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Research article

Hypergeometric Euler numbers

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Abstract: In this paper, we introduce the hypergeometric Euler number as an analogue of the hypergeometric Bernoulli number and the hypergeometric Cauchy number. We study several expressions and sums of products of hypergeometric Euler numbers. We also introduce complementary hypergeometric Euler numbers and give some characteristic properties. There are strong reasons why these hypergeometric numbers are important. The hypergeometric numbers have one of the advantages that yield the natural extensions of determinant expressions of the numbers, though many kinds of generalizations of the Euler numbers have been considered by many authors.

Keywords: hypergeometric Euler numbers; Euler numbers; Bernoulli numbers; Hasse-Teichmüller derivative; sums of products; determinants
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1. Introduction

In this paper, Euler numbers E_n are defined by the generating function

$$\frac{1}{\cosh t}=\sum_{n=0}^\infty E_n\frac{t^n}{n!}\,. \tag{1.1}$$

One of the different definitions is

$$\frac{2}{e^t+1}=\sum_{n=0}^\infty E_n^*\frac{t^n}{n!}$$

(see e.g. [1]). Here, E_n^* are sometimes called the zig numbers or secant numbers. There have been many generalizations of Euler numbers from the different view points. For example, one kind of poly-Euler numbers is a typical generalization, in the aspect of L-functions ([2–4]). Other generalizations can be found in [5, 6] and the reference therein.

Bernoulli numbers and Cauchy numbers also have many generalizations. Universal Bernoulli numbers were studied in [7, 8], and particularly, some universal Kummer congruences were established in [7, 8]. In this paper, we focus on the generalizations based upon hypergeometric functions. For $N \ge 1$, define hypergeometric Bernoulli numbers $B_{N,n}$ (see [9–11]) by

$$\frac{1}{{}_1F_1(1;N+1;t)} = \frac{t^N/N!}{e^t - \sum_{n=0}^{N-1} t^n/n!} = \sum_{n=0}^{\infty} B_{N,n}\frac{t^n}{n!}\,,$$

where

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}}\frac{z^n}{n!}$$

is the confluent hypergeometric function with $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ $(n \ge 1)$ and $(x)^{(0)} = 1$. When $N = 1$, $B_n = B_{1,n}$ are classical Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}\,.$$

In addition, define hypergeometric Cauchy numbers $c_{N,n}$ (see [12]) by

$$\frac{1}{{}_2F_1(1,N;N+1;-t)} = \frac{(-1)^{N-1}t^N/N}{\log(1+t) - \sum_{n=1}^{N-1}(-1)^{n-1}t^n/n} = \sum_{n=0}^{\infty} c_{N,n}\frac{t^n}{n!}\,,$$

where

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!}$$

is the Gauss hypergeometric function. When $N = 1$, $c_n = c_{1,n}$ are classical Cauchy numbers defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n\frac{t^n}{n!}\,.$$

Some of the similar generalizations can be found in [13] (see also references therein), but their generating functions are not related to hypergeometric functions. There are several advantages for these so-called hypergeometric numbers. For example, as shown in Section 2, a naturally generalized expression is possible for hypergeometric numbers, but not for poly numbers like poly-Bernoulli or poly-Cauchy numbers, which are differently directed generalized Bernoulli or Cauchy numbers. Recently, poly-Euler numbers [4] are proposed and studied as one kind of poly-numbers. On the contrary, in this paper, we consider a generalization for Euler numbers by using hypergeometric functions. Then we study their characteristic or combinatorial properties.

For $N \ge 0$ define *hypergeometric Euler numbers* $E_{N,n}$ $(n = 0, 1, 2, \dots)$ by

$$\frac{1}{{}_1F_2(1;N+1,(2N+1)/2;t^2/4)} = \sum_{n=0}^{\infty} E_{N,n}\frac{t^n}{n!}\,, \tag{1.2}$$

where ${}_1F_2(a; b, c; z)$ is the hypergeometric function defined by

$${}_1F_2(a; b, c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}} \frac{z^n}{n!}\,.$$

It is seen that

$$\begin{aligned}\cosh t - \sum_{n=0}^{N-1} \frac{t^{2n}}{(2n)!} &= \frac{t^{2N}}{(2N)!} \sum_{n=0}^{\infty} \frac{(2N)!n!}{(2n+2N)!} \frac{(t^2)^n}{n!} \\ &= \frac{t^{2N}}{(2N)!}\,{}_1F_2\Big(1; N+1, \frac{2N+1}{2}; \frac{t^2}{4}\Big)\,. \end{aligned} \tag{1.3}$$

When $N = 0$, then $E_n = E_{0,n}$ are classical Euler numbers defined in (1.1). In [14], the truncated Euler polynomial $E_{m,n}(x)$ is introduced as a generalization of the classical Euler polynomial $E_n(x)$. The concept is similar but without hypergeometric functions.

We list the numbers $E_{N,n}$ for $0 \le N \le 6$ and $0 \le n \le 12$ in Table 1 in Appendix. From (1.3) we see that $E_{N,n} = 0$ if n is odd. Similarly to poly-Euler numbers ([2–4]), hypergeometric Euler numbers are rational numbers, though the classical Euler numbers are integers.

From (1.2) and (1.3), we have

$$\begin{aligned}\frac{t^{2N}}{(2N)!} &= \left(\sum_{n=N}^{\infty} \frac{t^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} E_{N,n} \frac{t^n}{n!}\right) \\ &= t^{2N}\left(\sum_{n=0}^{\infty} \frac{\frac{1+(-1)^n}{2} t^n}{(n+2N)!}\right)\left(\sum_{n=0}^{\infty} E_{N,n} \frac{t^n}{n!}\right) \\ &= t^{2N} \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{\frac{1+(-1)^{n-i}}{2}}{(2N+n-i)!} \frac{E_{N,i}}{i!}\right) t^n\,. \end{aligned}$$

Hence, for $n \ge 1$, we have

$$\sum_{i=0}^{n} \frac{1+(-1)^{n-i}}{(2N+n-i)!i!} E_{N,i} = 0\,.$$

Thus, we have the following proposition. Note that $E_{N,n} = 0$ when n is odd.

Proposition 1.

$$\sum_{i=0}^{n/2} \frac{1}{(2N+n-2i)!(2i)!} E_{N,2i} = 0 \quad (n \ge 2 \textit{ is even})$$

and $E_{N,0} = 1$.

By using the identity in Proposition 1 or the identity

$$E_{N,n} = -n!(2N)! \sum_{i=0}^{n/2-1} \frac{E_{N,2i}}{(2N+n-2i)!(2i)!}\,, \tag{1.4}$$

we can obtain the values of $E_{N,n}$ $(n = 0, 2, 4, \dots)$. We record the first few values of $E_{N,n}$:

$$E_{N,2} = -\frac{2}{(2N+1)(2N+2)}\,,$$

$$E_{N,4} = \frac{2\cdot 4!(4N+5)}{(2N+1)^2(2N+2)^2(2N+3)(2N+4)},$$
$$E_{N,6} = \frac{4\cdot 6!(8N^3-2N^2-65N-61)}{(2N+1)^3(2N+2)^3(2N+3)(2N+4)(2N+5)(2N+6)},$$
$$E_{N,8} = \frac{16\cdot 8!}{(2N+1)^4(2N+2)^4(2N+3)^2(2N+4)^2(2N+6)(2N+7)(2N+8)} \times (16N^6-44N^5-516N^4-667N^3+1283N^2+3126N+1662).$$

We have an explicit expression of $E_{N,n}$ for each even n:

Theorem 1. *For $N \ge 0$ and $n \ge 1$ we have*

$$E_{N,2n} = (2n)!\sum_{r=1}^{n}(-1)^r \sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r\ge 1}} \frac{((2N)!)^r}{(2N+2i_1)!\cdots(2N+2i_r)!}.$$

Proof. The proof is done by induction for n. If $n=1$, then

$$E_{N,2} = 2!(-1)\frac{(2N)!}{(2N+2)!} = -\frac{2}{(2N+1)(2N+2)}.$$

Assume that the result is valid up to $n-1$. Then by Proposition 1

$$\begin{aligned}
E_{N,2n} &= -(2n)!(2N)!\sum_{i=0}^{n-1}\frac{E_{N,2i}}{(2N+2n-2i)!(2i)!}\\
&= -(2n)!(2N)!\sum_{i=1}^{n-1}\frac{1}{(2N+2n-2i)!}\sum_{r=1}^{i}(-1)^r\sum_{\substack{i_1+\cdots+i_r=i\\ i_1,\dots,i_r\ge 1}}\frac{((2N)!)^r}{(2N+2i_1)!\cdots(2N+2i_r)!}\\
&\quad - (2n)!(2N)!\frac{1}{(2N+2n)!}\\
&= -(2n)!(2N)!\sum_{r=1}^{n-1}(-1)^r((2N)!)^r\sum_{i=r}^{n-1}\frac{1}{(2N+2n-2i)!}\sum_{\substack{i_1+\cdots+i_r=i\\ i_1,\dots,i_r\ge 1}}\frac{1}{(2N+2i_1)!\cdots(2N+2i_r)!}\\
&\quad - \frac{(2n)!(2N)!}{(2N+2n)!}\\
&= -(2n)!(2N)!\sum_{r=2}^{n}(-1)^{r-1}((2N)!)^{r-1}\sum_{i=r-1}^{n-1}\frac{1}{(2N+2n-2i)!}\\
&\quad \times \sum_{\substack{i_1+\cdots+i_{r-1}=i\\ i_1,\dots,i_{r-1}\ge 1}}\frac{1}{(2N+2i_1)!\cdots(2N+2i_{r-1})!}\\
&\quad - \frac{(2n)!(2N)!}{(2N+2n)!}\\
&= (2n)!\sum_{r=2}^{n}(-1)^r((2N)!)^r\sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r\ge 1}}\frac{1}{(2N+2i_1)!\cdots(2N+2i_r)!}
\end{aligned}$$

$$-\frac{(2n)!(2N)!}{(2N+2n)!} \quad (n-i=i_r)$$
$$=(2n)!\sum_{r=1}^{n}(-1)^r\sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r\ge 1}}\frac{((2N)!)^r}{(2N+2i_1)!\cdots(2N+2i_r)!}\,.$$

□

2. Determinant expressions of hypergeometric numbers

These hypergeometric numbers have one of the advantages that yield the natural extensions of determinant expressions of the numbers, though many kinds of generalizations of the Euler numbers have been considered by many authors.

By using Proposition 1 or the relation (1.4), we have a determinant expression of hypergeometric Euler numbers ([15]).

Proposition 2. *The hypergeometric Euler numbers $E_{N,2n}$ ($N \ge 0$, $n \ge 1$) can be expressed as*

$$E_{N,2n}=(-1)^n(2n)!\begin{vmatrix}\frac{(2N)!}{(2N+2)!} & 1 & & 0\\ \frac{(2N)!}{(2N+4)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1\\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!}\end{vmatrix}.$$

In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli, Cauchy and Euler numbers. When $N=0$, the determinant in Proposition (2) is reduced to a famous determinant expression of Euler numbers (*cf.* [16, p.52]):

$$E_{2n}=(-1)^n(2n)!\begin{vmatrix}\frac{1}{2!} & 1 & & & 0\\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1\\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix}.$$

In [17], the hypergeometric Bernoulli numbers $B_{N,n}$ ($N \ge 1$, $n \ge 1$) can be expressed as

$$B_{N,n}=(-1)^n n!\begin{vmatrix}\frac{N!}{(N+1)!} & 1 & & & 0\\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1\\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!}\end{vmatrix}.$$

When $N = 1$, we have a determinant expression of Bernoulli numbers ([16, p.53]):

$$B_n = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & & & 0 \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} . \tag{2.1}$$

In [18], the hypergeometric Cauchy numbers $c_{N,n}$ ($N \geq 1$, $n \geq 1$) can be expressed as

$$c_{N,n} = n! \begin{vmatrix} \frac{N}{N+1} & 1 & & & 0 \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix} .$$

When $N = 1$, we have a determinant expression of Cauchy numbers ([16, p.50]):

$$c_n = n! \begin{vmatrix} \frac{1}{2} & 1 & & & 0 \\ \frac{1}{3} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix} . \tag{2.2}$$

In [15], the complementary Euler numbers $\widehat{E}_n$ and their hypergeometric generalizations (defined below) have also determinant expressions.

3. Hasse-Teichmüller derivative

We define the Hasse-Teichmüller derivative $H^{(n)}$ of order n by

$$H^{(n)}\left(\sum_{m=R}^{\infty} c_m z^m\right) = \sum_{m=R}^{\infty} c_m \binom{m}{n} z^{m-n}$$

for $\sum_{m=R}^{\infty} c_m z^m \in \mathbb{F}((z))$, where R is an integer and $c_m \in \mathbb{F}$ for any $m \geq R$.

The Hasse-Teichmüller derivatives satisfy the product rule [19], the quotient rule [20] and the chain rule [21]. One of the product rules can be described as follows.

Lemma 1. *For $f_i \in \mathbb{F}[[z]]$ ($i = 1, \ldots, k$) with $k \geq 2$ and for $n \geq 1$, we have*

$$H^{(n)}(f_1 \cdots f_k) = \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} H^{(i_1)}(f_1) \cdots H^{(i_k)}(f_k) .$$

The quotient rules can be described as follows.

Lemma 2. *For $f \in \mathbb{F}[[z]]\backslash\{0\}$ and $n \geq 1$, we have*

$$H^{(n)}\left(\frac{1}{f}\right) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1,\dots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} H^{(i_1)}(f)\cdots H^{(i_k)}(f) \tag{3.1}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1,\dots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} H^{(i_1)}(f)\cdots H^{(i_k)}(f)\,. \tag{3.2}$$

By using the Hasse-Teichmüller derivative of order n, we shall obtain some explicit expressions of the hypergeometric Euler numbers.

Another proof of Theorem 1. Put

$$\begin{aligned} F :&= {}_1F_2(1; N+1, \frac{2N+1}{2}; \frac{t^2}{4}) \\ &= \sum_{n=0}^{\infty} \frac{(2N)!}{(2N+2n)!} t^{2n} \end{aligned}$$

for simplicity. Note that

$$H^{(i)}(F)\Big|_{t=0} = \sum_{j=0}^{\infty} \frac{(2N)!}{(2N+2j)!}\binom{2j}{i} t^{2j-i}\Bigg|_{t=0} = \begin{cases} (2N)!/(2N+i)! & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Hence, by using Lemma 2 (3.1), we have

$$\begin{aligned} \frac{E_{N,n}}{n!} &= H^{(n)}\left(\frac{1}{F}\right)\Bigg|_{t=0} \\ &= \sum_{k=1}^{n} \frac{(-1)^k}{F^{k+1}}\Bigg|_{t=0} \sum_{\substack{i_1,\dots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} H^{(i_1)}(F)\Big|_{t=0}\cdots H^{(i_k)}(F)\Big|_{t=0} \\ &= \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,\dots,i_k \geq 1 \\ 2(i_1+\cdots+i_k)=n}} \frac{((2N)!)^k}{(2N+2i_1)!\cdots(2N+2i_k)!}\,. \end{aligned}$$

□

We can express the hypergeometric Euler numbers also in terms of the binomial coefficients. In fact, by using Lemma 2 (3.2) instead of Lemma 2 (3.1) in the above proof, we obtain a little different expression from one in Theorem 1.

Proposition 3. *For $N \geq 0$ and even $n \geq 2$,*

$$E_{N,n} = n! \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \sum_{\substack{i_1,\dots,i_k \geq 0 \\ i_1+\cdots+i_k=n/2}} \frac{((2N)!)^k}{(2N+2i_1)!\cdots(2N+2i_k)!}\,.$$

For example, when $n = 4$, we get

$$\begin{aligned}
E_4 &= 4!\left(-\binom{5}{2}\frac{1}{4!}+\binom{5}{3}\left(\frac{2}{4!}+\frac{1}{2!2!}\right)\right.\\
&\qquad\left.-\binom{5}{4}\left(\frac{3}{4!}+\frac{3}{2!2!}\right)+\binom{5}{5}\left(\frac{4}{4!}+\frac{6}{2!2!}\right)\right)\\
&= 5\,,\\
E_{1,4} &= 4!\left(-\binom{5}{2}2\frac{1}{6!}+\binom{5}{3}2^2\left(\frac{2}{6!2!}+\frac{1}{4!4!}\right)\right.\\
&\qquad\left.-\binom{5}{4}2^3\left(\frac{3}{6!2!2!}+\frac{3}{4!4!2!}\right)+\binom{5}{5}2^4\left(\frac{4}{6!2!2!2!}+\frac{6}{4!4!2!2!}\right)\right)\\
&= \frac{1}{10}\,,\\
E_{2,4} &= 4!\left(-\binom{5}{2}4!\frac{1}{8!}+\binom{5}{3}(4!)^2\left(\frac{2}{8!4!}+\frac{1}{6!6!}\right)\right.\\
&\qquad\left.-\binom{5}{4}(4!)^3\left(\frac{3}{8!4!4!}+\frac{3}{6!6!4!}\right)+\binom{5}{5}(4!)^4\left(\frac{4}{8!4!4!4!}+\frac{6}{6!6!4!4!}\right)\right)\\
&= \frac{13}{1050}\,,\\
E_{3,4} &= 4!\left(-\binom{5}{2}6!\frac{1}{10!}+\binom{5}{3}(6!)^2\left(\frac{2}{10!6!}+\frac{1}{8!8!}\right)\right.\\
&\qquad\left.-\binom{5}{4}(6!)^3\left(\frac{3}{10!6!6!}+\frac{3}{8!8!6!}\right)+\binom{5}{5}(6!)^4\left(\frac{4}{10!6!6!6!}+\frac{6}{8!8!6!6!}\right)\right)\\
&= \frac{17}{5880}\,.
\end{aligned}$$

4. Some hypergeometric Euler numbers

If $N = 1$, we have the following relation between hypergeometric Euler numbers and Bernoulli numbers.

Theorem 2. *For $n \ge 1$ we have*

$$E_{1,n} = -(n-1)B_n\,.$$

Proof. The result is clear for $n = 0, 1$ and odd numbers n. By using the following Lemma 3 and Proposition 1, we get the result. □

Lemma 3. *For $n \ge 1$ we have*

$$\sum_{i=0}^{n}\frac{(i-1)B_i}{(n-i+2)!i!}=\begin{cases}0 & \text{if } n \text{ is even};\\ -B_{n+1}/n! & \text{if } n \text{ is odd}.\end{cases}$$

Proof. Firstly,

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(i-1)B_i}{(n-i+2)!i!}x^n&=\left(\sum_{k=0}^{\infty}\frac{x^k}{(k+2)!}\right)\left(\sum_{i=0}^{\infty}(i-1)B_i\frac{x^i}{i!}\right)\\&=\left(\frac{1}{x^2}\sum_{k=0}^{\infty}\frac{x^{k+2}}{(k+2)!}\right)\left(-2\sum_{i=0}^{\infty}B_i\frac{x^i}{i!}+\frac{d}{dx}\sum_{i=0}^{\infty}B_i\frac{x^{i+1}}{i!}\right)\\&=\frac{e^x-1-x}{x^2}\left(-\frac{2x}{e^x-1}+\frac{2x(e^x-1)-x^2e^x}{(e^x-1)^2}\right)\\&=\frac{e^x(x+1-e^x)}{(e^x-1)^2}\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}-\frac{1}{2}-\sum_{n=0}^{\infty}B_{2n+2}\frac{x^{2n+1}}{(2n+1)!}&=-\frac{1}{2}-\frac{d}{dx}\left(\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}-B_0-B_1x\right)\\&=-\frac{1}{2}-\frac{d}{dx}\left(\frac{x}{e^x-1}-1+\frac{x}{2}\right)\\&=\frac{e^x(x+1-e^x)}{(e^x-1)^2}\,.\end{aligned}$$

Comparing the coefficients of x^n, we get the result. □

5. Sums of products of hypergeometric Euler numbers

It is known that

$$\sum_{i=0}^{n}\binom{2n}{2i}E_{2i}=0$$

with $E_0=1$, and $E_{2i-1}=0$ $(i\ge 1)$.

First, let us consider the sums of products of hypergeometric Euler numbers:

$$Y_{N,2}(n)=\sum_{i=0}^{n}\binom{2n}{2i}E_{N,2i}E_{N,2n-2i}\,.$$

It is clear that

$$\sum_{i=0}^{n}\binom{n}{i}E_{N,i}E_{N,n-i}=0$$

if n is odd.

If $N=0$, then

$$Y_{0,2}(n)=\frac{2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{2n+2}\quad(n\ge 0)\,.$$

Indeed,

$$\{Y_{0,2}(n)\}_{n\ge 0}=1,-2,16,-272,7936,-353792,22368256,-1903757312,\dots.$$

The numbers taking their absolute value are called the *tangent numbers* or the *zag numbers* ([22, A000182]). Thus, we also have

$$Y_{0,2}(n) = \sum_{k=1}^{2n+2} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{j+1}(k-2j)^{2n+2}}{2^k \sqrt{-1}^k k}.$$

In other words, they appear as numerators in the Maclaurin series of $\tan x$:

$$\tan x = \sum_{n=0}^{\infty} (-1)^n Y_{0,2}(n) \frac{x^{2n+1}}{(2n+1)!}.$$

Put

$$\begin{aligned} F : &= {}_1F_2(1; N+1, \frac{2N+1}{2}; \frac{t^2}{4}) \\ &= \sum_{n=0}^{\infty} \frac{(2N)!}{(2N+2n)!} t^{2n} \end{aligned}$$

for simplicity again. Then by

$$\frac{d}{dt} F = \sum_{n=0}^{\infty} \frac{(2n)(2N)!}{(2N+2n)!} t^{2n-1},$$

we have

$$2NF + t\frac{d}{dt}F = 2N \cdot {}_1F_2(1; N, \frac{2N+1}{2}; \frac{t^2}{4}). \tag{5.1}$$

For further simplicity, we put for $k = 1, 2, \dots, 2N$

$$F_{(2N-k)} = {}_1F_2\left(1; \left\lfloor \frac{k+2}{2} \right\rfloor, \left\lfloor \frac{k+1}{2} \right\rfloor + \frac{1}{2}; \frac{t^2}{4}\right)$$

with $F_{(0)} = F$. Then, in general, we obtain for $k = 1, 2, \dots, 2N$

$$kF_{(2N-k)} + t\frac{d}{dt}F_{(2N-k)} = kF_{(2N-k+1)}. \tag{5.2}$$

Proposition 4. *For $k = 0, 1, \dots, 2N$ we have*

$$\cosh t = \sum_{i=0}^{k} \frac{t^i}{i!} \binom{k}{i} \frac{d^i}{dt^i} F_{(2N-k)}.$$

Proof. For $k = 0$, we get

$$F_{(2N)} = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \cosh t.$$

Assume that the result holds for some $k \geq 0$. Then by (5.2)

$$\sum_{i=0}^{k} \frac{t^i}{i!} \binom{k}{i} \frac{d^i}{dt^i} F_{(2N-k)}$$

$$
\begin{aligned}
&= \sum_{i=0}^{k} \frac{t^i}{i!}\binom{k}{i}\frac{d^i}{dt^i}\left(F_{(2N-k-1)} + \frac{t}{k+1}\frac{d}{dt}F_{(2N-k-1)}\right)\\
&= \sum_{i=0}^{k} \frac{t^i}{i!}\binom{k}{i}\left(\frac{d^i}{dt^i}F_{(2N-k-1)} + \frac{i}{k+1}\frac{d^i}{dt^i}F_{(2N-k-1)} + \frac{t}{k+1}\frac{d^{i+1}}{dt^{i+1}}F_{(2N-k-1)}\right)\\
&= \sum_{i=0}^{k} \frac{t^i}{i!}\binom{k}{i}\frac{k+i+1}{k+1}\frac{d^i}{dt^i}F_{(2N-k-1)} + \sum_{i=1}^{k+1}\frac{t^{i-1}}{(i-1)!}\binom{k}{i-1}\frac{t}{k+1}\frac{d^i}{dt^i}F_{(2N-k-1)}\\
&= \sum_{i=0}^{k+1} \frac{t^i}{i!}\binom{k+1}{i}\frac{d^i}{dt^i}F_{(2N-k-1)}\,.
\end{aligned}
$$

□

We introduce the *complementary hypergeometric Euler numbers* $\widehat{E}_{N,n}$ by

$$
\frac{t^{2N+1}/(2N+1)!}{\sinh t - \sum_{n=0}^{N-1} t^{2n+1}/(2n+1)!} = \sum_{n=0}^{\infty} \widehat{E}_{N,n}\frac{t^n}{n!}
$$

as an analogue of (1.2). When $N = 0$, $\widehat{E}_n = \widehat{E}_{0,n}$ are the *complementary Euler numbers* defined by

$$
\frac{t}{\sinh t} = \sum_{n=0}^{\infty} \widehat{E}_n \frac{t^n}{n!}
$$

as an analogue of (1.1). In [23], they are called *weighted Bernoulli numbers*, but this naming means different in other literatures. Since

$$
\begin{aligned}
F^* &:= {}_1F_2\left(1; N, \frac{2N+1}{2}; \frac{t^2}{4}\right)\\
&= \sum_{n=0}^{\infty} \frac{(2N-1)!}{(2N+2n-1)!} t^{2n}
\end{aligned}
$$

and

$$
\frac{d}{dt}F = -F^2 \frac{d}{dt}\frac{1}{F}\,, \tag{5.3}
$$

by (5.1) we have

$$
\frac{1}{F^2} = \frac{1}{F^*}\left(\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F}\right). \tag{5.4}
$$

Since

$$
\begin{aligned}
\frac{1}{F^*} &= \frac{t^{2N-1}}{(2N-1)!\sum_{n=0}^{\infty} t^{2N+2n-1}/(2N+2n-1)!}\\
&= \sum_{n=0}^{\infty} \widehat{E}_{N-1,n}\frac{t^n}{n!}
\end{aligned}
$$

and

$$\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F} = \sum_{n=0}^{\infty} E_{N,n}\frac{t^n}{n!} - \frac{t}{2N}\sum_{n=1}^{\infty} E_{N,n}\frac{t^{n-1}}{(n-1)!}$$
$$= \sum_{n=0}^{\infty}\frac{2N-n}{2N}E_{N,n}\frac{t^n}{n!},$$

we have

$$\frac{1}{F^*}\left(\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F}\right) = \left(\sum_{m=0}^{\infty}\widehat{E}_{N-1,m}\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\frac{2N-k}{2N}E_{N,k}\frac{t^k}{k!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,n-k}\frac{t^n}{n!}.$$

Comparing the coefficients, we obtain a result about the sums of products.

Theorem 3. *For $N \geq 1$ and $n \geq 0$,*

$$\sum_{i=0}^{n}\binom{n}{i}E_{N,i}E_{N,n-i} = \sum_{k=0}^{n}\binom{n}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,n-k}.$$

Using (5.3) and (5.4) again, we have

$$\frac{1}{F^3} = \frac{1}{F^*}\left(\frac{1}{F^2} - \frac{t}{2N}\frac{1}{F}\frac{d}{dt}\frac{1}{F}\right)$$
$$= \frac{1}{F^*}\left(\frac{1}{F^2} - \frac{t}{4N}\frac{d}{dt}\frac{1}{F^2}\right).$$

Since

$$\frac{1}{F^2} - \frac{t}{4N}\frac{d}{dt}\frac{1}{F^2} = \sum_{n=0}^{\infty}\frac{4N-n}{4N}\sum_{k=0}^{n}\binom{n}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,n-k}\frac{t^n}{n!},$$

we have

$$\frac{1}{F^*}\left(\frac{1}{F^2} - \frac{t}{4N}\frac{d}{dt}\frac{1}{F^2}\right)$$
$$= \left(\sum_{i=0}^{\infty}\widehat{E}_{N-1,i}\frac{t^i}{i!}\right)\left(\sum_{m=0}^{\infty}\frac{4N-m}{4N}\sum_{k=0}^{m}\binom{m}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,m-k}\frac{t^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}\frac{(4N-m)(2N-k)}{8N^2}E_{N,k}\widehat{E}_{N-1,n-m}\widehat{E}_{N-1,m-k}\frac{t^n}{n!}.$$

Comparing the coefficients, we get a result about the sums of products for trinomial coefficients.

Theorem 4. *For $N \geq 1$ and $n \geq 0$,*

$$\sum_{\substack{i_1+i_2+i_3=n\\ i_1,i_2,i_3\geq 0}}\binom{n}{i_1,i_2,i_3}E_{N,i_1}E_{N,i_2}E_{N,i_3} = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}\frac{(4N-m)(2N-k)}{8N^2}E_{N,k}\widehat{E}_{N-1,n-m}\widehat{E}_{N-1,m-k}.$$

Complementary hypergeometric Euler numbers

By using the similar methods in previous sections, the complementary hypergeometric Euler numbers satisfy the recurrence relation for even n

$$\sum_{i=0}^{n/2}\frac{\widehat{E}_{N,2i}}{(2N+n-2i+1)!(2i)!}=0$$

or

$$\widehat{E}_{N,n}=-n!(2N+1)!\sum_{i=0}^{n/2-1}\frac{\widehat{E}_{N,2i}}{(2N+n-2i+1)!(2i)!}\,.$$

By using the Hasse-Teichmüller derivative or by proving by induction, we have

Theorem 5. *For $N\ge 0$ and $n\ge 1$ we have*

$$\begin{aligned}\widehat{E}_{N,n}&=n!\sum_{k=1}^{n}(-1)^k\sum_{\substack{i_1,\dots,i_k\ge 1\\ i_1+\dots+i_k=n/2}}\frac{\big((2N+1)!\big)^k}{(2N+2i_1+1)!\cdots(2N+2i_k+1)!}\\&=n!\sum_{k=1}^{n}(-1)^k\binom{n+1}{k+1}\sum_{\substack{i_1,\dots,i_k\ge 0\\ i_1+\dots+i_k=n/2}}\frac{\big((2N+1)!\big)^k}{(2N+2i_1+1)!\cdots(2N+2i_k+1)!}\,.\end{aligned}$$

Some initial values of $\widehat{E}_{N,n}$ $(n=0,2,4,\dots)$, we have

$$\begin{aligned}\widehat{E}_{N,2}&=-\frac{2}{(2N+2)(2N+3)}\,,\\ \widehat{E}_{N,4}&=\frac{2\cdot 4!(4N+7)}{(2N+2)^2(2N+3)^2(2N+4)(2N+5)}\,,\\ \widehat{E}_{N,6}&=\frac{4\cdot 6!(8N^3+10N^2-61N-93))}{(2N+2)^3(2N+3)^3(2N+4)(2N+5)(2N+6)(2N+7)}\,,\\ \widehat{E}_{N,8}&=\frac{8\cdot 8!}{(2N+2)^4(2N+3)^4(2N+4)^2(2N+5)^2(2N+7)(2N+8)(2N+9)}\\&\quad\times(32N^6+8N^5-1132N^4-3538N^3-1063N^2+7280N+6858)\,.\end{aligned}$$

Put

$$\widehat{F}=\sum_{n=0}^{\infty}\frac{(2N+1)!}{(2N+2n+1)!}t^{2n}$$

so that

$$\frac{1}{\widehat{F}}=\sum_{n=0}^{\infty}\widehat{E}_{N,n}\frac{t^n}{n!}\,.$$

Since

$$(2N+1)\widehat{F}+t\frac{d}{dt}\widehat{F}=(2N+1)F\,,$$

we have

$$\frac{1}{\widehat{F}^2}=\frac{1}{F}\left(\frac{1}{\widehat{F}}-\frac{t}{2N+1}\frac{d}{dt}\frac{1}{\widehat{F}}\right)$$

$$= \left(\sum_{m=0}^{\infty} E_{N,m}\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}\frac{t^k}{k!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}\frac{t^n}{n!}\,.$$

Hence, as an analogue of Theorem 3, we have the following.

Theorem 6. *For $N \ge 1$ and $n \ge 0$,*

$$\sum_{i=0}^{n}\binom{n}{i}\widehat{E}_{N,i}\widehat{E}_{N,n-i} = \sum_{k=0}^{n}\binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}\,.$$

We then have

$$\frac{1}{\widehat{F}^3} = \frac{1}{F}\left(\frac{1}{\widehat{F}^2} - \frac{t}{2(2N+1)}\frac{d}{dt}\frac{1}{\widehat{F}^2}\right).$$

Since

$$\frac{1}{\widehat{F}^2} - \frac{t}{2(2N+1)}\frac{d}{dt}\frac{1}{\widehat{F}^2} = \sum_{n=0}^{\infty}\frac{4N-n+2}{2(2N+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}\frac{t^n}{n!}\,,$$

we have the following result as an analogue of Theorem 4.

Theorem 7. *For $N \ge 1$ and $n \ge 0$,*

$$\sum_{\substack{i_1+i_2+i_3=n\\ i_1,i_2,i_3\ge 0}}\binom{n}{i_1,i_2,i_3}\widehat{E}_{N,i_1}\widehat{E}_{N,i_2}\widehat{E}_{N,i_3} = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}\frac{(4N-m+2)(2N-k+1)}{2(2N+1)^2}\widehat{E}_{N,k}E_{N,n-m}E_{N,m-k}\,.$$

One can continue to obtain the sum of four or more products, though the results seem to become more complicated.

6. Applications by Trudi's formula

We shall use Trudi's formula to obtain different explicit expressions for the hypergeometric Euler numbers $E_{N,n}$.

Lemma 4 (Trudi's formula [24, 25]). *For a positive integer m, we have*

$$\begin{vmatrix} a_1 & a_2 & \cdots & & a_m\\ a_0 & a_1 & \cdots & & \\ \vdots & \vdots & \ddots & & \vdots\\ 0 & 0 & \cdots & a_1 & a_2\\ 0 & 0 & \cdots & a_0 & a_1 \end{vmatrix} = \sum_{t_1+2t_2+\cdots+mt_m=m}\binom{t_1+\cdots+t_m}{t_1,\dots,t_m}(-a_0)^{m-t_1-\cdots-t_m}a_1^{t_1}a_2^{t_2}\cdots a_m^{t_m},$$

where $\binom{t_1+\cdots+t_m}{t_1,\dots,t_m} = \frac{(t_1+\cdots+t_m)!}{t_1!\cdots t_m!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [26, Vol.3, p.214], [25] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [27], [26, Vol.3, pp.208–209].

In addition, there exists the following inversion formula (see, e.g. [24]), which is based upon the relation:

$$\sum_{k=0}^{n}(-1)^{n-k}\alpha_k D(n-k) = 0 \quad (n \geq 1)\,.$$

Lemma 5. *If $\{\alpha_n\}_{n\geq 0}$ is a sequence defined by $\alpha_0 = 1$ and*

$$\alpha_n = \begin{vmatrix} R(1) & 1 & & 0 \\ R(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ R(n) & \cdots & R(2) & R(1) \end{vmatrix}, \text{ then } R(n) = \begin{vmatrix} \alpha_1 & 1 & & 0 \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

Moreover, if

$$A = \begin{pmatrix} 1 & & & 0 \\ \alpha_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ \alpha_n & \cdots & \alpha_1 & 1 \end{pmatrix}, \text{ then } A^{-1} = \begin{pmatrix} 1 & & & 0 \\ R(1) & 1 & & \\ \vdots & \ddots & \ddots & \\ R(n) & \cdots & R(1) & 1 \end{pmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}(-1)^{n-t_1-\cdots-t_n} R(1)^{t_1} R(2)^{t_2}\cdots R(n)^{t_n}\,.$$

By applying these lemmata to Proposition 2, we obtain an explicit expression for the hypergeometric Euler numbers $E_{N,n}$.

Theorem 8. *For $N \geq 0$ and $n \geq 1$,*

$$E_{N,2n} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}(-1)^{t_1+\cdots+t_n}\left(\frac{(2N)!}{(2N+2)!}\right)^{t_1}\left(\frac{(2N)!}{(2N+4)!}\right)^{t_2}\cdots\left(\frac{(2N)!}{(2N+2n)!}\right)^{t_n}.$$

Moreover,

$$\frac{(-1)^n(2N)!}{(2N+2n)!} = \begin{vmatrix} \frac{E_{N,2}}{2!} & 1 & & 0 \\ \frac{E_{N,4}}{4!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{E_{N,2n}}{(2n)!} & \cdots & \frac{E_{N,4}}{4!} & \frac{E_{N,2}}{2!} \end{vmatrix},$$

and

$$\begin{pmatrix} 1 & & & & 0 \\ -\frac{E_{N,2}}{2!} & 1 & & & \\ \frac{E_{N,4}}{4!} & -\frac{E_{N,2}}{2!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(-1)^n E_{N,2n}}{(2n)!} & \cdots & \frac{E_{N,4}}{4!} & -\frac{E_{N,2}}{2!} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & 0 \\ \frac{(2N)!}{(2N+2)!} & 1 & & & \\ \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} & 1 \end{pmatrix}.$$

When $N = 0$ in Theorem 8, we have a different expression for the classical Euler numbers E_n.

Corollary 1. *For* $n \ge 1$

$$E_{2n} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{t_1+\cdots+t_n} \left(\frac{1}{2!}\right)^{t_1} \left(\frac{1}{4!}\right)^{t_2} \cdots \left(\frac{1}{(2n)!}\right)^{t_n}.$$

Moreover,

$$\frac{(-1)^n}{(2n)!} = \begin{vmatrix} \frac{E_2}{2!} & 1 & & 0 \\ \frac{E_4}{4!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{E_{2n}}{(2n)!} & \cdots & \frac{E_4}{4!} & \frac{E_2}{2!} \end{vmatrix}.$$

Similarly, by the results in [15], after applying Lemmata 4 and 5, we have a new expression of the complementary hypergeometric Euler numbers $\widehat{E}_{N,n}$.

Theorem 9. *For* $N \ge 0$ *and* $n \ge 1$,

$$\begin{aligned} &\widehat{E}_{N,2n}^{(r)} \\ &= (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{t_1+\cdots+t_n} \left(\frac{(2N+1)!}{(2N+3)!}\right)^{t_1} \left(\frac{(2N+1)!}{(2N+5)!}\right)^{t_2} \cdots \left(\frac{(2N+1)!}{(2N+2n+1)!}\right)^{t_n}. \end{aligned}$$

Moreover,

$$\frac{(-1)^n (2N+1)!}{(2N+2n+1)!} = \begin{vmatrix} \frac{\widehat{E}_{N,2}}{2!} & 1 & & 0 \\ \frac{\widehat{E}_{N,4}}{4!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{\widehat{E}_{N,2n}}{(2n)!} & \cdots & \frac{\widehat{E}_{N,4}}{4!} & \frac{\widehat{E}_{N,2}}{2!} \end{vmatrix},$$

and

$$\begin{pmatrix} 1 & & & & 0 \\ -\frac{\widehat{E}_{N,2}}{2!} & 1 & & & \\ \frac{\widehat{E}_{N,4}}{4!} & -\frac{\widehat{E}_{N,2}}{2!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(-1)^n \widehat{E}_{N,2n}}{(2n)!} & \cdots & \frac{\widehat{E}_{N,4}}{4!} & -\frac{\widehat{E}_{N,2}}{2!} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & 0 \\ \frac{(2N+1)!}{(2N+3)!} & 1 & & & \\ \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(2N+1)!}{(2N+2n+1)!} & \cdots & \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!} & 1 \end{pmatrix}.$$

When $N = 0$ in Theorem 9, we have a different expression for the original complementary Euler numbers $\widehat{E}_n$.

Corollary 2. *For* $n \ge 1$

$$\widehat{E}_{2n} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{t_1+\cdots+t_n} \left(\frac{1}{3!}\right)^{t_1} \left(\frac{1}{5!}\right)^{t_2} \cdots \left(\frac{1}{(2n+1)!}\right)^{t_n}.$$

Moreover,

$$\frac{(-1)^n}{(2n+1)!} = \begin{vmatrix} \frac{\widehat{E}_2}{2!} & 1 & & 0 \\ \frac{\widehat{E}_4}{4!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{\widehat{E}_{2n}}{(2n)!} & \cdots & \frac{\widehat{E}_4}{4!} & \frac{\widehat{E}_2}{2!} \end{vmatrix}.$$

7. Conclusions

There are more advantages and applications for so-called hypergeometric numbers. For example, we can show the following continued fraction expansion of the generating function of hypergeometric Euler numbers.

$$\sum_{n=0}^{\infty} E_{N,n}\frac{t^n}{n!} = 1 - \cfrac{t^2}{(2N+1)(2N+2)+t^2-\cfrac{(2N+1)(2N+2)t^2}{(2N+3)(2N+4)+t^2-\cfrac{(2N+3)(2N+4)t^2}{(2N+5)(2N+6)+t^2-\ddots}}}.$$

When $N = 0$, we get a continued fraction expansion of the classical Euler numbers.

$$\begin{aligned}\sum_{n=0}^{\infty} E_n\frac{t^n}{n!} &= \frac{1}{\cosh t} \\ &= 1 - \cfrac{t^2}{1\cdot 2+t^2-\cfrac{1\cdot 2t^2}{3\cdot 4+t^2-\cfrac{3\cdot 4t^2}{5\cdot 6+t^2-\ddots}}}.\end{aligned}$$

Similarly, one of the continued fraction expansions of the generating function of complementary hypergeometric Euler numbers is given by

$$\sum_{n=0}^{\infty} \widehat{E}_{N,n}\frac{t^n}{n!} = 1 - \cfrac{t^2}{(2N+2)(2N+3)+t^2-\cfrac{(2N+2)(2N+3)t^2}{(2N+4)(2N+5)+t^2-\cfrac{(2N+4)(2N+5)t^2}{(2N+6)(2N+7)+t^2-\ddots}}}.$$

When $N = 0$, we get a continued fraction expansion of the complementary Euler numbers.

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{E}_n\frac{t^n}{n!} &= \frac{t}{\sinh t} \\ &= 1 - \cfrac{t^2}{2\cdot 3+t^2-\cfrac{2\cdot 3t^2}{4\cdot 5+t^2-\cfrac{4\cdot 5t^2}{6\cdot 7+t^2-\ddots}}}.\end{aligned}$$

However, so-called poly-numbers do not have such natural generalizations in continued fractions. The more detailed and more general results including other hypergeometric numbers will be discussed in other papers.

In addition, hypergeometric numbers can be discussed on the rational function fields. They will be naturally generalized from Bernoullli-Carlitz, Cauchy-Carlitz or Euler-Carlitz numbers. Their details and structures will be also studied in other papers (e.g., see [28, 29]).

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Conflict of interest

The authors declare no conflict of interest.

Appendix

Table 1. The numbers $E_{N,n}$ for $0 \le N \le 6$ and $0 \le n \le 14$.

n	0	2	4	6	8	10
$E_{0,n}$	1	−1	5	−61	1385	−50521
$E_{1,n}$	1	−1/6	1/10	−5/42	7/30	−15/22
$E_{2,n}$	1	−1/15	13/1050	−1/350	−31/173250	1343/750750
$E_{3,n}$	1	−1/28	17/5880	−29/362208	−863/6420960	6499/131843712
$E_{4,n}$	1	−1/45	7/7425	53/2027025	−443/22052250	−10157/4873547250
$E_{5,n}$	1	−1/66	25/66066	47/2906904	−16945/5300012718	−475767/492312292472
$E_{6,n}$	1	−1/91	29/165620	1205/153728484	−2279/4467168888	−6430761/25339270989032

12	14
2702765	−199360981
7601/2730	−91/6
−6137/2388750	3499/6693750
6997213/156894017280	−68936107/917226562560
558599021/126395447928750	39045649/62503243481250
71844089/268802511689712	1162911301/4483980359834976
−17675104079/4917799642149532320	837165624457/24588998210747661600

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