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Research article

Solitary wave solutions of few nonlinear evolution equations

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Abstract: The solitary wave solutions of nonlinear evolution equations, in the recent years is being attractive in the field of physical sciences and engineering. In this article, we have investigated further general solitary wave solutions of three important nonlinear evolution equations, via the simplified MCH equation, the Pochhammer-Chree equation and the Schrödinger-Hirota equation by using modified simple equation method. These equations play an important role in the study of nonlinear sciences. The obtained solutions are expressed in terms of exponential and trigonometric functions including kink, singular kink and periodic soliton solutions. It is shown that the obtained solutions are more general and fresh and can be helpful to analyze the intricate physical incident in mathematical physics.

Keywords: modified simple equation method; simplified MCH equation; Pochhammer-Chree equation; Schrödinger-Hirota equation; solitary wave solutions
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1. Introduction

Now it is well recognized that nonlinear evolution equations (NLEEs) and its solutions are the most embracing way to describe the physical significance of nonlinear phenomena appearing in the field of science and engineering. In particular, the soliton solutions are most remarkable in the study of the nonlinear physical sciences, as for instance the wave phenomena are observed in fluid mechanics, optical fibers, biophysics, high-energy physics, chemical kinematics etc. But, the nonlinear processes are one of the basic challenges and not easy to control, because the nonlinear characteristic of the system sharply changes due to small changes of valid parameters including time. Thus the issue becomes more intricate and hence ultimate solution is needed. Therefore, searching solitary wave solutions to NLEEs is becoming increasingly attractive field in nonlinear sciences day by day. There are lot of NLEEs that can be solved using different mathematical methods. For these physical problems, soliton solutions, compactons, singular solitons and the other solutions have been originated. However, not all equations posed of these models are solvable. Thus, the methods for deriving exact solutions for the governing equations have to be developed. As a result, significant improvements have been made for searching solitary wave solutions to NLEEs and many effective and powerful methods have been established to examine the NLEEs, such as the nonlinear transformation method [1], the Hirota's bilinear transformation method [2,3], the first integral method [4], the sine-cosine method [5,6], the Jacobi-elliptic function expansion method [7,8], the functional variable method [9], the Adomian decomposition method [10], the modified Exp-function method [11], the generalized Riccati equation method [12], the Exp-function method [13,14], the bifurcation theory [15], the exp(-Φ(η))-expansion method [16,17], the extended direct algebraic method [18–20], the (G'/G)-expansion method and its different variant [21–28], the variational method [29–31], the generalized Kudryashov method [32], the ansatz method [33–38], the modified simple equation (MSE) [39–42] method, the modified extended direct algebraic method [43,44], the modified extended auxiliary equation method [45], the modified auxiliary equation method [46,47], the generalized exponential rational function method [48], the bilinear forms [49], the generalized unified method [50,51], the extended unified method [52,53] etc. The modified simple equation method is a recently developed method and getting popularity in use because of its straight forward calculation procedure. The objective of this article is to contrivance the modified simple equation method to construct solitary wave solutions to the simplified modified Camassa-Holm (MCH) equation, the Pochhammer-Chree (PC) equation and the Schrödinger-Hirota equation. The rest of the article is organized as follows: In section 2, we summarize the description of the method. In section 3, the MSE method is applied to extract exact soliton solutions to the NLEEs stated earlier. In section 4, explanation and physical interpretation of the solutions are presented and in section 5, we have drawn our conclusions.

2. The modified simple equation (MSE) method

To describe the MSE, let us consider a nonlinear evolution equation in two independent variables x and t in the form:

$$\mathcal{F}\left(u, u_{t,} u_{x,} u_{tt,} u_{xx,} u_{xt,} \dots\right) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function and $\mathcal{F}$ is a polynomial of $u(x,t)$ and its partial derivatives wherein the highest order derivatives and nonlinear terms are involved and the subscripts are used for partial derivatives. The essential steps of this method are presented as the following:

Step 1: Initiating a compound variable ξ, we combine the real variables x and t:

$$u(x,y,t) = u(\xi), \quad \xi = k\,x + l\,y \pm c\,t, \tag{2.2}$$

where c is the speed of the traveling wave.

The traveling wave transformations (2.2) allow us in reducing Eq. (2.1) into an ODE for $u = u(\xi)$ in the form:

$$\mathcal{R}(\mathrm{u}, u', u'', u'''\dots) = 0, \tag{2.3}$$

where $\mathcal{R}$ is a polynomial in $u(\xi)$ and its derivatives with respect to ξ.

Step 2: Affording the MSE method, it is assume that the solution of (2.3) can be expressed in the subsequent form:

$$u(\xi) = \sum_{i=0}^{N} a_i \left(\frac{\psi'(\xi)}{\psi(\xi)}\right)^i, \tag{2.4}$$

where $a_i (i = 0, 1, 2, 3, \dots N)$ are arbitrary constants to be determined such that $a_N \neq 0$ and $\psi(\xi)$ is an unknown function to be evaluated later, such that $\psi'(\xi) \neq 0$. The attribute and uniqueness of this method is that, it is not possible to assumption in advance what kind of solutions one may obtain through this method. Thus, it might be possible to achieve some fresh solution by this method.

Step 3: The positive integer N arises in (2.4) can be found by the balancing principle of the highest order of linear and nonlinear terms appearing in (2.3).

Step 4: Compute the necessary derivatives $u', u'', \dots$ and insert Eq. (2.4) into (2.3) and then we account the function $\psi(\xi)$. The above procedure yields a polynomial in $(1/\psi(\xi))$. Equating the coefficients of same power of this polynomial to zero delivers a system of algebraic and differential equations that can be solved to get $a_i (i = 0, 1, 2, 3, \dots N)$ and $\psi(\xi)$. This completes the determination of solutions of Eq. (2.1).

3. Determination of the solutions

In this section, we will investigate the solitary wave solutions leading to the simplified MCH equation, the PC equation and the Schrödinger-Hirota equation using the MSE method.

3.1. The simplified MCH equation

In this subsection, the MSE method has been put in use to examine the closed form soliton solutions and then the solitary wave solution to the simplified MCH equation of the form [19]:

$$u_t + 2\alpha u_x - u_{xxt} + \beta u^2 u_x = 0. \ \text{ where } \alpha \in \Re, \beta > 0. \tag{3.1}$$

where α and β are constants. Camassa and Holm derived a completely integrable wave equation namely CH equation for water waves by retaining two terms that are usually neglected in the small amplitude, shallow water limit [54]. Tian and Song [55] has investigated MCH equation and obtained peaked solitary wave solutions. Wazwaz [56] investigated a modified form of the Camassa-Holm equation, which is simplified from of MCH equation and Eq. (3.1) is obtained by considering $n = 2$ is called the simplified MCH equation. More details can be found in references [14,36,54,55].

The traveling wave transformation $u(x,t) = u(\xi)$, $\xi = kx - ct$, converts Eq. (3.1) to the following form

$$-cu' + 2\alpha k u' + ck^2 u''' + \beta k u^2 u' = 0, \tag{3.2}$$

where c is the wave speed.
Integrating (3.2) with respect to ξ once and the setting the constant of integration to zero, we obtain the following result

$$(2\alpha k - c)u + ck^2u'' + \frac{\beta k}{3}u^3 = 0. \quad (3.3)$$

Since, solitary waves are localized and they decay as $\xi \to \pm\infty$ and we are probing solitary wave solutions, therefore the boundary conditions must be $u(\xi) \to 0,\ u'(\xi) \to 0,\ u''^{(\xi)} \to 0, \ldots$ etc. as $\xi \to \pm\infty$ and these boundary conditions yield zero constant.

Balancing between the terms u'' and u^3 yield $N = 1$. Therefore, the solution of Eq. (3.3) becomes

$$u(\xi) = a_0 + a_1\left(\frac{\psi'}{\psi}\right), \quad (3.4)$$

where a_0 and a_1 are constants, such that $a_1 \neq 0$ and $\psi(\xi)$ is an unknown function to be calculated. Inserting (3.4) and its derivatives into (3.3) yield a polynomial and equating the coefficients of ψ^0, ψ^{-1}, ψ^{-2}, ψ^{-3} to zero, we achieve the successive algebraic and differential equations

$$\frac{\beta k}{3}a_0^3 + 2\alpha k a_0 - c a_0 = 0 \quad (3.5)$$

$$-ca_1\psi''' + \beta a_0^2 a_1 \psi' + 2ka_1\psi' - ca_1\psi' = 0 \quad (3.6)$$

$$-3k^2 c a_1 \psi'\psi'' + \beta k a_0 a_1^2 {\psi'}^2 = 0 \quad (3.7)$$

$$2k^2 c a_1 {\psi'}^3 + \frac{\beta k}{3}a_1^3{\psi'}^3 = 0 \quad (3.8)$$

From Eq. (3.5) and Eq. (3.8), we obtain $a_0 = 0, \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}$ and $a_1 = \pm\frac{\sqrt{-6kc}}{\sqrt{\beta}}$, since $a_1 \neq 0$.

From Eq. (3.7), it can be deduced that

$$\frac{\psi''}{\psi'} = \frac{\beta a_0 a_1}{3kc} = \theta \quad (3.9)$$

Integrating (3.9) with respect to ξ, yields

$$\psi' = c_1 e^{\theta\xi} \quad (3.10)$$

and $\quad \psi = \frac{c_1 e^{\theta\xi}}{\theta} + c_2, \quad (3.11)$

where c_1 and c_2 are constants of integration and $\theta = \frac{\beta a_0 a_1}{3kc}$.

Case 1: When $a_0 = \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}$ and $a_1 = \pm\frac{\sqrt{-6kc}}{\sqrt{\beta}}$, solving Eqs. (3.6) and (3.7) with (3.10) and (3.11), provides $c = c$ and $\theta = \pm\frac{\sqrt{2(2\alpha k - c)}}{k\sqrt{c}}$. Making use of the values of a_0, a_1 and c in (3.4), we found the subsequent general solution

$$u(\xi) = \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}\left(1 - \frac{2c_1}{c_1 + c_2\theta e^{-\theta\xi}}\right), \quad (3.12)$$

where $\xi = (kx - ct)$ and $\theta = \pm\frac{\sqrt{2(2\alpha k - c)}}{k\sqrt{c}}$.

Thus, in (x, t) variables, the general closed form traveling wave solution of the simplified MCH equation is obtained as follows:

$$u(x,t) = \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}\left(1 - \frac{2c_1}{c_1 + c_2\theta e^{-(kx-ct)\theta}}\right) \quad (3.13)$$

The exponential solution (3.13) can be transformed to the closed form hyperbolic function solution as

$$u(x,t) = \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}\left(1 - \frac{2c_1\left(cosh\left(\frac{(kx-ct)\theta}{2}\right) + sinh\left(\frac{(kx-ct)\theta}{2}\right)\right)}{(c_1+c_2\theta)\, cosh\left(\frac{(kx-ct)\theta}{2}\right) + (c_1-c_2\theta)\, sinh\left(\frac{(kx-ct)\theta}{2}\right)}\right) \quad (3.14)$$

Since c_1 and c_2 are integral constants, we may generally pick their values, Therefore, if we put $c_1 = \theta$ and $c_2 = 1$ into solution (3.14), we attain the following kink shape soliton solution to the simplified MCH equation

$$u(x,t) = \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}\left(tanh\left(\frac{(kx-ct)\theta}{2}\right)\right) \quad (3.15)$$

On the other hand, if we put $c_1 = -\theta$ and $c_2 = 1$ into solution (3.14), we obtain the following closed form singular kink type solution to the simplified MCH equation

$$u(x,t) = \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}\left(coth\left(\frac{(kx-ct)\theta}{2}\right)\right) \quad (3.16)$$

Using hyperbolic function identities, solutions (3.15) and (3.16) can be rewritten as

$$u(x,t) = \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}\left(itan\left(\frac{i(kx-ct)\theta}{2}\right)\right) \quad (3.17)$$

and $$u(x,t) = \pm\frac{\sqrt{3(c-2\alpha k)}}{\sqrt{\beta k}}\left(icot\left(\frac{i(kx-ct)\theta}{2}\right)\right) \quad (3.18)$$

Case 2: When $a_0 = 0$ and $a_1 = \pm\frac{\sqrt{-6kc}}{\sqrt{\beta}}$, solving Eqs. (3.6) and (3.7) together with (3.10) and (3.11), we achieve $c = 2\alpha k$ and $\theta = 0$. Introducing these values into solution (3.4), we achieve the next rational function solution

$$u(\xi) = \pm\frac{2k\sqrt{3\alpha}}{\sqrt{\beta}}\left(\frac{1}{\xi}\right), \quad (3.19)$$

where $\xi = (kx - 2\alpha kt)$.

Thus, in (x, t) variables, the general closed form traveling wave solution of the simplified MCH equation is obtained as follows:

$$u(x,t) = \pm\frac{2\sqrt{3\alpha}}{\sqrt{\beta}}.\frac{1}{(x-2\alpha t)} \quad (3.20)$$

3.2. The Pochhammer-Chree equation

In this subsection, we will put in use the method described in section 2 to extract the closed form solutions of the Pochhammer-Chree (PC) equation of the form [27]:

$$u_{tt} - u_{ttxx} - (\alpha u + \beta u^3 + \gamma u^5)_{xx} = 0, \tag{3.21}$$

where α, β and γ are constants and the equation describes the nonlinear model for longitudinal wave propagation in elastic rods. Li *et al*. [15] and Zhang *et al*. [56] derived some explicit solitary wave solution to the generalized PC equation of the form $u_{tt} - u_{ttxx} - (\alpha u + \beta u^{n+1} + \gamma u^{2n+1})_{xx} = 0,\ n \geq 1$, considering $n = 1$. In this article, we will study Eq. (3.21) considering $n = 2$. For details see the references [15,56–57].

The traveling wave transformation $u(x,t) = u(\xi),\ \xi = x - ct$, where c is the wave speed to be determined latter, converts (3.21) to the ODE in the form

$$c^2 u'' - c^2 u^{(iv)} - (\alpha u + \beta u^3 + \gamma u^5)'' = 0, \tag{3.22}$$

here $u^{(iv)}$stands for the fourth derivative and u'' indicate the second derivative of u with respect to ξ. Eq. (3.22) is integrable, therefore integrating twice and setting constant of integration to zero, we obtain

$$(c^2 - \alpha)u - c^2 u'' - \beta u^3 - \gamma u^5 = 0. \tag{3.23}$$

Taking homogeneous balance between u'' and u^5 yields $n = 1/2$. To establish a closed form analytic solution through an ansatz method n should be an integer. This requires the use of the transformation $u(\xi) = v(\xi)^{\frac{1}{2}}$. This transformation converts Eq. (3.23) to the following equation:

$$4(c^2 - \alpha)v^2 - 2c^2 v v'' + c^2 (v')^2 - 4\beta v^3 - 4\gamma v^4 = 0. \tag{3.24}$$

Balancing $vv''$and v^4 gives $N = 1$. Therefore, the solution structure of Eq. (3.24) is identical to solution (3.4). Substituting solution (3.4) and its derivatives into Eq. (3.24) and completing the analogous process described in subsection 3.1, we achieve the successive algebraic and differential equations,

$$4c^2 a_0^2 - 4\beta a_0^3 - 4\alpha a_0^2 - 4\gamma a_0^4 = 0 \tag{3.25}$$

$$(8c^2 a_0 a_1 - 8\alpha a_0 a_1 - 12\beta a_0^2 a_1 - 16\gamma a_0^3 a_1)\psi' + 2c^2 a_0 a_1 \psi''' = 0 \tag{3.26}$$

$$c^2 a_1^2 {\psi''}^2 + 4a_1^2(c^2 - \alpha - 3\beta a_0 - 6\gamma a_0^2){\psi'}^2 + 6c^2 a_0 a_1 \psi'' \psi' - 2c^2 a_1^2 \psi' \psi''' = 0 \tag{3.27}$$

$$4c^2 a_1^2 {\psi'}^2 \psi'' - (4c^2 a_0 a_1 + 4\beta a_1^3 + 16\gamma a_0 a_1^3){\psi'}^3 = 0 \tag{3.28}$$

$$-4\gamma a_1^4 {\psi'}^4 - 3c^2 a_1^2 {\psi'}^4 = 0 \tag{3.29}$$

From Eqs. (3.25) and (3.29), we achieve $a_0 = 0,\ \frac{-\beta \pm \sqrt{\beta^2 + 4\gamma(c^2 - \alpha)}}{2\gamma}$ and $a_1 = \pm \frac{c\sqrt{-3}}{2\sqrt{\gamma}}$, since $a_1 \neq 0$. From Eq. (3.26) it can be figure out that

$$\frac{\psi''}{\psi'} = \lambda \tag{3.30}$$

Integrating (3.30) with respect to ξ, yields

$$\psi' = c_1 e^{\lambda\xi}, \tag{3.31}$$

and $$\psi = \frac{c_1 e^{\lambda\xi}}{\lambda} + c_2, \tag{3.32}$$

where $\lambda = \frac{a_0 c^2 + a_1^2(\beta + 4\gamma a_0)}{a_1 c^2}$ and c_1, c_2 are integral constants.

Case 1: When $a_0 = \frac{-\beta \pm \sqrt{\beta^2 + 4\gamma(c^2 - \alpha)}}{2\gamma}$ and $a_1 = \pm \frac{c\sqrt{-3}}{2\sqrt{\gamma}}$, solving Eqs. (3.26) and (3.27) with (3.31) and (3.32), we achieve $c = \pm \frac{\sqrt{16\gamma\alpha - 3\beta^2}}{4\sqrt{\gamma}}$ and $\lambda = \frac{2\beta\sqrt{-3}}{\sqrt{16\gamma\alpha - 3\beta^2)}}$. Embedding the values of a_0, a_1 and c into (3.4) provides

$$v(\xi) = \frac{3\beta}{4\gamma}\left(-1 + \frac{c_1}{c_1 + c_2\lambda e^{-\lambda\xi}}\right), \tag{3.33}$$

where $\xi = x - ct$ and $\lambda = \frac{2\beta\sqrt{-3}}{\sqrt{16\gamma\alpha - 3\beta^2)}}$.

Thus, in (x, t) variables, the general solitary wave solution of the PC equation is obtained as follows:

$$u(x,t) = \sqrt{\frac{3\beta}{4\gamma}}\left(-1 + \frac{c_1}{c_1 + c_2\lambda e^{-\lambda(x-ct)}}\right)^{\frac{1}{2}} \tag{3.34}$$

Simplifying the exponential solution to the hyperbolic function, the solitary wave solution of the Eq. (3.34) turns into

$$u(x,t) = \sqrt{\frac{3\beta}{4\gamma}}\left(-1 + \frac{c_1\left(cosh\left(\frac{(x-ct)\lambda}{2}\right) + sinh\left(\frac{(x-ct)\lambda}{2}\right)\right)}{\left((c_1 + c_2\lambda)\, cosh\left(\frac{(x-ct)\lambda}{2}\right) + (c_1 - c_2\lambda)\, sinh\left(\frac{(x-ct)\lambda}{2}\right)\right)}\right)^{\frac{1}{2}} \tag{3.35}$$

Since c_1 and c_2 are integral constants, we might spontaneously choose their values, Therefore, if we choose $c_1 = \lambda$ and $c_2 = 1$, from solution (3.35) we attain the following closed form solution to the PC equation:

$$u(x,t) = \sqrt{\frac{3\beta}{16\gamma}}\left(-1 + tanh\left(\frac{(x-ct)\lambda}{2}\right)\right)^{\frac{1}{2}}. \tag{3.36}$$

Setting $c_1 = -\lambda$ and $c_2 = 1$ into solution (3.35), we arrive at the following solitary wave solution to the PC equation:

$$u(x,t) = \sqrt{\frac{3\beta}{16\gamma}}\left(-1 + coth\left(\frac{(x-ct)\lambda}{2}\right)\right)^{\frac{1}{2}}. \tag{3.37}$$

Using hyperbolic function identities, solutions (3.36) and (3.37) can be rewritten as

$$u(x,t) = \sqrt{\frac{3\beta}{16\gamma}}\left(-1 - itan\left(\frac{i(x-ct)\lambda}{2}\right)\right)^{\frac{1}{2}}, \tag{3.38}$$

and $$u(x,t) = \sqrt{\frac{3\beta}{16\gamma}}\left(-1 + icot\left(\frac{i(x-ct)\lambda}{2}\right)\right)^{\frac{1}{2}}, \tag{3.39}$$

where $c = \pm\frac{\sqrt{16\gamma\alpha-3\beta^2}}{4\sqrt{\gamma}}$ and $\lambda = \frac{2\beta\sqrt{-3}}{\sqrt{16\gamma\alpha-3\beta^2)}}$.

Case 2: When $a_0 = 0$ and $a_1 = \pm\frac{c\sqrt{-3}}{2\sqrt{\gamma}}$, solving Eq. (3.26) and (3.27) with (3.31) and (3.32), we achieve $c = \pm\frac{\sqrt{16\gamma\alpha-3\beta^2}}{4\sqrt{\gamma}}$ and $\lambda = \frac{2\beta\sqrt{-3}}{\sqrt{16\gamma\alpha-3\beta^2)}}$. Replace the values of a_0, a_1, c and λ into solution (3.4) provides

$$v(\xi) = -\frac{3\beta}{4\gamma}\left(\frac{c_1 e^{\lambda\xi}}{c_1 e^{\lambda\xi}+c_2\lambda}\right), \tag{3.40}$$

where $\xi = x - ct$ and $\lambda = \frac{2\beta\sqrt{-3}}{\sqrt{16\gamma\alpha-3\beta^2)}}$.

In (x,t) variables, the general solitary wave solution to the PC equation becomes:

$$u(x,t) = \sqrt{-\frac{3\beta}{4\gamma}\left(\frac{c_1 e^{(x-ct)\lambda}}{c_1 e^{(x-ct)\lambda}+c_2\lambda}\right)^{\frac{1}{2}}} \tag{3.41}$$

Changing the exponential solution into the hyperbolic function solution, the solitary wave solution of the Eq. (3.41) turns into

$$u(x,t) = \sqrt{-\frac{3\beta}{4\gamma}}\left(\frac{c_1\left(cosh\left(\frac{(x-ct)\lambda}{2}\right)+sinh\left(\frac{(x-ct)\lambda}{2}\right)\right)}{\left((c_1+c_2\lambda)\,cosh\left(\frac{(x-ct)\lambda}{2}\right)+(c_1-c_2\lambda)\,sinh\left(\frac{(x-ct)\lambda}{2}\right)\right)}\right)^{\frac{1}{2}} \tag{3.42}$$

Inasmuch as c_1 and c_2 are arbitrary constants, we may arbitrarily select their values, Therefore, if we select $c_1 = \lambda$ and $c_2 = 1$, solution (3.42) to the PC equation turns into the stable kink type solution as follows:

$$u(x,t) = \sqrt{-\frac{3\beta}{16\gamma}}\left(1 + tanh\left(\frac{(x-ct)\lambda}{2}\right)\right)^{\frac{1}{2}} \tag{3.43}$$

Setting $c_1 = -\lambda$ and $c_2 = -1$ into solution (3.42), we arrive to the following solution to the PC equation:

$$u(x,t) = \sqrt{-\frac{3\beta}{16\gamma}}\left(1 + coth\left(\frac{(x-ct)\lambda}{2}\right)\right)^{\frac{1}{2}}. \tag{3.44}$$

Using hyperbolic functions identities, solutions (3.43) and (3.44) can be rewritten as

$$u(x,t) = \sqrt{-\frac{3\beta}{16\gamma}}\left(1 - itan\left(\frac{i(x-ct)\lambda}{2}\right)\right)^{\frac{1}{2}}, \tag{3.45}$$

and $$u(x,t)=\sqrt{-\frac{3\beta}{16\gamma}}\left(1+icot\left(\frac{i(x-ct)\lambda}{2}\right)\right)^{\frac{1}{2}},\tag{3.46}$$

where $c=\pm\frac{\sqrt{16\gamma\alpha-3\beta^2}}{4\sqrt{\gamma}}$ and $\lambda=\frac{2\beta\sqrt{-3}}{\sqrt{16\gamma\alpha-3\beta^2)}}$.

3.3. The nonlinear Schrödinger-Hirota equation

Let us consider the nonlinear Schrödinger-Hirota Equation [6]:

$$iq_t+\frac{1}{2}q_{xx}+i\lambda q_{xxx}+|q|^2q=0.\tag{3.47}$$

The Eq. (3.47) analyzes the propagation of optical soliton in a dispersive optical fiber. Here q represents the wave profile and λ is the third order dispersion coefficient. The first term represents the evolution, while the second term is the group velocity dispersion and the fourth term is the Kerr law of nonlinearity that arises when the intensity of the light is dependent on the refractive index of the material. The third order dispersion term is taken into account when the group velocity dispersion is small so that there is performance enhancement during pulse propagation across transoceanic and trans-continental distances [37].

The complex transformation $q(x,t)=e^{i(\alpha x+\beta t)}u(\xi)$, $\xi=k(x-2\alpha t$, where α,β,k and ω are real constants reduces Eq. (3.47) to an ordinary differential equation of the form:

$$\left(\frac{k^2}{2}-3\alpha\lambda k^2\right)u''-ik(2\alpha-\alpha+3\lambda\alpha^2)u'-\left(\beta-\frac{\alpha^2}{2}+\lambda\alpha^3\right)u+i\lambda k^3u'''+u^3=0\tag{3.48}$$

From the above Eq. (3.48), we obtain $\alpha=-1/3\,\lambda$ and $u(\xi)$ satisfy the differential equation

$$\frac{3k^2}{2}u''-\left(\beta+\frac{5}{54\lambda^2}\right)u+u^3=0.\tag{3.49}$$

Eq. (3.49) can be rewritten as

$$u''+\mu_1u^3-\mu_2u=0,\tag{3.50}$$

where $\mu_1=\frac{2}{3k^2}$ and $\mu_2=\frac{2}{3k^2}\left(\beta+\frac{5}{54\lambda^2}\right)$.

Taking homogeneous balance between linear term u'' and nonlinear term u^3 yields $N=1$. Therefore, the solution structure of Eq. (3.50) is same as the solution (3.4). Hence substituting solution (3.4) and its derivatives into Eq. (3.50), and completing the similar procedure described in subsection 3.1, we achieve the successive algebraic and differential equations

$$\mu_1a_0^3-\mu_2a_0=0\tag{3.51}$$

$$a_1\psi'''+3\mu_1a_0^2a_1\psi'-\mu_2a_1\psi'=0\tag{3.52}$$

$$-3a_1\psi'\psi''+3\mu_1a_0a_1^2\psi'^2=0\tag{3.53}$$

$$2a_1\psi'^3+\mu_1a_1^3\psi'^3=0\tag{3.54}$$

From Eq. (3.51) and Eq. (3.54), we obtain $a_0 = 0, \pm\sqrt{\frac{\mu_2}{\mu_1}}$ and $a_1 = \pm\sqrt{\frac{-2}{\mu_1}}$, since $a_1 \neq 0$. And from Eq. (3.53), we attain

$$\frac{\psi''}{\psi'} = \mu_1 a_0 a_1 = \theta \tag{3.55}$$

Integrating (3.55) with respect to ξ, yields

$$\psi' = c_1 \mathrm{e}^{\theta\xi}, \tag{3.56}$$

and $$\psi = \frac{c_1 \mathrm{e}^{\theta\xi}}{\theta} + c_2, \tag{3.57}$$

where c_1 and c_2 are arbitrary constants and $\theta = \mu_1 a_0 a_1$.

Case 1: When $a_0 = \pm\sqrt{\frac{\mu_2}{\mu_1}}$ and $a_1 = \pm\sqrt{\frac{-2}{\mu_1}}$, and $\alpha = -1/3\,\lambda$, substitute the values of a_0, a_1 and α into solution (3.4), we determine

$$u(\xi) = \pm\sqrt{\frac{\mu_2}{\mu_1}}\left(1 - \frac{2c_1}{c_1 + c_2\theta e^{-\theta\xi}}\right), \tag{3.58}$$

where $\xi = k(x - \frac{2}{3\lambda}t)$ and $\theta = \pm\sqrt{-2\mu_2}$.

Forasmuch as c_1 and c_2 are arbitrary constants, we may freely accept their values. Therefore, if we set $c_1 = \theta$ and $c_2 = 1$ into solution (3.58) and simplifying the exponential solution to hyperbolic function, we attain the following solitary wave solution of the Eq. (3.47)

$$u(\xi) = \mp\sqrt{\frac{\mu_2}{\mu_1}}\left(tanh\left(\frac{\theta\xi}{2}\right)\right) \tag{3.59}$$

Moreover, setting $c_1 = -\theta$ and $c_2 = 1$ into solution (3.58), we arrive at the following solitary wave solution of the nonlinear Schrödinger-Hirota equation:

$$u(\xi) = \mp\sqrt{\frac{\mu_2}{\mu_1}}\left(coth\left(\frac{\theta\xi}{2}\right)\right) \tag{3.60}$$

Thus, in (x, t) variables, the general solitary wave solution to the nonlinear Schrödinger-Hirota equation is obtained as follows:

$$q(x,t) = \mp\sqrt{\left(\beta + \frac{5}{54\lambda^2}\right)}\left(tanh\left(\frac{\sqrt{\frac{-4}{3k^2}\left(\beta + \frac{5}{54\lambda^2}\right)}\left[k\left(x - \frac{2}{3\lambda}t\right)\right]}{2}\right)\right), \tag{3.61}$$

and $$q(x,t) = \mp\sqrt{\left(\beta + \frac{5}{54\lambda^2}\right)}\left(coth\left(\frac{\sqrt{\frac{-4}{3k^2}\left(\beta + \frac{5}{54\lambda^2}\right)}\left[k\left(x - \frac{2}{3\lambda}t\right)\right]}{2}\right)\right). \tag{3.62}$$

Case 2: When $a_0 = 0$ and $a_1 = \pm\sqrt{\frac{-2}{\mu_1}}$, and $\alpha = -1/3\,\lambda$, making use of the values of a_0, a_1 and α into (3.4), it does not satisfy the Eq. (3.53) and hence the solution must be rejected.

4. Explanation and physical interpretation of the solutions

In this section, we have discussed about the obtained solution of the simplified MCH equation, the PC equation and the Schrödinger-Hirota equation. Using the MSE method, we get the traveling wave solutions assembled from Eqs. (3.12) to (3.20) to the simplified MCH equation. The solutions are general solitary wave solutions which are periodic wave solution, kink shape soliton and singular kink shape soliton respectively. From the above solution, it has been detected that the solutions (3.12) and (3.13) provides periodic wave solution where the solutions (3.15), (3.17) and (3.19) gives kink shape wave solution. The solutions (3.14), (3.16) and (3.20) present singular kink solutions. The kink shape wave solution (3.15) is represented in Figure 1 for $\alpha = 1, \beta = 1, k = 1$ and $\omega = 4$. The singular kink solutions (3.16) and (3.20) for $\alpha = 1, \beta = 1, k = 1, \omega = 4$ and for $\alpha = 1$, $\beta = 1$ are plotted in the Figures 2 and 3 respectively. From the solutions to the PC equation, it is observed that the solutions (3.34) and (3.35) show the nature of singular kink, solutions (3.36), (3.38), (3.43) and (3.45) represent the kink shape soliton and solutions (3.37), (3.39), (3.44) and (3.46) are singular solution. Singular solitons can be connected to solitary waves when the center position of the solitary wave is imaginary [58]. This solution has spike and therefore it can probably provide an explanation to the formation of Rogue waves [38]. The kink shape solution (3.43) for $\alpha = \frac{1}{2}, \beta = 2$ and $\gamma = -1$ is represented in Figure 4. From the solutions of the Schrödinger-Hirota equation, the solutions (3.59)–(3.62) are categorized to the character of singular periodic solution and (3.58) represents periodic solution. Periodic traveling waves play an important role in numerous physical phenomena, including reaction-diffusion-advection systems, self-reinforcing systems, impulsive systems etc. Mathematical modeling of many intricate physical events, for instance physics, mathematical physics, engineering and many more phenomena resemble periodic traveling wave solutions. The singular periodic solutions (3.59) for $\beta = 1, k = 1, \lambda = 1/3$ and (3.60) for $\beta = 1, k = 1, \lambda = 1/3$ are given in Figures 5 and 6 respectively. The figures of other solutions are similar mentioned above and ignored these figures for simplicity.

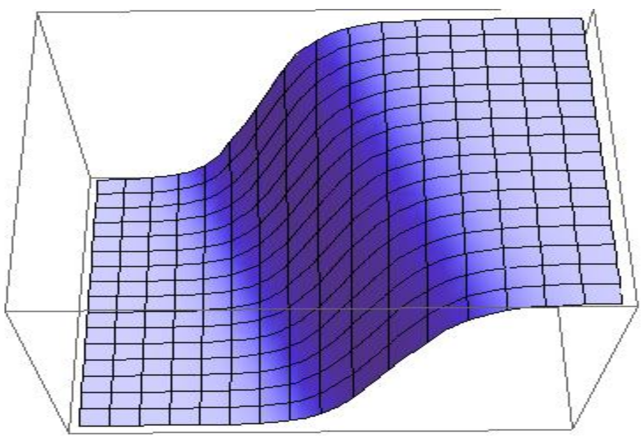

Figure 1. Plot of kink shape soliton of solution (3.15) of simplified MCH equation for $\alpha = 1, \beta = 1, k = 1$ and $\omega = 4$.

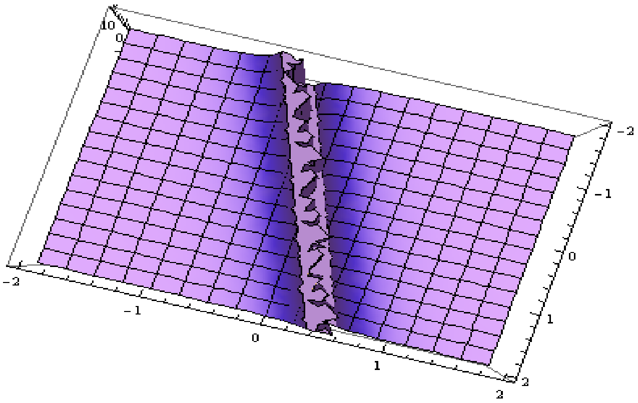

Figure 2. Plot of singular kink soliton of solution (3.16) of simplified MCH equation for $\alpha = 1, \beta = 1, k = 1$ and $\omega = 4$.

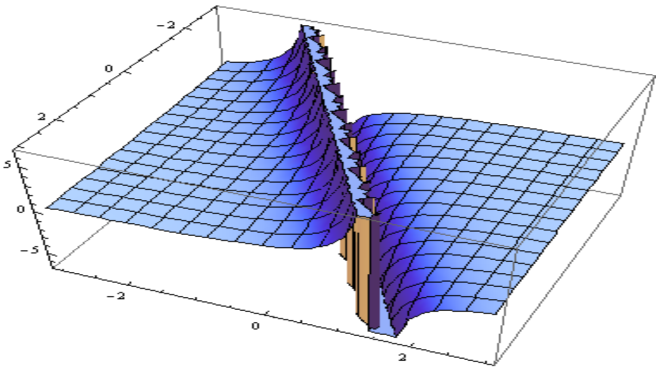

Figure 3. Plot of singular kink solution (3.20) of the simplified MCH equation for $\alpha = 1$ and $\beta = 1$.

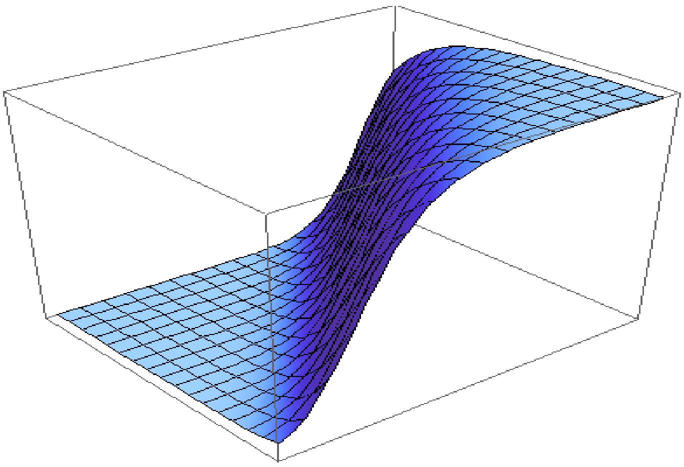

Figure 4. Plot of kink shape soliton of solution (3.43) of PC equation for $\alpha = \frac{1}{2}, \beta = 2$ and $\gamma = -1$.

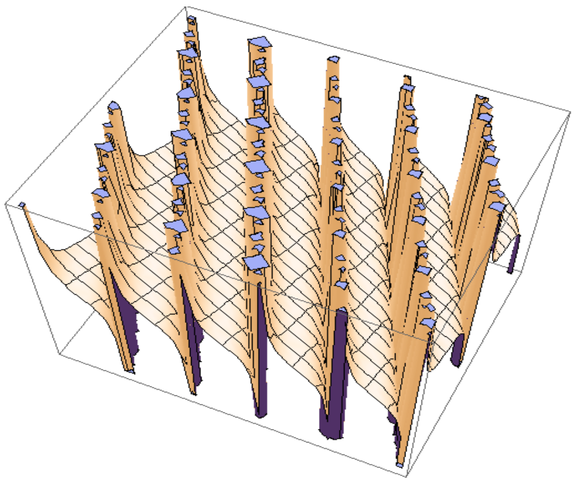

Figure 5. Plot of singular periodic solution (3.60) of Schrödinger-Hirota equation for $\beta = 1, k = 1, \lambda = 1/3$.

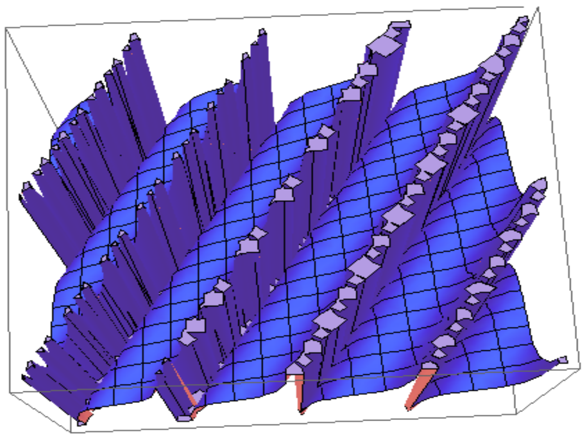

Figure 6. Plot of singular periodic solution (3.62) of Schrödinger-Hirota equation for $\beta = 1, k = 1, \lambda = 1/3$.

5. Conclusion

In this article, the modified simple equation method has successfully been used to establish the solitary wave solutions to the simplified MCH equation, the Pochhammer-Chree equation and the Schrödinger-Hirota equation. The attribute and uniqueness of this method is that the considered function $\psi(\xi)$ is not an early known function. So is not possible to presume in advance what kind of solutions one may obtain through this method. Therefore, the obtained solutions are more general and fresh and important to analyze the inner mechanism of these nonlinear phenomena. The solutions are confirmed through checking the correctness by inserting them into the original equations and found correct. The results show that the method is reliable and effective. The used method has several advantages: it is straightforward and its calculation procedure is concise. Therefore this efficient method could be more effectively used to solve various NLEEs which regularly arise in science, engineering and other technical arenas.

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Conflict of interest

The authors declare no conflict of interest.

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