Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(2): 1247–1259.
DOI: 10.3934/math.2020085
Received: 15 October 2019
Accepted: 16 December 2019
Published: 17 January 2020

Research article

On the integral operators pertaining to a family of incomplete I-functions

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Abstract: This paper introduces a new incomplete $I-$functions. The incomplete $I-$function is an extension of the $I-$function given by Saxena [1] which is a extension of a familiar Fox's $H-$function. Next, we find the several interesting classical integral transform of these functions and also find the some basic properties of incomplete $I-$function. Further, numerous special cases are obtained from our main results among which some are explicitly indicated. Incomplete special functions thus obtained consisting of probability theory has many potential applications which are also presented. Finally, we find the solution of non-homogeneous heat conduction equation in terms of Incomplete $I-$function.

Keywords: Gamma function; incomplete Gamma functions; incomplete I-functions; Mellin-Barnes type contour; integral transforms
Mathematics Subject Classification: Primary: 33B20, 44A10; Secondary: 33E20, 44A40

1. Introduction and preliminaries

The classical definition of Gamma function $\Gamma(\mathfrak{I})$ is defined as follows:

$$\Gamma(\mathfrak{I}) = \begin{cases} \int_0^\infty e^{-u} u^{\mathfrak{I}-1} du & (\Re(\mathfrak{I}) > 0) \\ \frac{\Gamma(\mathfrak{I}+\mathfrak{K})}{(\mathfrak{I})_{\mathfrak{K}}} & (\mathfrak{I} \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathfrak{K} \in \mathbb{N}_0), \end{cases} \tag{1.1}$$

where $(\mathfrak{I})_{\mathfrak{K}}$ denotes the Pochhammer symbol defined (for $\mathfrak{I}, \mathfrak{K} \in \mathbb{C}$) by

$$(\mathfrak{I})_{\mathfrak{K}} := \frac{\Gamma(\mathfrak{I}+\mathfrak{K})}{\Gamma(\mathfrak{I})} = \begin{cases} 1 & (\mathfrak{K} = 0;\ \mathfrak{I} \in \mathbb{C} \setminus \{0\}) \\ \mathfrak{I}(\mathfrak{I}+1)\cdots(\mathfrak{I}+s-1) & (\mathfrak{K} = s \in \mathbb{N};\ \mathfrak{I} \in \mathbb{C}), \end{cases} \tag{1.2}$$

provided that the Gamma quotient exists.
The well known incomplete Gamma functions (IGFs) $\gamma(\Im, x)$ and $\Gamma(\Im, x)$ are defined as follows

$$\gamma(\Im, x) = \int_0^x u^{\Im-1} e^{-u} du \qquad (\Re(\Im) > 0;\ x \geq 0), \tag{1.3}$$

and

$$\Gamma(\Im, x) = \int_x^\infty u^{\Im-1} e^{-u} du \qquad (x \geq 0;\ \Re(\Im) > 0 \text{ when } x = 0), \tag{1.4}$$

respectively, holds the subsequent result:

$$\gamma(\Im, x) + \Gamma(\Im, x) = \Gamma(\Im), \qquad (\Re(\Im) > 0). \tag{1.5}$$

The gamma function $\Gamma(\Im)$ and IGFs $\gamma(\Im, x)$ and $\Gamma(\Im, x)$, which is defined in (1.1), (1.3) and (1.4), respectively, are play main role in the field of science and engineering (see, for example, [2–5]; see also the recent papers [6–21]). Incomplete special functions thus obtained consisting of probability theory has many potential applications which are also presented. We obtained the solution of non-homogeneous heat conduction equation in terms of Incomplete I−function.

We now introduce the incomplete I−functions ${}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)$ containing the IGFs $\gamma(\Im, x)$ and $\Gamma(\Im, x)$ as follows:

$${}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z) = {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,n}, (a_{j\ell}, A_{j\ell})_{n+1,p_\ell} \\ \\ (\mathrm{g}_j, \mathsf{G}_j)_{1,m}, (\mathrm{g}_{j\ell}, \mathsf{G}_{j\ell})_{m+1,q_\ell} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathfrak{L}} \mathbb{K}(\xi, x) z^{-\xi} d\xi \tag{1.6}$$

where

$$\mathbb{K}(\xi, x) = \frac{\Gamma(1 - a_1 - A_1\xi, x) \prod_{j=1}^{m} \Gamma(\mathrm{g}_j + \mathsf{G}_j\xi) \prod_{j=2}^{n} \Gamma(1 - a_j - A_j\xi)}{\sum_{\ell=1}^{r} \left[\prod_{j=m+1}^{q_\ell} \Gamma(1 - \mathrm{g}_{j\ell} - \mathsf{G}_{j\ell}\xi) \prod_{j=n+1}^{p_\ell} \Gamma(a_{j\ell} + A_{j\ell}\xi) \right]}. \tag{1.7}$$

and

$${}^{(\gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z) = {}^{(\gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,n}, (a_{j\ell}, A_{j\ell})_{n+1,p_\ell} \\ \\ (\mathrm{g}_j, \mathsf{G}_j)_{1,m}, (\mathrm{g}_{j\ell}, \mathsf{G}_{j\ell})_{m+1,q_\ell} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathfrak{L}} \mathbb{L}(\xi, x) z^{-\xi} d\xi \tag{1.8}$$

where

$$\mathbb{L}(\xi, x) = \frac{\gamma(1 - a_1 - A_1\xi, x) \prod_{j=1}^{m} \Gamma(\mathrm{g}_j + \mathsf{G}_j\xi) \prod_{j=2}^{n} \Gamma(1 - a_j - A_j\xi)}{\sum_{\ell=1}^{r} \left[\prod_{j=m+1}^{q_\ell} \Gamma(1 - \mathrm{g}_{j\ell} - \mathsf{G}_{j\ell}\xi) \prod_{j=n+1}^{p_\ell} \Gamma(a_{j\ell} + A_{j\ell}\xi) \right]}. \tag{1.9}$$

The incomplete I−functions ${}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$ in (1.6) and (1.8) exist for $x \geq 0$ under the following set of conditions stated.

The Mellin Barnes contour integral $\mathfrak{L}$ is extend from $\gamma - i\infty$ to $\gamma + i\infty$, $\gamma \in \mathbb{R}$, and poles of the gamma functions $\Gamma(1 - a_j - A_j\xi)$, $j = \overline{1, n}$ do not exactly match with the poles of the gamma functions $\Gamma(\mathrm{g}_j + \mathsf{G}_j\xi)$, $j = \overline{1, m}$. The parameters m, n, p_ℓ, q_ℓ are non negative integers satisfying $0 \leq n \leq p_\ell$, $0 \leq m \leq q_\ell$ for $i = \overline{1, r}$. The parameters $A_j, \mathsf{G}_j, A_{j\ell}, \mathsf{G}_{j\ell}$ are positive numbers and $a_j, \mathrm{g}_j, a_{j\ell}, \mathrm{g}_{j\ell}$ are complex. All poles of $\mathbb{K}(\xi, x)$ and $\mathbb{L}(\xi, x)$ are supposed to be simple, and the empty product is treated as unity.

$$\mathfrak{H}_i > 0, \qquad |\arg(z)| < \frac{\pi}{2}\mathfrak{H}_i \qquad i = \overline{1, r} \tag{1.10}$$

$$\mathfrak{H}_i \geq 0, \qquad |\arg(z)| < \frac{\pi}{2}\mathfrak{H}_i \qquad \text{and} \qquad \Re(\zeta_i) + 1 < 0 \tag{1.11}$$

where

$$\mathfrak{H}_i = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} \mathsf{G}_j - \sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=m+1}^{q_i} \mathsf{G}_{ji}, \tag{1.12}$$

$$\zeta_i = \sum_{j=1}^{m} \mathrm{g}_j - \sum_{j=1}^{n} a_j + \sum_{j=m+1}^{q_i} A_{ji} - \sum_{j=n+1}^{p_i} \mathsf{G}_{ji} + \frac{1}{2}(p_i - q_i) \qquad i = \overline{1, r} \tag{1.13}$$

We are require the following results in the section 4:
(a) The orthogonal property of the Jacobi Polynomials [22, p.806, Eq (7.391.1)]

$$\int_{-1}^{1} (1-u)^{\alpha}(1-u)^{\beta} P_w^{(\alpha,\beta)}(u) P_{\mathsf{k}}^{(\alpha,\beta)}(u)\,du = h_w \delta_{w\mathsf{k}}, \qquad (\Re(\alpha) > -1, \Re(\beta) > -1) \tag{1.14}$$

where

$$h_w = \frac{2^{\alpha+\beta+1}\Gamma(\alpha + w + 1)\Gamma(\beta + w + 1)}{w!(\alpha + \beta + 1 + 2w)\Gamma(\alpha + \beta + 1 + w)}, \qquad (w = \mathsf{k}).$$

and $\delta_{w\mathsf{k}}$ is a Kronecker delta.
(b) The definite integral

$$\int_{-1}^{1} (1-u)^{\rho}(1+u)^{\beta} P_w^{(\alpha,\beta)}(u)\, {}^{(\Gamma)}I_{p_\ell, q_\ell, r}^{m,n}\left[z\left(\frac{1-u}{2}\right)^{\sigma} \,\middle|\, \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,n}, (a_{j\ell}, A_{j\ell})_{n+1,p_\ell} \\ (\mathrm{g}_j, \mathsf{G}_j)_{1,m}, (\mathrm{g}_{j\ell}, \mathsf{G}_{j\ell})_{m+1,q_\ell} \end{matrix} \right] du$$
$$= 2^{\rho+\beta+1} \frac{\Gamma(\beta + w + 1)}{w!} {}^{(\Gamma)}I_{p_\ell+2, q_\ell+2, r}^{m+1,n+1}\left[z \,\middle|\, \begin{matrix} (a_1, A_1, x), (-\rho, \sigma), (a_j, A_j)_{2,n}, (a_{j\ell}, A_{j\ell})_{n+1,p_\ell}, (\alpha - \rho, \sigma) \\ (\alpha - \rho + w, \sigma), (\mathrm{g}_j, \mathsf{G}_j)_{1,m}, (\mathrm{g}_{j\ell}, \mathsf{G}_{j\ell})_{m+1,q_\ell}, (-1 - \beta - \rho - w, \sigma) \end{matrix} \right] \tag{1.15}$$

above definite integral is valid under the following set of conditions:
(i) $\Re\left(\rho + \sigma\frac{\mathrm{g}_j}{\mathsf{G}_j}\right) > -1$, $j = 1, \cdots, m$.
(ii) $\Re(\rho) > -1, \Re(\beta) > -1$.
(iii) Eqs (1.10) to (1.13) are exist.

2. Some properties of the incomplete $I-$Functions

In this part, we present some basic properties and derivative formula for the incomplete $I-$functions:

Theorem 2.1. *The following reduction formulas holds for the incomplete I−function:*

$$ {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell-1},(a_2,A_2)\end{matrix}\right.\right] = {}^{(\Gamma)}I^{m,n-1}_{p_\ell-1,q_\ell-1,r}\left[z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{3,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell-1}\end{matrix}\right.\right], \tag{2.1}$$

and

$$ {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell-1},(a_2,A_2)\end{matrix}\right.\right] = \sigma\,{}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z^\sigma\left|\begin{matrix}(a_1,\sigma A_1,x),(a_j,\sigma A_j)_{2,n},(a_{j\ell},\sigma A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\sigma\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\sigma\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right]. \tag{2.2}$$

provided that each member in (2.1) and (2.2) *exists with* $\sigma > 0$.

Theorem 2.2. *The following derivative formula holds for the incomplete I−function:*

$$\left(\frac{d}{dz}\right)^\kappa\left\{z^{\lambda-1}\,{}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[cz^\mu\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right]\right\}$$
$$= z^{\lambda-\kappa-1}\,{}^{(\Gamma)}I^{m,n+1}_{p_\ell+1,q_\ell+1,r}\left[cz^\mu\left|\begin{matrix}(a_1,A_1,x),(1-\lambda,\mu),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(1-\lambda+\kappa,\mu),(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right] \tag{2.3}$$

provided that each member in (2.3) exists.

Proof. By differentiating the left hand side of (2.3) κ times with respect to z, we get

$$\left(\frac{d}{dz}\right)^\kappa\left\{z^{\lambda-1}\,{}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[cz^\mu\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right]\right\} = \frac{1}{2\pi i}\int_{\mathfrak{L}}\mathbb{K}(\xi,x)c^{-\xi}\left(\frac{d}{dz}\right)^\kappa\left(z^{\lambda-\mu\xi-1}\right)d\xi$$
$$= \frac{z^{\lambda-\kappa-1}}{2\pi i}\int_{\mathfrak{L}}\mathbb{K}(\xi,x)c^{-\xi}\frac{\Gamma(\lambda-\mu\xi)}{\Gamma(\lambda-\kappa-\mu\xi)}z^{-\mu\xi}d\xi$$

with the help of (1.6) and (1.7), we obtain the desired result after a little simplification. □

3. Well known integral transforms of ${}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$

In this section, we find the several well known integral transform like as *Mellin, Laplace, Hankel and Euler Beta Transform*, of the our introduce function in (1.6).

3.1. Mellin transform

The well known *Mellin transform* of a function $f(z)$ is defined by [23, p.340, Eq (8.2.5)]

$$\mathfrak{M}\{f(z);\mathfrak{p}\}=\int_0^\infty z^{\mathfrak{p}-1}f(z)dz, \qquad (\Re(\mathfrak{p})>0) \tag{3.1}$$

provided that the improper integral exists.

Theorem 3.1. *If*

$$\mathfrak{H}_i>0, \qquad \mu>0, \qquad |\arg(z)|<\frac{\pi}{2}\mathfrak{H}_i, \qquad \Re(\zeta_i)+1<0 \qquad i=\overline{1,r}$$

$$-\mu\min_{1\leqq j\leqq m}\Re\left(\frac{\mathrm{g}_j}{\mathsf{G}_j}\right)<\Re(\mathfrak{p})<\mu\min_{1\leqq j\leqq n}\Re\left(\frac{1-a_j}{A_j}\right), \qquad c>0 \qquad \textit{and} \qquad x\geqq 0$$

Then the Mellin transform of incomplete $I-$function defined as follows:

$$\mathfrak{M}\left\{{}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[cz^\mu\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right];\mathfrak{p}\right\}=\frac{c^{-\mathfrak{p}}}{\mu}\mathbb{K}\left(\frac{\mathfrak{p}}{\mu},x\right) \tag{3.2}$$

provided that each member of the assertions (3.2) *exists and* $\mathbb{K}(\xi,x)$ *is given in* (1.7).

Proof. The Mellin transform of (1.6) is based upon the Mellin Inversion Theorem as well as the Mellin-Barnes type contour integral in (1.7) which defines the incomplete $I-$function ${}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$. □

3.2. Laplace transform

The Classical *Laplace transform* of a function $f(z)$ is defined by [23, p.134, Eq (3.2.5)]

$$\mathcal{L}\{f(z);\mathfrak{p}\}=\int_0^\infty e^{-\mathfrak{p}z}f(z)dz, \qquad (\Re(\mathfrak{p})>0) \tag{3.3}$$

provided that the improper integral exists.

Theorem 3.2. *If*

$$\mathfrak{H}_i>0, \qquad \mu>0, \qquad |\arg(z)|<\frac{\pi}{2}\mathfrak{H}_i, \qquad \Re(\zeta_i)+1<0 \qquad i=\overline{1,r}$$

$$-\mu\min_{1\leqq j\leqq m}\Re\left(\frac{\mathrm{g}_j}{\mathsf{G}_j}\right)<\Re(\lambda), \qquad \Re(\mathfrak{p})>0, \qquad c>0 \qquad \textit{and} \qquad x\geqq 0$$

Then the Laplace transform of incomplete $I-$function defined as follows:

$$\begin{aligned}&\mathcal{L}\left\{z^{\lambda-1}\,{}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[cz^\mu\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right];\mathfrak{p}\right\}\\ &=\mathfrak{p}^{-\lambda}\,{}^{(\Gamma)}I^{m,n+1}_{p_\ell+1,q_\ell,r}\left[c\mathfrak{p}^{-\mu}\,\middle|\,\begin{matrix}(a_1,A_1,x),(1-\lambda,\mu),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right]\end{aligned} \tag{3.4}$$

provided that each member in (3.4) *exist.*

Proof. To prove the left hand side of (3.4), by taking the Laplace transform given in (3.3) of (1.6), we get

$$\mathcal{L}\left\{z^{\lambda-1(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[cz^{\mu}\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right];\mathfrak{p}\right\}=\mathcal{L}\left\{z^{\lambda-1}\int_{\mathfrak{L}}\mathbb{K}(\xi,x)(cz^{\mu})^{-\xi};\mathfrak{p}\right\}$$

where $\mathbb{K}(\xi, x)$ is given in (1.7).
Further, on interchanging the order of integral and contour integral (which is admissible under the conditions presented), it yields

$$\begin{aligned}\mathcal{L}\left\{z^{\lambda-1}\int_{\mathfrak{L}}\mathbb{K}(\xi,x)(cz^{\mu})^{-\xi};\mathfrak{p}\right\}&=\int_{\mathfrak{L}}\mathbb{K}(\xi,x)c^{-\xi}\mathcal{L}\left\{z^{\lambda-\mu\xi-1};\mathfrak{p}\right\}d\xi\\&=\int_{\mathfrak{L}}\mathbb{K}(\xi,x)c^{-\xi}\frac{\Gamma(\lambda-\mu\xi)}{\mathfrak{p}^{\lambda-\mu\xi}}d\xi\end{aligned}$$

Finally, with help of (1.6) and (1.7), we get the right hand side of (3.4) after a little simplification. □

3.3. Hankel transform

The *Hankel transform* of a function $f(z)$ is defined by [23, p.317, Eq (7.2.8)]

$$\mathcal{H}_{\alpha}\{f(z);\mathfrak{p}\}=\int_0^{\infty}z\mathbb{J}_{\alpha}(\mathfrak{p}z)f(z)dz,\qquad(\Re(\mathfrak{p})>0)\tag{3.5}$$

provided that the integral in (3.5) exists, $\mathbb{J}_\alpha$ is the Bessel function of order α. Now, we establish an integral which involving the Bessel function $\mathbb{J}_\alpha(z)$ and our introduce incomplete $I-$function, which can easily be reduces to *Hankel transform* of function ${}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$.

Theorem 3.3. *If*

$$\begin{aligned}&\mathfrak{H}_i>0,\qquad \mu>0,\qquad |\arg(z)|<\frac{\pi}{2}\mathfrak{H}_i,\qquad \Re(\zeta_i)+1<0\qquad i=\overline{1,r}\\&-1<\Re(\lambda+\alpha)+\mu\min_{1\leqq j\leqq m}\Re\left(\frac{\mathrm{g}_j}{\mathsf{G}_j}\right)<\Re(\lambda+\alpha)+\mu\min_{1\leqq j\leqq n}\Re\left(\frac{1-a_j}{A_j}\right),\\&c>0,\qquad \Re(\mathfrak{p})>0\qquad and\qquad x\geqq 0\end{aligned}$$

Then the Hankel type transform of incomplete I−function defined as follows:

$$\begin{aligned}&\int_0^{\infty}z^{\lambda-1}\mathbb{J}_{\alpha}(\mathfrak{p}z)^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[cz^{\mu}\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right]dz\\&=\frac{2^{\lambda-1}}{\mathfrak{p}^{\lambda}}{}^{(\Gamma)}I^{m,n+1}_{p_\ell+2,q_\ell,r}\left[c\left(\frac{2}{\mathfrak{p}}\right)^{\mu}\left|\begin{matrix}(a_1,A_1,x),(1-\frac{\lambda-\alpha}{2},\frac{\mu}{2}),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell},(1+\frac{\alpha-\lambda}{2},\frac{\mu}{2})\end{matrix}\right.\right]\end{aligned}\tag{3.6}$$

provided that both sides member in (3.6) *exist.*

Proof. To prove the assertion (3.6), incomplete $I-$function, which is define in (1.6) and (1.7), express in terms of Mellin-Barnes type contour integral, we get (say Ω)

$$\Omega = \frac{1}{2\pi i} \int_0^\infty z^{\lambda-1} \mathbb{J}_\alpha(\mathfrak{p}z) \int_{\mathfrak{L}} \mathbb{K}(\xi, x)(cz^\mu)^{-\xi} d\xi dz$$

where $\mathbb{K}(\xi, x)$ is given in (1.7).
Further, interchanging the order of integrals, which can be valid under the given conditions, to find

$$\Omega = \frac{1}{2\pi i} \int_{\mathfrak{L}} \mathbb{K}(\xi, x) c^{-\xi} \left\{ \int_0^\infty z^{\lambda-\mu\xi-1} \mathbb{J}_\alpha(\mathfrak{p}z) d\xi \right\} dz$$

Next, using the known formula [24, Vol. II, p.49, Eq 7.3.3(19)], we get

$$\Omega = \frac{2^{\lambda-1}\mathfrak{p}^{-\lambda}}{2\pi i} \int_{\mathfrak{L}} \mathbb{K}(\xi, x) c^{-\xi} \frac{2^{-\mu\xi}}{\mathfrak{p}^{-\mu\xi}} \frac{\Gamma\left(\frac{\lambda+\alpha-\mu\xi}{2}\right)}{\Gamma\left(1 + \frac{\alpha-\lambda+\mu\xi}{2}\right)} d\xi$$

Finally, with help of (1.6) and (1.7), we get the desired result after interpreting the last identites. □

3.4. *Euler's Beta transform*

The *Euler's Beta transform* of a function $f(z)$ is defined by [25]

$$\mathcal{B}\{f(z) : \alpha, \beta\} = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} f(z) dz, \qquad (\Re(\alpha) > 0, \Re(\beta) > 0) \tag{3.7}$$

Now, we give the following *Euler's Beta transform* of the incomplete $I-$function.

Theorem 3.4. *If*

$$\mathfrak{H}_i > 0, \qquad \mu > 0, \qquad |\arg(z)| < \frac{\pi}{2}\mathfrak{H}_i, \qquad \Re(\zeta_i) + 1 < 0 \qquad i = \overline{1, r}$$

$$\Re(\alpha) + \mu \min_{1 \leqq j \leqq m} \Re\left(\frac{\mathrm{g}_j}{\mathsf{G}_j}\right) > 0, \qquad \Re(\beta) > 0, \qquad c > 0.$$

Then the following Beta transform holds for $x \geqq 0$*:*

$$\mathcal{B}\left\{ {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r} \left[cz^\mu \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,n}, (a_{j\ell}, A_{j\ell})_{n+1,p_\ell} \\ \\ (\mathrm{g}_j, \mathsf{G}_j)_{1,m}, (\mathrm{g}_{j\ell}, \mathsf{G}_{j\ell})_{m+1,q_\ell} \end{matrix} \right. \right] : \alpha, \beta \right\}$$

$$= \Gamma(\beta)\, {}^{(\Gamma)}I^{m,n+1}_{p_\ell+1,q_\ell+1,r} \left[c \left| \begin{matrix} (a_1, A_1, x), (1-\alpha, \mu), (a_j, A_j)_{2,n}, (a_{j\ell}, A_{j\ell})_{n+1,p_\ell} \\ \\ (\mathrm{g}_j, \mathsf{G}_j)_{1,m}, (\mathrm{g}_{j\ell}, \mathsf{G}_{j\ell})_{m+1,q_\ell}, (1-\alpha-\beta, \mu) \end{matrix} \right. \right] \tag{3.8}$$

Proof. First, we write the Mellin-Barnes contour integral of the incomplete $I-$function in (1.6) and (1.7), interchange the order of integrals and then apply the well known definition of Beta function. We get the right hand side of (3.8). □

Remark 1. *It may be remarked that the above integral transforms of the incomplete $I-$function reduces incomplete $H-$function, Fox's $H-$function and many other special function.*

4. Applications of the incomplete I-Functions

The incomplete I-functions ${}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$ defined in (1.6) and (1.8) reduce to the several familiar special function (for example: Fox's H-function, Incomplete H-function, I-function, etc.) as follows:

If we set $x = 0$, then (1.6) and (1.8) reduces to the $I-$function introduced by Saxena [1]:

$$ {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z\left|\begin{matrix}(a_1,A_1,0),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right] = I^{m,n}_{p_\ell,q_\ell,r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right]. \tag{4.1} $$

Again setting $r = 1$ in (1.6) and (1.8), then it's reduces to the Incomplete $H-$functions introduced by Srivastva [26](see also, [27]):

$$ {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,1}\left[z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right] = \Gamma^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,q}\end{matrix}\right.\right], \tag{4.2} $$

and

$$ {}^{(\gamma)}I^{m,n}_{p_\ell,q_\ell,1}\left[z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right] = \gamma^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,q}\end{matrix}\right.\right]. \tag{4.3} $$

A complete description of Incomplete $H-$functions can be found in the article [26].

Further, we take $x = 0$ and $r = 1$ in (1.6), the Incomplete $I-$function reduces to the familiar Fox's $H-$function which were defined and represented in the following manner (see, for example, [28, p. 10]):

$$ \begin{aligned} {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,1}\left[z\left|\begin{matrix}(a_1,A_1,0),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell}\\ \\ (\mathrm{g}_j,\mathsf{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathsf{G}_{j\ell})_{m+1,q_\ell}\end{matrix}\right.\right] &= H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),\cdots,(a_p,A_p)\\ \\ (\mathrm{g}_1,\mathsf{G}_1),\cdots,(\mathrm{g}_q,\mathsf{G}_q)\end{matrix}\right.\right] \\ &:= \frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}}\Theta(\mathfrak{s})z^{\mathfrak{s}}\,d\mathfrak{s}, \end{aligned} \tag{4.4} $$

where $\mathrm{i} = \sqrt{-1}$, $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$ \Theta(\mathfrak{s}) = \frac{\prod_{j=1}^{m}\Gamma(\mathrm{g}_j - \mathsf{G}_j\mathfrak{s})\prod_{j=1}^{n}\Gamma(1-a_j+A_j\mathfrak{s})}{\prod_{j=m+1}^{q}\Gamma(1-\mathrm{g}_j+\mathsf{G}_j\mathfrak{s})\prod_{j=n+1}^{p}\Gamma(a_j-A_j\mathfrak{s})}, $$

and

$$ 1 \leqq m \leqq q \qquad \text{and} \qquad 0 \leqq n \leqq p \qquad (m,q \in \mathbb{N} = \{1,2,3,\cdots\};\ n,p \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}), $$

an empty product being treated to be unity. A complete details can be found in the text book (see, for details, [28, 29]).

4.1. Application of the incomplete I-Functions in probability theory

Several applications of extended Gauss hypergeometric function, incomplete gamma function, fox's H-function, etc. in communication theory, statistical distribution theory, groundwater pumping modeling, quantum physics, Velocity distribution in an ideal gas and solution of fractional advection dispersion equation in terms of Fox's H-function.It is believed that the incomplete I-Functions ${}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$ and ${}^{(\gamma)}I^{m,n}_{p_\ell,q_\ell,r}(z)$, which we have studied here, have the potential for applications in the extended forms of similar and other situations. For example, in probability theory, the incomplete I-functions finds uses in the analytic investigation of the survival and cumulative probability density functions along the lines given by Chaudhry and Qadir [30] who made use of the incomplete exponential functions presented by

$$e((u,v);\rho) = \sum_{s=0}^{\infty} \frac{\gamma(\rho+s,u)}{\Gamma(\rho+s)} \frac{v^s}{s!} = {}_1\gamma_1 \left[\begin{array}{c} (\rho,u); \\ \rho; \end{array} v \right] \tag{4.5}$$

$$E((u,v);\rho) = \sum_{s=0}^{\infty} \frac{\gamma(\rho+s,u)}{\Gamma(\rho+s)} \frac{v^s}{s!} = {}_1\Gamma_1 \left[\begin{array}{c} (\rho,u); \\ \rho; \end{array} v \right] \tag{4.6}$$

In fact, the incomplete I-function representations of the above-defined incomplete exponential $e((u,v);\rho)$ and $E((u,v);\rho)$ functions are given by

$$e((u,v);\rho) = {}^{(\gamma)}I^{1,1}_{1,2,1} \left[-v \,\middle|\, \begin{array}{c} (1-\rho,u,1) \\ \\ (0,1),(1-\rho,1) \end{array} \right] \tag{4.7}$$

$$E((u,v);\rho) = {}^{(\Gamma)}I^{1,1}_{1,2,1} \left[-v \,\middle|\, \begin{array}{c} (1-\rho,u,1) \\ \\ (0,1),(1-\rho,1) \end{array} \right] \tag{4.8}$$

4.2. Application of the incomplete I-Function in heat conduction

We are driving a solution $y(u,t)$ for temperature distribution in a insulated non-homogeneous bar with thermal conductivity varies as $(1-u^2)$ and having ends at $u = \pm 1$. The function $y(u,t)$ satisfies the following partial differential equation of heat conduction [31, p.197, Eq (8)]:

$$\frac{\partial y}{\partial t} = \lambda \frac{\partial}{\partial u} \left[(1-u^2) \frac{\partial y}{\partial u} \right] \tag{4.9}$$

where λ is a constant which is treated as thermal coefficient.
At $u = \pm 1$, both ends of a bar are insulated due to the conductivity zero there, is a boundary conditions and the initial condition:

$$y(u,0) = f(u), \qquad -1 < u < 1 \tag{4.10}$$

Now, we consider

$$f(u) = (1-u)^{\rho} \cdot {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z\left(\frac{1-u}{2}\right)^{\sigma} \middle| \begin{matrix} (a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell} \\ \\ (\mathrm{g}_j,\mathrm{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathrm{G}_{j\ell})_{m+1,q_\ell} \end{matrix} \right] \tag{4.11}$$

Let the solution of (4.9) can be represented in the following form

$$y(u,t) = \sum_{\mathrm{k}=0}^{\infty} R_{\mathrm{k}}\, e^{-\lambda \mathrm{k}(\mathrm{k}+1)t} P^{(\alpha,\beta)}_{\mathrm{k}}(u) \tag{4.12}$$

putting $t = 0$ in (4.12) and using (4.11), we have obtain that

$$f(u) = (1-u)^{\rho} \cdot {}^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z\left(\frac{1-u}{2}\right)^{\sigma} \middle| \begin{matrix} (a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell} \\ \\ (\mathrm{g}_j,\mathrm{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathrm{G}_{j\ell})_{m+1,q_\ell} \end{matrix} \right] = \sum_{\mathrm{k}=0}^{\infty} R_{\mathrm{k}}\, P^{(\alpha,\beta)}_{\mathrm{k}}(u) \tag{4.13}$$

where $P^{(\alpha,\beta)}_{\mathrm{k}}(u)$ is a Jacobi Polynomial (see, for details, [32, p. 59, Eq (4.1.3)] and [33, p.35, Eq (34)]). Equation (4.13) is valid because $f(u)$ is continuous in $u \in [-1,1]$ and has a piecewise continuous derivative there, then with $\alpha > -1, \beta > -1$, the Jacobi series (4.13) converges uniformaly to $f(u)$ in $u \in [-1+\in, 1+\in]$, $0 <\in< 1$.
Now, Eq (4.13) multiply by $(1-u)^{\alpha}(1+u)^{\beta}P^{(\alpha,\beta)}_{w}(u)$ and integrate -1 to 1, we get

$$A_w = h_w^{-1}\int_{-1}^{1}(1-u)^{\rho+\alpha}(1+u)^{\beta}P^{(\alpha,\beta)}_{w}(u)^{(\Gamma)}I^{m,n}_{p_\ell,q_\ell,r}\left[z\left(\frac{1-u}{2}\right)^{\sigma} \middle| \begin{matrix} (a_1,A_1,x),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell} \\ \\ (\mathrm{g}_j,\mathrm{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathrm{G}_{j\ell})_{m+1,q_\ell} \end{matrix} \right] du \tag{4.14}$$

where h_w is calculate with the help of (1.14), we get

$$h_w = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+w+1)\Gamma(\beta+w+1)}{w!(\alpha+\beta+1+2w)\Gamma(\alpha+\beta+1+w)}$$

Now with the help of result (1.15), we obtain

$$A_w = \frac{2^{\rho}(2w+\alpha+\beta+1)\Gamma(w+\alpha+\beta+1)}{\Gamma(w+\alpha+1)}{}^{(\Gamma)}I^{m+1,n+1}_{p_\ell+2,q_\ell+2,r}\left[z \middle| \begin{matrix} A^* \\ \\ B^* \end{matrix} \right]$$

where

$$A^* = (a_1,A_1,x),(-\rho-\alpha,\sigma),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell},(-\rho,\sigma)$$
$$B^* = (-\rho+w,\sigma),(\mathrm{g}_j,\mathrm{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathrm{G}_{j\ell})_{m+1,q_\ell},(-1-\beta-\rho-\alpha-w,\sigma)$$

Next, substituting the value of R_k in (4.12), we arrive at the desired solution

$$y(u,t) = 2^{\rho}\sum_{\mathrm{k}=0}^{\infty} f(\mathrm{k})e^{-\lambda\mathrm{k}(\mathrm{k}+1)t(\Gamma)}I^{m+1,n+1}_{p_\ell+2,q_\ell+2,r}\left[z \middle| \begin{matrix} (a_1,A_1,x),(-\rho-\alpha,\sigma),(a_j,A_j)_{2,n},(a_{j\ell},A_{j\ell})_{n+1,p_\ell},(-\rho,\sigma) \\ \\ (-\rho+\mathrm{k},\sigma),(\mathrm{g}_j,\mathrm{G}_j)_{1,m},(\mathrm{g}_{j\ell},\mathrm{G}_{j\ell})_{m+1,q_\ell},(-1-\beta-\rho-\alpha-\mathrm{k},\sigma) \end{matrix} \right] \tag{4.15}$$

where

$$f(\mathsf{k}) = \frac{2^{\rho}\Gamma(2\mathsf{k}+\alpha+\beta+1)\Gamma(\mathsf{k}+\alpha+\beta+1)}{\Gamma(\mathsf{k}+\alpha+1)}$$

Remark 2. *If incomplete $I-$function reduces into $H-$function in* (4.11) *then, we get the result obtained by Chaurasia [34].*

5. Conclusions

In this work, we introduce a new incomplete $I-$functions which . The incomplete $I-$function is an extension of the $I-$function given by Saxena [1] which is a extension of a familiar Fox's $H-$function. Next, we find the several interesting classical integral transforms of incomplete $I-$function and also find the some basic properties of incomplete $I-$function. Further, numerous special cases are obtained from our main results among which some are explicitly indicated. Finally, we find the solution of non-homogeneous heat conduction equation in terms of Incomplete $I-$function.

Acknowledgments

The authors would like to express their deep-felt thanks for the reviewers' valuable comments to improve this paper as it stands. The present investigation was supported, in part, by the TEQIP-III under CRS Grant 1-5730065311.

Conflict of interest

The authors declare no conflict of interest.

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