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Research article

On n-Polynomial convexity and some related inequalities

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Abstract: In this paper, we introduce and study the concept of n-polynomial convexity functions and their some algebric properties. We prove two Hermite-Hadamard type inequalities for the newly introduced class of functions. In addition, we obtain some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is n-polynomial convexity. Also, we compare the results obtained with both Hölder, Hölder-İşcan inequalities and power-mean, improved-power-mean integral inequalities and show that the result obtained with Hölder-İşcan and improved power-mean inequalities give better approach than the others. Some applications to special means of real numbers are also given.

Keywords: convex function; n-polynomial convexity; Hölder integral inequality; Hölder-İşcan integral inequality; Hermite-Hadamard inequality
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1. Preliminaries

In this paper, a new class of functions, which is a special state of n-polynomial convexity, has been defined and some algebraic properties of this class of function have been investigated. In addition, some Hermite-Hadamard type inequalities were obtained.

A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [1–5] and the references therein.

Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double inequality is well known as the Hermite-Hadamard inequality [6]. Some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [7, 8]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f.

Definition 1. *[9] Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0, 1)$ we have*

$$f(\alpha x + (1-\alpha)y) \le h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [10, 11] for studies on h-convexity.

In [12], İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 1 (Hölder-İşcan integral inequality [12])**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \quad &\le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}\end{aligned} \tag{1.2}$$

An refinement of power-mean integral inequality as a different version of the Hölder-İşcan integral inequality can be given as follows:

Theorem 2 (Improved power-mean integral inequality [13])**.** *Let $q \ge 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a, b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \quad &\le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}\end{aligned}$$

The main purpose of this paper is to introduce the concept of n-polynomial convex functions and establish some results connected with the right-hand side of new inequalities similar to the Hermite-hadamard inequality for these classes of functions. Some applications to special means of positive real numbers are also given.

2. The definition of n-polynomial convex functions

In this section, we introduce a new concept, which is called n-polynomial convexity and we give by setting some algebraic properties for the n-polynomial convex functions, as follows:

Definition 2. *Let $n \in \mathbb{N}$. A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial convex function if for every $x, y \in I$ and $t \in [0,1]$,*

$$f(tx + (1-t)y) \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s] f(x) + \frac{1}{n}\sum_{s=1}^{n}[1-t^s] f(y). \tag{2.1}$$

We will denote by $POLC(I)$ the class of all n-polynomial convex functions on interval I.

We note that, every n-polynomial convex function is an h-convex function with the function $h(t) = \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]$. Therefore, if $f, g \in POLC(I)$, then

i.) $f + g \in POLC(I)$ *and for* $c \in \mathbb{R}$ *(*$c \geq 0$*)* $cf \in POLC(I)$ *(see [9], Proposition 9).*

ii.) if f and g be a similarly ordered functions on I , then $fg \in POLC(I)$ *.(see [9], Proposition 10).*

Also, if $f : I \to J$ is a convex and $g \in POLC(J)$ and nondecreasing, then $g \circ f \in POLC(I)$ *(see [9], Theorem 15).*

Remark 1. *We especially note that; if we take $n = 1$ in the inequality (2.1), then the 1-polynomial convexity reduces to the clasical convexity.*

Remark 2. *Let the function $f : I \subset \mathbb{R} \to [0,\infty)$ be a 2-polynomial convex function if for every $x, y \in I$ and $t \in [0,1]$,*

$$f(tx + (1-t)y) \leq \frac{3t-t^2}{2} f(x) + \frac{2-t-t^2}{2} f(y).$$

It is easily seen that

$$t \leq \frac{3t-t^2}{2} \text{ and } 1-t \leq \frac{2-t-t^2}{2}$$

for all $t \in [0,1]$. This shows that every nonnegative convex function is also a 2-polynomial convex function.

More generally, we can give the following remark together with proof:

Remark 3. *Every nonnegative convex function is also an n-polynomial convex function. Indeed, this case is clear from the following inequalities*

$$t \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s] \text{ and } 1-t \leq \frac{1}{n}\sum_{s=1}^{n}[1-t^s]$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. Now, we will prove that the inequality

$$t \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]$$

holds for all $t \in [0,1]$ and $n \in \mathbb{N}$: The following inequality is well known as Bernoulli's inequality in mathematical analysis $(1-t)^n \geq 1 - nt$ for all $t \in [0,1]$ and $n \in \mathbb{N}$. From the above inequality, we get

$$\frac{1}{n}\sum_{s=1}^{n}(1-t)^{s-1} = \frac{1-(1-t)^s}{nt} \leq 1$$

and thus

$$n(1-t)\left[-1+\frac{1}{n}\sum_{s=1}^{n}(1-t)^{s-1}\right]=-n(1-t)+\sum_{s=1}^{n}(1-t)^{s}\leq 0$$

then we have

$$t\leq\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^{s}\right].$$

The cases of $t=0$ *and* $t=1$ *are clear.*

Example 1. *In case of* $n=2$*, the function* $f:[0,\infty)\to\mathbb{R}$*,* $f(x)=x$ *is a* 2*-polynomial convex.*

Theorem 3. *Let* $b>0$ *and* $f_\alpha:[a,b]\to\mathbb{R}$ *be an arbitrary family of* n*-polynomial convex functions and let* $f(x)=\sup_\alpha f_\alpha(x)$*. If* $J=\{u\in[a,b]:f(u)<\infty\}$ *is nonempty, then* J *is an interval and* f *is an* n*-polynomial convex function on* J*.*

Proof. Let $t\in[0,1]$ and $x,y\in J$ be arbitrary. Then

$$\begin{aligned}
f(tx+(1-t)y) &= \sup_\alpha f_\alpha(tx+(1-t)y)\\
&\leq \sup_\alpha\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^{s}\right]f_\alpha(x)+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^{s}\right]f_\alpha(y)\right]\\
&\leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^{s}\right]\sup_\alpha f_\alpha(x)+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^{s}\right]\sup_\alpha f_\alpha(y)\\
&= \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^{s}\right]f(x)+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^{s}\right]f(y)\\
&< \infty.
\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is an n-polynomial convex function on J. This completes the proof of theorem. □

3. Hermite-Hadamard inequality for n-polynomial convex functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for n-polynomial convex functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 4. *Let* $f:[a,b]\to\mathbb{R}$ *be an* n*-polynomial convex function. If* $a<b$ *and* $f\in L[a,b]$*, then the following Hermite-Hadaamrd type inequalities hold:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\left(\frac{f(a)+f(b)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}. \tag{3.1}$$

Proof. From the propery of the n-polynomial convex function of f, we get

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{[ta+(1-t)b]+[(1-t)a+tb]}{2}\right)$$

$$
\begin{aligned}
&= f\left(\frac{1}{2}[ta+(1-t)b]+\frac{1}{2}[(1-t)a+tb]\right)\\
&\le \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(1-\frac{1}{2}\right)^{s}\right]f(ta+(1-t)b)+\frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^{s}\right]f((1-t)a+tb)\\
&= \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^{s}\right][f(ta+(1-t)b)+f((1-t)a+tb)].
\end{aligned}
$$

By taking integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^{s}\right]\left[\int_0^1 f(ta+(1-t)b)\,dt+\int_0^1 f((1-t)a+tb)\,dt\right]\\
&= \frac{2}{b-a}\left(\frac{n+2^{-n}-1}{n}\right)\int_a^b f(x)dx.
\end{aligned}
$$

By using the property of the n-polynomial convex function f, if the variable is changed as $x = ta + (1-t)b$, then

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)du &= \int_0^1 f(ta+(1-t)b)\,dt\\
&\le \int_0^1\left[\frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^{s}]f(a)+\frac{1}{n}\sum_{s=1}^{n}[1-t^{s}]f(b)\right]dt\\
&= \frac{f(a)}{n}\int_0^1\sum_{s=1}^{n}[1-(1-t)^{s}]\,dt+\frac{f(b)}{n}\int_0^1\sum_{s=1}^{n}[1-t^{s}]\,dt\\
&= \frac{f(a)}{n+1}\sum_{s=1}^{n}\int_0^1[1-(1-t)^{s}]\,dt+\frac{f(b)}{n}\sum_{s=1}^{n}\int_0^1[1-t^{s}]\,dt\\
&= \frac{f(a)}{n}\sum_{s=1}^{n}\frac{s}{s+1}+\frac{f(b)}{n}\sum_{s=1}^{n}\frac{s}{s+1}\\
&= \left[\frac{f(a)+f(b)}{n}\right]\sum_{s=1}^{n}\frac{s}{s+1},
\end{aligned}
$$

where

$$
\int_0^1[1-(1-t)^{s}]\,dt=\int_0^1[1-t^{s}]=\frac{s}{s+1}.
$$

This completes the proof of theorem. □

Remark 4. *In case of* $n = 1$*, the inequality (3.1) coincides with the the inequality (1.1)*

4. New inequalities for *n*-polynomial convex functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is n-polynomial convex function. Dragomir and Agarwal [14] used the following lemma:

Lemma 1 ([14]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following identity holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt.$$

Theorem 5. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is an n-polynomial convex function on interval $[a, b]$, then the following inequality holds for $t \in [0, 1]$.*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right] A\left(|f'(a)|, |f'(b)|\right). \quad (4.1)$$

Proof. Using Lemma 1 and the inequality

$$|f'(ta+(1-t)b)| \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\,|f'(a)| + \frac{1}{n}\sum_{s=1}^{n}[1-t^s]\,|f'(b)|,$$

we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\left(\frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\,|f'(a)| + \frac{1}{n}\sum_{s=1}^{n}[1-t^s]\,|f'(b)|\right)dt \\
\leq\ & \frac{b-a}{2n}\left(|f'(a)|\int_0^1 |1-2t|\sum_{s=1}^{n}[1-(1-t)^s]\,dt + |f'(b)|\int_0^1 |1-2t|\sum_{s=1}^{n}[1-t^s]\,dt\right) \\
=\ & \frac{b-a}{2n}\left(|f'(a)|\sum_{s=1}^{n}\int_0^1 |1-2t|\,[1-(1-t)^s]\,dt + |f'(b)|\sum_{s=1}^{n}\int_0^1 |1-2t|\,[1-t^s]\,dt\right) \\
=\ & \frac{b-a}{2n}\left(|f'(a)|\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right] + |f'(b)|\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right]\right) \\
=\ & \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right]\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \\
=\ & \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right] A\left(|f'(a)|, |f'(b)|\right)
\end{aligned}$$

where

$$\int_0^1 |1-2t|\,[1-(1-t)^s]\,dt = \int_0^1 |1-2t|\,[1-t^s]\,dt = \frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}$$

and A is the arithmetic mean. This completes the proof of theorem. □

Corollary 1. *If we take $n = 1$ in the inequality (4.1), we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right).$$

This inequality coincides with the inequality in [14].

Theorem 6. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and assume that $f' \in L[a,b]$. If $|f'|^q$ is an n-polynomial convex function on interval $[a,b]$, then the following inequality holds for $t \in [0,1]$.*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right) \tag{4.2}$$

Proof. Using Lemma 1, Hölder's integral inequality and the following inequality

$$\left|f'(ta+(1-t)b)\right|^q \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]|f'(a)|^q+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right]|f'(b)|^q$$

which is the n-polynomial convex function of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|dt \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1\left[1-(1-t)^s\right]dt+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1\left[1-t^s\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(|f'(a)|^q\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}+|f'(b)|^q\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right)
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 |1-2t|^p\,dt &= \frac{1}{p+1} \\
\int_0^1 \left[1-(1-t)^s\right]dt &= \int_0^1\left[1-t^s\right]dt = \frac{s}{s+1}
\end{aligned}$$

and A is the arithmetic mean. This completes the proof of theorem. □

Corollary 2. *If we take $n = 1$ in the inequality (4.2), we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).$$

This inequality coincides with the inequality in [14].

Theorem 7. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $q \geq 1$, and assume that $f' \in L[a,b]$. If $|f'|^q$ is an n-polynomial convex function on the interval $[a,b]$, then the following inequality holds for $t \in [0,1]$.*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\; & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).\end{aligned} \tag{4.3}$$

Proof. Assume first that $q > 1$. From Lemma 1, Hölder integral inequality and the property of the n-polynomial convex function of $|f'|^q$, we obtain

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\ &\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]|f'(a)|^q\right.\right. \\ &\quad \left.\left.+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right]|f'(b)|^q\,dt\right]\right)^{\frac{1}{q}} \\ &= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 |1-2t|\left[1-(1-t)^s\right]dt\right. \\ &\quad \left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 |1-2t|\left[1-t^s\right]dt\right]^{\frac{1}{q}} \\ &= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right. \\ &\quad \left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right]^{\frac{1}{q}} \\ &= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\end{aligned}$$

where

$$\int_0^1 |1-2t|dt = \frac{1}{2}$$

$$\int_0^1 |1-2t|\left[1-(1-t)^s\right]dt = \int_0^1 |1-2t|\left[1-(1-t)^s\right]dt = \frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}.$$

For $q=1$ we use the estimates from the proof of Theorem 5, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 3. *Under the assumption of Theorem 7 with $q=1$, we get the conclusion of Theorem 5.*

Corollary 4. *If we take $n=1$ in the inequality (4.3), we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).$$

If we take $q=1$ in the above inequality, then obtained inequality coincides with the inequality in [14].

Now, we will prove the Theorem 6 by using Hölder-İşcan integral inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 6.

Theorem 8. *Let $f: I \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and assume that $f' \in L[a,b]$. If $|f'|^q$ is an n-polynomial convex function on interval $[a,b]$, then the following inequality holds for $t \in [0,1]$.*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (4.4)\\
\leq\ & \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^n \frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^n \frac{s(s+3)}{2(s+1)(s+2)}\right)^{\frac{1}{q}}\\
&+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^n \frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^n \frac{s}{2(s+2)}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Using Lemma 1, Hölder-İşcan integral inequality and the following inequality

$$|f'(ta+(1-t)b)|^q \leq \frac{1}{n}\sum_{s=1}^n \left[1-(1-t)^s\right]|f'(a)|^q + \frac{1}{n}\sum_{s=1}^n \left[1-t^s\right]|f'(b)|^q$$

which is the n-polynomial convex function of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\,|f'\left(ta+(1-t)b\right)|^q\,dt\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{2}\left(\int_0^1 t\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\,|f'\left(ta+(1-t)b\right)|^q\,dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 (1-t)\,[1-(1-t)^s]\,dt\right. \\
&\quad\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 (1-t)\,[1-t^s]\,dt\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 t\,[1-(1-t)^s]\,dt\right. \\
&\quad\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 t\,[1-t^s]\,dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 (1-t)\,|1-2t|^p\,dt &= \int_0^1 t\,|1-2t|^p\,dt = \frac{1}{2\,(p+1)}, \\
\int_0^1 (1-t)\,[1-(1-t)^s]\,dt &= \int_0^1 t\,[1-t^s]\,dt = \frac{s}{2(s+2)}, \\
\int_0^1 (1-t)\,[1-t^s]\,dt &= \int_0^1 t\,[1-(1-t)^s]\,dt = \frac{s(s+3)}{2(s+1)(s+2)}
\end{aligned}$$

This completes the proof of theorem. □

Corollary 5. *If we take $n = 1$ in the inequality (4.4), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+2\,|f'(b)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\,|f'(a)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This inequality coincides with the inequality of Theorem 3.2 in [12].

Remark 5. *The inequality (4.4) gives better results than the inequality (4.2). Let us show that*

$$\frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right)$$

Using concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\leq 1$ by sample calculation we get

$$\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}2\left[\frac{1}{2}\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}+\frac{1}{2}\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right]^{\frac{1}{q}}$$

$$=\frac{b-a}{2}2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(a)|^q\right)$$

which is the required.

Theorem 9. *Let $f:I\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $q\geq 1$, and assume that $f'\in L[a,b]$. If $|f'|^q$ is an n-polynomial convex function on the interval $[a,b]$, then the following inequality holds for $t\in[0,1]$.*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)\right)^{\frac{1}{q}} \quad (4.5)$$

where

$$K_1(s) := \int_0^1 (1-t)\,|1-2t|\,[1-(1-t)^s]\,dt=\int_0^1 t\,|1-2t|\,[1-t^s]\,dt = \frac{\left(s^2+s+2\right)2^s-2}{2^{s+2}(s+2)(s+3)},$$

$$K_2(s) := \int_0^1 t\,|1-2t|\,[1-(1-t)^s]\,dt=\int_0^1 (1-t)\,|1-2t|\,[1-t^s]\,dt = \frac{(s+5)\left[\left(s^2+s+2\right)2^s-2\right]}{2^{s+2}(s+1)(s+2)(s+3)}.$$

Proof. Assume first that $q>1$. From Lemma 1, improved power-mean integral inequality and the property of the n-polynomial convex function of $|f'|^q$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt$$

$$
\begin{aligned}
&\leq \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{2}\left(\int_0^1 t\,|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 (1-t)\,|1-2t|\,[1-(1-t)^s]\,dt\right. \\
&\quad\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 (1-t)\,|1-2t|\,[1-t^s]\,dt\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 t\,|1-2t|\,[1-(1-t)^s]\,dt\right. \\
&\quad\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 t\,|1-2t|\,[1-t^s]\,dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)\right)^{\frac{1}{q}}
\end{aligned}
$$

where

$$
\int_0^1 (1-t)\,|1-2t|\,dt = \int_0^1 t\,|1-2t|\,dt = \frac{1}{4}.
$$

For $q = 1$ we use the estimates from the proof of Theorem 5, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 6. *If we take $n = 1$ in the inequality (4.5), we get the following inequality:*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left(\frac{|f'(a)|^q}{4}+\frac{3\,|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\,|f'(a)|^q}{4}+\frac{|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].
$$

Remark 6. *The inequality (4.5) gives better result than the inequality (4.3). Let us show that*

$$
\begin{aligned}
&\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).
\end{aligned}
$$

If we use the concavity of the function $h : [0, \infty) \to \mathbb{R}$*,* $h(x) = x^{\lambda}, 0 < \lambda \leq 1$*, we get*

$$\begin{aligned} & \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)\right)^{\frac{1}{q}} \\ & +\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)\right)^{\frac{1}{q}} \\ \leq\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}[K_1(s)+K_2(s)]\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right), \end{aligned}$$

where

$$K_1(s)+K_2(s)=\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}$$

which completes the proof of remark.

5. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a, b with $b > a$:

1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a,b \geq 0,$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a,b \geq 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0,$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases} ; \ a,b > 0$$

5. The p-logaritmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R}\backslash\{-1,0\} \\ a, & a = b \end{cases} ; \ a,b > 0.$$

6.The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b > 0,$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature: $H \leq G \leq L \leq I \leq A$. It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 1. *Let* $a, b \in [0, \infty)$ *with* $a < b$ *and* $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$. *Then, the following inequalities are obtained:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)A^n(a,b) \leq L_n^n(a,b) \leq A(a^n, b^n)\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}.$$

Proof. The assertion follows from the inequalities (3.1) for the function $f(x) = x^n$, $x \in [0, \infty)$. □

Proposition 2. *Let* $a, b \in (0, \infty)$ *with* $a < b$ *. Then, the following inequalities are obtained:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)A^{-1}(a,b) \leq L^{-1}(a,b) \leq \frac{2}{n}H^{-1}(a,b)\sum_{s=1}^{n}\frac{s}{s+1}.$$

Proof. The assertion follows from (3.1) for the function $f(x) = x^{-1}$, $x \in (0, \infty)$. □

Proposition 3. *Let* $a, b \in (0, 1]$ *with* $a < b$. *Then, the following inequalities are obtained:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\ln G \leq \ln I \leq \frac{\ln A}{n}\sum_{s=1}^{n}\frac{s}{s+1}.$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$f(x) = -\ln x, \quad x \in (0, 1].$$

□

6. Conclusion

We established some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is n-polynomial convexity. Similar method can be applied to the different type of convex functions.

Conflict of interest

The authors declare no conflict of interest in this paper.

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