show the above results are true when $\mu = 1$, we need the following remark and lemmas.

Remark 2.2. Assume that (V_1) and $(f_0)-(f_2)$ hold. Then there exist $\{\mu_n\}\subset[\frac{1}{2},1]$ and $\{u_{\mu_n}\}\subset X\setminus\{0\}$ such that $\lim_{n\to+\infty}\mu_n=1$, $u_{\mu_n}>0$, $J_{\mu_n,\lambda}(u_{\mu_n})=c_{\mu_n}\le c_{\frac{1}{2}}$ and $J'_{\mu_n,\lambda}(u_{\mu_n})=0$.

Lemma 2.5.(See [12]) If $u\in X$ is a critical point of $J_{\mu,\lambda}$ and (V_1) holds, then

$$
\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{3}{2}\int_{\mathbb{R}^3}V(x)u^2dx+\frac{5}{4}\int_{\mathbb{R}^3}\phi_u u^2dx&\\
+\frac{1}{2}\int_{\mathbb{R}^3}\nabla V(x)\cdot xu^2dx-3\mu\int_{\mathbb{R}^3}\lambda G(u)dx=0.& \qquad (2.9)
\end{aligned}
$$

Lemma 2.6. *Assume that (V_2) and $(f_0)-(f_3)$ hold. Then the sequence $\{u_{\mu_n}\}$ obtained in Remark 2.2 is bounded with respect to μ_n in X.*

Proof. Using the fact $J_{\mu_n,\lambda}(u_{\mu_n})\le c_{\frac{1}{2}}$, $\langle J'_{\mu_n,\lambda}(u_{\mu_n}),u_{\mu_n}\rangle=0$ and Lemma 2.5, we have

$$
\left(3-\frac{\gamma}{2}\right)c_{\frac{1}{2}}\ge\left(3-\frac{\gamma}{2}\right)J_{\mu_n,\lambda}(u_{\mu_n})-\langle J'_{\mu_n,\lambda}(u_{\mu_n}),u_{\mu_n}\rangle+\left(\frac{\gamma}{2}-1\right)\cdot(2.9)_L
$$

and

$$
\begin{aligned}
(3-\frac{\gamma}{2})c_{\frac{1}{2}}&\ge(\frac{\gamma}{4}-\frac{1}{2})\int_{\mathbb{R}^3}(2V(x)+\nabla V(x)\cdot x)u_{\mu_n}^2dx\\
&\quad+(\frac{\gamma}{2}-\frac{3}{2})\int_{\mathbb{R}^3}\phi_{u_{\mu_n}}u_{\mu_n}^2dx+\lambda\int_{\mathbb{R}^3}\left(u_{\mu_n}g(u_{\mu_n})-\gamma G(u_{\mu_n})\right)dx \qquad (2.10)\\
&\ge(\frac{\gamma}{2}-\frac{3}{2})\int_{\mathbb{R}^3}\phi_{u_{\mu_n}}u_{\mu_n}^2dx.
\end{aligned}
$$

Using (V_2), Lemma 2.1 and the fact that $3<\gamma\le 4$, it implies that $\{\int_{\mathbb{R}^3}\phi_{u_{\mu_n}}u_{\mu_n}^2dx\}$ is bounded.

Next we prove that $\|u_{\mu_n}\|$ is bounded. By $\langle J'_{\mu_n,\lambda}(u_{\mu_n}),u_{\mu_n}\rangle=0$, we obtain

$$
\begin{aligned}
\gamma c_{\frac{1}{2}}&\ge\left(\frac{\gamma}{2}-1\right)\int_{\mathbb{R}^3}\left(|\nabla u_{\mu_n}|^2+V(x)u_{\mu_n}\right)dx+(\frac{\gamma}{4}-1)\int_{\mathbb{R}^3}\phi_{u_{\mu_n}}u_{\mu_n}^2dx\\
&\quad+\int_{\mathbb{R}^3}\left(u_{\mu_n}g(u_{\mu_n})-\gamma G(u_{\mu_n})\right)dx\\
&\ge\left(\frac{\gamma}{2}-1\right)\int_{\mathbb{R}^3}\left(|\nabla u_{\mu_n}|^2+V(x)u_{\mu_n}\right)dx-\frac{(4-\gamma)(6-\gamma)}{4\gamma-12}c_{\frac{1}{2}},
\end{aligned}
$$

then we complete the proof. □

Lemma 2.7. *Assume that* (V_1), (V_2) *and* $(f_0)-(f_3)$ *hold. Then problem (2.2) has at least one positive solution.*

Proof. Using Remark 2.2 and Lemma 2.6, there exist $\{\mu_n\} \subset [\frac{1}{2}, 1]$ and a bounded sequence $\{u_{\mu_n}\} \subset X \setminus \{0\}$ such that

$$\lim_{n\to+\infty} \mu_n = 1, \;\; J_{\mu_n,\lambda}(u_{\mu_n}) = c_{\mu_n}, \;\; J'_{\mu_n,\lambda}(u_{\mu_n}) = 0.$$

Furthermore

$$\lim_{n\to\infty} J_\lambda(u_{\mu_n}) = \lim_{n\to\infty}\left(J_{\mu_n,\lambda}(u_{\mu_n}) + (\mu_n - 1)\int_{\mathbb{R}^3} \lambda G(u_{\mu_n})dx\right) = \lim_{n\to\infty} c_{\mu_n} = c_1,$$

where we used the fact that $\mu \mapsto c_\mu$ is continuous from the left. By the similar argument, we get

$$J'_\lambda(u_{\mu_n}) \to 0 \text{ in } X^*.$$

Thus $\{u_{\mu_n}\}$ is a bounded Palais-Smale sequence for J_λ and $\lim_{n\to\infty} J_\lambda(u_{\mu_n}) = c_1$. By the argument of Lemma 2.4 again, we complete the proof. □

Indeed, the critical points of J_λ with L^∞-norm not more than δ are also the weak solutions of problem (1.1). So next we shall study the L^∞-estimates for solutions of (2.2), which is essentially contained in the work of Brezis-Kato.

Lemma 2.8. *Assume* (V_1), (V_2), $(f_0)-(f_3)$ *hold and* $u \in X$ *is a weak solution of problem (2.2). Then* $u \in L^\infty(\mathbb{R}^3)$. *Moreover,*

$$|u|_\infty \le C_9 \lambda^{\frac{1}{2^*-\alpha}} \|u\|^{\frac{2^*-2}{2^*-\alpha}}, \tag{2.11}$$

where $C_9 > 0$ *only depends on* α.

Proof. Let $u \in X$ be a weak solution of

$$\begin{cases} -\Delta u + V(x)u + \phi u = \lambda g(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \; \lim\limits_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{2.12}$$

which is equivalent to

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla\varphi dx + \int_{\mathbb{R}^3} V(x)u\varphi dx + \int_{\mathbb{R}^3} \phi_u u\varphi dx - \lambda\int_{\mathbb{R}^3} g(u)\varphi dx = 0, \text{ for all } \varphi \in X. \tag{2.13}$$

From the above Lemma 2.7, we know that $u > 0$. Let $T > 0$, and define

$$u_T = \begin{cases} u, \text{ if } 0 < u \le T, \\ T, \text{ if } u \ge T. \end{cases}$$

Choosing $\varphi = u_T^{2(\eta-1)}u$ in (2.13), where $\eta > 1$, we get

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u|^2 \cdot u_T^{2(\eta-1)} dx + 2(\eta-1)\int_{\{x|u(x)<T\}} u_T^{2(\eta-1)-1} u|\nabla u|^2 dx \\ &+ \int_{\mathbb{R}^3} \phi_u(x)u^2 u_T^{2(\eta-1)} dx + \int_{\mathbb{R}^3} V(x)u^2 u_T^{2(\eta-1)} dx \\ =&\lambda\int_{\mathbb{R}^3} g(u)u_T^{2(\eta-1)} u dx. \end{aligned}$$

Combining Lemma 2.1-(3) and the nonnegativity of the second, the third and the fourth terms in the left side of the above equation, we obtain

$$\int_{\mathbb{R}^3} |\nabla u|^2 \cdot u_T^{2(\eta-1)} dx \le \lambda \int_{\mathbb{R}^3} g(u) u_T^{2(\eta-1)} u dx \le \lambda C_5 \int_{\mathbb{R}^3} u^\alpha u_T^{2(\eta-1)} dx. \tag{2.14}$$

On the other hand, by the Sobolev inequality, we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^3} (u u_T^{\eta-1})^{2^*} dx \right)^{\frac{2}{2^*}} &\le C_{10} \int_{\mathbb{R}^3} |\nabla (u u_T^{\eta-1})|^2 dx \\ &\le C_{11} \int_{\mathbb{R}^3} |\nabla u|^2 u_T^{2(\eta-1)} dx + C_{10}(\eta-1)^2 \int_{\mathbb{R}^3} |\nabla u|^2 u_T^{2(\eta-1)} dx \\ &\le C_{12} \eta^2 \int_{\mathbb{R}^3} |\nabla u|^2 u_T^{2(\eta-1)} dx, \end{aligned}$$

where we used the fact that $(a+b)^2 \le 2(a^2+b^2)$.

By (2.14), the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^3} (u u_T^{\eta-1})^{2^*} dx \right)^{\frac{2}{2^*}} &\le \lambda C_{13} \eta^2 \int_{\mathbb{R}^3} u^{\alpha-2} u^2 u_T^{2(\eta-1)} dx \\ &\le \lambda C_{13} \eta^2 \left(\int_{\mathbb{R}^3} u^{2^*} dx \right)^{\frac{\alpha-2}{2^*}} \left(\int_{\mathbb{R}^3} (u u_T^{\eta-1})^{\frac{22^*}{2^*-\alpha+2}} dx \right)^{\frac{2^*-\alpha+2}{2^*}} \\ &\le \lambda C_{14} \eta^2 \|u\|^{\alpha-2} \left(\int_{\mathbb{R}^3} u^{\frac{\eta 22^*}{2^*-\alpha+2}} dx \right)^{\frac{2^*-\alpha+2}{2^*}}, \end{aligned}$$

where we used the fact that $0 \le u_T \le u$.

In what follows, taking $\zeta = \frac{22^*}{2^*-\alpha+2}$, we get

$$\left(\int_{\mathbb{R}^3} (u u_T^{\eta-1})^{2^*} dx \right)^{\frac{2}{2^*}} \le \lambda C_{14} \eta^2 \|u\|^{\alpha-2} |u|_{\eta\zeta}^{2\eta}.$$

Using the Fatou's lemma, letting $T \to +\infty$, it follows that

$$|u|_{\eta 2^*} \le (\lambda C_{14} \eta^2 \|u\|^{\alpha-2})^{\frac{1}{2\eta}} |u|_{\eta\zeta}. \tag{2.15}$$

Define $\eta_{n+1}\zeta = 2^*\eta_n$, where $n = 0, 1, 2, \ldots$ and $\eta_0 = \frac{2^*+2-\alpha}{2}$. By (2.15) we have

$$\begin{aligned} |u|_{\eta_1 2^*} &\le (\lambda C_{14} \eta_1^2 \|u\|^{\alpha-2})^{\frac{1}{2\eta_1}} |u|_{2^*\eta_0} \\ &\le (\lambda C_{14} \|u\|^{\alpha-2})^{\frac{1}{2\eta_1}+\frac{1}{2\eta_0}} \eta_0^{\frac{1}{\eta_0}} \eta_1^{\frac{1}{\eta_1}} |u|_{2^*}. \end{aligned}$$

By iteration we have

$$|u|_{\eta_n 2^*} \le (\lambda C_{14} \|u\|^{\alpha-2})^{\frac{1}{2\eta_0} \sum_{i=0}^n (\frac{\zeta}{2^*})^i} (\eta_0)^{\frac{1}{\eta_0} \sum_{i=0}^n (\frac{\zeta}{2^*})^i} (\frac{2^*}{\zeta})^{\frac{1}{\eta_0} \sum_{i=0}^n i (\frac{\zeta}{2^*})^i} |u|_{2^*}.$$

Thus, we obtain $|u|_\infty \le C_9 \lambda^{\frac{1}{2^*-\alpha}} \|u\|^{\frac{2^*-2}{2^*-\alpha}}$. □

By the similar argument in Lemma 2.6, we can obtian the following lemma.

Lemma 2.9. *Let $\lambda > \frac{V_L}{2}$ and u_λ be a critical point of J_λ with $J_\lambda(u_\lambda) = c_1$. Then there exists $C_{16} > 0$ (independent of λ) such that*

$$||u_\lambda||^2 \leq C_{16}c_1. \tag{2.16}$$

Proof of Theorem 1.1. Let $u_0 \in C_0^\infty(\mathbb{R}^N) \cap X \setminus \{0\}$ be a nonnegative function such that $J_\lambda(u_0) < 0$. Then, the functional J_λ has the mountain pass geometry and we can define

$$d_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \ \gamma(1) = u_0\}$. By Lemma 2.7, there exists a positive critical point u_λ of J_λ with $J_\lambda(u_\lambda) = d_\lambda$. And more remarkable $d_\lambda = c_1$.

Furthermore, taking $T = |u_0|_\infty$, from Lemma 2.1-(4), we obtain

$$\begin{aligned}
d_\lambda &\leq \max_{t\in[0,1]} J_\lambda(tu_0) \\
&\leq \max_{t\in[0,1]} \left(\frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla u_0|^2 + V(x)u_0^2)dx + \frac{t^4}{4} \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx - \lambda \int_{\mathbb{R}^3} G(tu_0)dx \right) \\
&\leq \max_{t\in[0,1]} \left(\frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla u_0|^2 + V(x)u_0^2 + \frac{1}{2}\phi_{u_0} u_0^2)dx - C_{17}\lambda t^\beta \int_{\mathbb{R}^3} u_0^\beta dx \right) \\
&\leq C_{18}\lambda^{-\frac{2}{\beta-2}}.
\end{aligned} \tag{2.17}$$

By (2.11), (2.16) and (2.17), we see that

$$|u_\lambda|_\infty \leq C_{19}\lambda^{\frac{\beta-2^*}{2^*-\alpha}}.$$

Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, we get

$$|u_\lambda|_\infty \leq \delta,$$

where δ is fixed in (2.1). Thus, the u_λ is a positive solution of the original problem (1.1).

3. Proof of Theorem 1.2

We start by finding that the conditions (f_1') and (f_2') imply the existence of positive constants C_{20}, C_{21} such that

$$F(t) \leq C_{20}|t|^\alpha \tag{3.1}$$

and

$$F(t) \geq C_{21}|t|^\beta \tag{3.2}$$

with $|t|$ small. Consider $\rho(t) \in C^1(\mathbb{R}, \mathbb{R})$ an even cut-off function satisfying:

$$\rho(t) = \begin{cases} 1, & \text{if } |t| \leq \delta, \\ 0, & \text{if } |t| \geq 2\delta, \end{cases}$$

$0 \leq \rho(t) \leq 1$, $t\rho'(t) \leq 0$ and $|t\rho'(t)| \leq \frac{2}{\delta}$, where $0 < \delta < \frac{1}{2}$ is chosen such that (3.1), (3.2) hold for $|t| \leq 2\delta$ and (f_4) holds for $|t| \leq \delta$. Define

$$H(t) = \rho(t)F(t) + (1 - \rho(t))\widetilde{F}_\infty(t) \ \text{ and } \ h(t) = H'(t),$$

where $\widetilde{F}_\infty(t) = C_{22}|t|^\alpha$. From (f_1') and the definitions of $\rho(t)$ and $h(t)$, for $u \in X$ we have

$$|h(u)| \leq C_{23}|u|^{\alpha-1}. \tag{3.3}$$

Lemma 3.1.(See [8]) If f satisfies $(f_1') - (f_3')$, then for all $t \neq 0$, we have

$$0 < \theta H(t) \leq th(t),$$

where $\theta = \min\{\alpha, \widetilde{\gamma}\}$.

We now consider another modified equation of (1.1) given by

$$\begin{cases} -\Delta u + V(x)u + \phi u = \lambda h(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \ \lim\limits_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{3.4}$$

By the definition of $H(t)$ and (3.3), the functional associated to (3.4) stated by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \lambda\int_{\mathbb{R}^3} H(u)dx, \ u \in X \tag{3.5}$$

is well-defined. It is well known that its critical points are the weak solutions of (3.4).

The goal of this section is to prove Theorem 1.2. To this end, we use the Lemma 3.1 to get the boundedness of Palais-Smale sequence. Moreover, by the similar argument in Lemma 2.4, it is easy to show that I_λ satisfies Palais-Smale condition. These are standard results which can be found in textbooks and no proof is given here.

Lemma 3.2. *Assume that $(f_1') - (f_3')$ are satisfied. If $u \in X$ is a critical point of I_λ, then*

$$||u||^2 \leq C_{24} I_\lambda(u), \tag{3.6}$$

where C_{24} depends on θ.

The proof of the above result is quite similar to the one used in Lemma 2.6 and so is omitted.

Since X is a real, reflexive, and separable Banach space, there exists $\{e_j\}_{j\in\mathbb{N}} \subset X$ such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, ...\}}.$$

We denote

$$Y_k = \text{span}\{e_1, ..., e_k\}, \qquad Z_k = \overline{\text{span}\{e_{k+1}, ...\}}.$$

Lemma 3.3. *Set $\theta_{k,\lambda} = \sup\limits_{u\in Y_k} I_\lambda(u)$. If $\lambda > 1$, then*

$$\theta_{k,\lambda} \leq C_{25}\lambda^{-\frac{2}{\beta-2}}, \tag{3.7}$$

where C_{25} depends on α, β and k.

Proof. We notice that $\delta > 0$ was chosen in Section 3 such that

$$H(t) = \widetilde{F}_\infty(t) = C_{22}|t|^\alpha \text{ for } |t| \geq 2\delta, \tag{3.8}$$

$$H(t) \geq F(t) \geq C_{26}|t|^\beta \text{ for } |t| \leq 2\delta. \tag{3.9}$$

For $u \in Y_k$, denote $\Omega_1 = \{x \in \mathbb{R}^3 : |u(x)| \geq 2\delta\}$, $\Omega_2 = \{x \in \mathbb{R}^3 : |u(x)| < 2\delta\}$, and let $u_1 = u|_{\Omega_1}$, $u_2 = u|_{\Omega_2}$. Since all norms in Y_k are equivalent, from (3.8) and (3.9) we obtain

$$\int_{\mathbb{R}^3} H(u_1)dx \geq \widehat{C}(k)|u_1|_2^{\alpha}$$

and

$$\int_{\mathbb{R}^3} H(u_2)dx \geq \widehat{C}(k)|u_2|_2^{\beta},$$

where $\widehat{C}(k)$ is a positive constant. By the same reason, we can define

$$\gamma_k = \sup\{\|u\| : u \in Y_k,\ |u|_2 = 1\} < \infty.$$

Then for $u \in Y_k$, it follows that

$$\begin{aligned}
I_\lambda(u) &\leq \frac{\gamma_k}{2}|u|_2^2 + \frac{C_7}{4}\|u\|^4 - \lambda\widehat{C}(k)|u_1|_2^{\alpha} - \lambda\widehat{C}(k)|u_2|_2^{\beta} \\
&\leq \frac{\gamma_k}{2}|u_1|_2^2 + \frac{C_7\gamma_k}{4}|u_1|_2^4 - \lambda\widehat{C}(k)|u_1|_2^{\alpha} + \frac{\gamma_k}{2}|u_2|_2^2 + \frac{C_7\gamma_k}{4}|u_2|_2^4 - \lambda\widehat{C}(k)|u_2|_2^{\beta} \\
&\leq \overline{C}_1(k)\lambda^{-\frac{2}{\alpha-2}} + \overline{C}_2(k)\lambda^{-\frac{2}{\beta-2}}.
\end{aligned}$$

For $\lambda > 1$, by $4 < \alpha \leq \beta < 2^*$, we have

$$\theta_{k,\lambda} \leq C_{25}\lambda^{-\frac{2}{\beta-2}},$$

where $C_{25} := \overline{C}_1(k) + \overline{C}_2(k)$. □

Lemma 3.4. *Assume that $u \in X$ is a weak solution of problem (3.4). Then $u \in L^\infty(\mathbb{R}^N)$. Moreover,*

$$|u|_\infty \leq C_{27}\lambda^{\frac{1}{2^*-\alpha}}\|u\|^{\frac{2^*-2}{2^*-\alpha}}, \tag{3.10}$$

where $C_{27} > 0$ only depends on α.

The proof of Lemma 3.4 is quite similar to Lemma 2.8 and so is omitted.

To prove Theorem 1.2, we will apply the following symmetric mountain pass theorem due to Rabinowitz [16].

Proposition 2. *Let X be an infinite dimensional Banach space, $J \in C^1(X, \mathbb{R})$ be even, satisfy (PS) condition and $J(0) = 0$. If $X = Y \bigoplus Z$ with $\dim Y < +\infty$, and J satisfies*
(1) there are constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho \cap Z} \geq \alpha$,
(2) for any finite dimensional subspace $W \subset X$, there is an $R = R(W)$ such that $J \leq 0$ on $W \setminus B_{R(W)}$,
then J has a sequence of critical values.

Remark 3.1. We point out that I_λ satisfies all assumptions of Proposition 2. By Proposition 2, then I_λ possesses a sequence of critical points.

Proof of Theorem 1.2. Fix an integer k. Choose $R > 0$ such that $I_\lambda(u) \leq 0$ for all $u \in Y_k$ with $\|u\| \geq R$, and for all $\lambda \geq 1$. For $B_R = \{u \in X : \ \|u\| < R\}$, let $D = B_R \cap Y_k$. Define

$$\Gamma := \{\gamma \in C(D, X) : \ \gamma \text{ is odd}, \gamma(u) = u, \text{if } \|u\| = R\}.$$

Let $i(A)$ be the genus of symmetric subset A. For $j \leq k$, we denote

$$\Theta_j = \{\gamma(\overline{D \setminus B}) : \ \gamma \in \Gamma,\ i(B) \leq k - j\}$$

and

$$c_{j,\lambda} = \inf_{A \in \Theta_j} \sup_{u \in A} I_\lambda(u).$$

From Proposition 2 and Remark 3.1, and under our conditions on h, we get

$$0 < c_{1,\lambda} \le c_{2,\lambda} \le ... \le c_{k,\lambda}.$$

Moreover, they are also critical values of I_λ and there exist at least $2k$ critical points $\{\pm u_{j,\lambda}\}_{j=1}^{k}$ at these critical values. Since $Id \in \Gamma$, the definition of Θ_j and Lemma 3.3, we obtain

$$c_{j,\lambda} \le \theta_{j,\lambda} \le C(j)\lambda^{-\frac{2}{\beta-2}}. \tag{3.11}$$

By Lemmas 3.2–3.4, we have

$$|u_{j,\lambda}|_\infty \le C_{28}\lambda^{\frac{\beta-2^*}{(\beta-2)(2^*-\alpha)}}.$$

Since $4 < \beta < 2^*$, there exists $\lambda_1 > 0$ such that $|u_{j,\lambda}|_\infty \le \delta$, for all $\lambda > \lambda_1$. Thus, $\pm u_{j,\lambda}(j = 1, 2, \cdots, k)$ are weak solutions of problem (1.1).

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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