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Research article

Probabilistic α-min Ciric type contraction results using a control function

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Abstract: The purpose of the paper is to propose some new probabilistic α-minimum Ciric type contraction results. Our results are established on probabilistic generalization of metric spaces or probabilistic metric spaces. The use of class of control fucntion Φ which was introduced by Choudhury et al. in 2008 helped us to deduce the result. We also get a corollary. Some illustrative examples are given here. Our results are supported by those examples. Lastly an application of integral equation is given. An important conclusion is also made at the end of the results.

Keywords: PM-space; Ciric type contraction; Cauchy sequence; fixed point; altering distance function
Mathematics Subject Classification: 47H10, 54E40, 54H25

1. Introduction

In this current paper, the probabilistic outcomes of Ciric contraction of α-min are considered. Probabilistic metric space are probabilistic generalization of metric spaces which was introduced by K. Menger in 1942 [20]. Distribution function plays the role of metric on these spaces. Menger spaces are the specific probabilistic metric spaces where the triangle inequality is postulated with the help of t-norm. Sehgal and Bharucha-Reid were the pesons who established Banach contraction mapping principle to probabilistic metric spaces in 1972. This result was done in their research works [27]. Schweizer and Sklar have described many aspect on these spaces in their book [26].

Being a control function, "altering distance function", alters the distance between two points in a metric space and Khan, Swaleh and Sessa in 1984 showed us the property in their paper [17]. Some generalized works in this line may be referred as [16, 18, 19, 21, 22, 24, 25, 28].

In recent time, the concept of altering distance function is extended to the context of Menger spaces in [6]. This control function is known as ϕ-function and very useful for proving fixed point results in Menger spaces. This concept is also applied to many other problem such as coincidence point problems in this line. Some recent works using ϕ-function are mentioned in [1–3, 7, 12, 13].

Main features of this paper are following:
(1) A new probabilistic α-min special Ciric type contraction result.
(2) For such contraction, unique fixed point is obtained.
(3) The use of control function to prove the theorems.
(4) A corollary.
(5) Two illustrative examples validating our theorems.
(6) An application of our results on integral calculus.
(7) An important conclusion which may incur new problems.

2. Definitions and mathematical requisits

Some important definitions and mathematical preliminaries are discussed before we want to prove our main results.

Definition 2.1. *[15, 26] A distribution function is a mapping $F : R \to R^+$ if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$, where R is the set of reals and R^+ is the set of non-negative reals respectively.*

Definition 2.2. ***t-norm [15, 26]*** *A function $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm, if it satisfies the following conditions for all $a, b, c, d \in [0, 1]$*

(i) $\Delta(1, a) = a$,

(ii) $\Delta(a, b) = \Delta(b, a)$,

(iii) $\Delta(c, d) \geq \Delta(a, b)$ *whenever* $c \geq a$ *and* $d \geq b$,

(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

The examples of t-norm are as follows:
(i) $\Delta = T_m$, which is the minimum t-norm and is defined by $T_m(a, b) = \min\{a, b\}$.
(ii) $\Delta = T_p$, which is the product t-norm and is defined by $T_p(a, b) = a.b$.

Definition 2.3. ***Menger space [15, 26]*** *A triplet (X, F, Δ) is called a Menger space where $X \neq \phi$, F is a function on $X \times X$ to the set of distribution functions and Δ is a t-norm, such that it satisfies the following conditions:*

(i) $F_{x,y}(0) = 0$ *for all* $x, y \in X$,

(ii) $F_{x,y}(s) = 1$ *for all* $s > 0$ *and* $x, y \in X$ *if and only if* $x = y$,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ *for all* $x, y \in X, s > 0$ *and*

(iv) $F_{x,y}(u + v) \geq \Delta(F_{x,z}(u), F_{z,y}(v))$ *for all* $u, v \geq 0$ *and* $x, y, z \in X$.

Definition 2.4. *[15, 26] A sequence* $\{x_n\} \subset X$ *is said to converge to some point* $x \in X$ *if given* $\epsilon > 0, 0 < \lambda < 1,$ *we can find a positive integer* $N_{\epsilon,\lambda}$ *such that for all* $n > N_{\epsilon,\lambda}$

$$F_{x_n,x}(\epsilon) \geq 1 - \lambda. \tag{2.2}$$

Definition 2.5. *[15, 26] A sequence* $\{x_n\}$ *is said to be a Cauchy sequence in X if given* $\epsilon > 0, 0 < \lambda < 1,$ *there exists a positive integer* $N_{\epsilon,\lambda}$ *such that*

$$F_{x_n,x_m}(\epsilon) \geq 1 - \lambda \text{ for all } m, n > N_{\epsilon,\lambda}. \tag{2.3}$$

The equivalent of Definition 2.4 and 2.5 is to replace ≥ with > in (2.2) and (2.3) respectively. They are not written in this conventional way. We have presently given them the evidence from our theorems for our convenience.

Definition 2.6. *[15, 26] A Menger space* (X, F, Δ) *is said to be complete if every Cauchy sequence is convergent in X.*

We use the following control function ϕ which Choudhury and Das presented [6].

Definition 2.7. Φ***-function [6]*** *A function* $\phi : R \to R^+$ *is said to be a* Φ*-function if it satisfies the following conditions:*

(i) $\phi(t) = 0$ *if and only if* $t = 0,$

(ii) $\phi(t)$ *is strictly monotone increasing and* $\phi(t) \to \infty$ *as* $t \to \infty,$

(iii) ϕ *is left continuous in* $(0, \infty),$

(iv) ϕ *is continuous at* $0.$

In numerous research works, many authors [4, 8–11] use this function.

3. Main results

We begin this section by introducing the concept of α-min Ciric type contraction and α-admissible mappings in Menger PM spaces.

Recent documents, such as [13, 14] motivated us.

Definition 3.1. *Let* (X, F, Δ) *be a PM-space and* $f : X \to X$ *be a mapping. We say that f is an* α*-min Ciric type mapping if there exists function* $\alpha : X \times X \times (0, \infty) \to R^+$ *satisfying the following inequality*

$$\alpha(x, y, t)\left(\frac{1}{F_{fx,fy}(\phi(t))} - 1\right) \leq \min\left(\frac{1}{F_{x,y}(\phi(\frac{t}{c}))} - 1, \frac{1}{F_{x,fx}(\phi(\frac{t}{c}))} - 1, \frac{1}{F_{y,fy}(\phi(\frac{t}{c}))} - 1\right) \tag{3.1}$$

for all $x, y \in X,\ t > 0,$ *where* $0 < c < 1,\ \phi \in \Phi.$

Definition 3.2. *([14]) Let* (X, F, Δ) *be a PM-space,* $f : X \to X$ *be a given mapping and* $\alpha : X \times X \times (0, \infty) \to R^+$ *be a function, we say that f is* α*-admissible if*
$x, y \in X,$ *for all* $t > 0,$

$$\alpha(x, y, t) \geq 1 \Rightarrow \alpha(fx, fy, t) \geq 1$$

Theorem 3.1. *Let (X, F, Δ) be a complete Menger space, Δ is a continuous t-norm and $f : X \to X$ be an α-min Ciric type mapping satisfying the following conditions.*

(i) f is α-admissible,

(ii)there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$, for all $t > 0$,

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in N$ and for all $t > 0$.

Then f has a fixed point , that is, there exists a point $u \in X$ such that $fu = u$.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$. Define a sequence $\{x_n\}$ in X so that $x_{n+1} = fx_n$, for all $n \in N$, where N is the set of natural numbers. Clearly, we suppose $x_{n+1} \neq x_n$ for all $n \in N$, otherwise f has trivially a fixed point.

Then by using the fact f is α-admissible, we write

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1 \Rightarrow \alpha(fx_0, fx_1, t) = \alpha(x_1, x_2, t) \geq 1,$$

and, by induction, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1, \text{ for all } n \in N \text{ and for all } t > 0.$$

From the properties of function ϕ, we can find $t > 0$ such that $F_{x_0,x_1}(\phi(t)) > 0$.

Now, we have from (3.1) for $t > 0$ and $c \in (0, 1)$,

$$\begin{aligned}
\frac{1}{F_{x_{n+1},x_n}(\phi(t))} - 1 &= \frac{1}{F_{fx_n,fx_{n-1}}(\phi(t))} - 1 \\
&\leq \alpha(x_n, x_{n-1}, t)\frac{1}{F_{fx_n,fx_{n-1}}(\phi(t))} - 1 \\
&\leq \min(\frac{1}{F_{x_n,x_{n-1}}(\phi(\frac{t}{c}))} - 1, \frac{1}{F_{x_n,fx_n}(\phi(\frac{t}{c}))} - 1, \frac{1}{F_{x_{n-1},fx_{n-1}}(\phi(\frac{t}{c}))} - 1) \\
&= \min(\frac{1}{F_{x_n,x_{n-1}}(\phi(\frac{t}{c}))} - 1, \frac{1}{F_{x_n,x_{n+1}}(\phi(\frac{t}{c}))} - 1, \frac{1}{F_{x_{n-1},x_n}(\phi(\frac{t}{c}))} - 1) \\
&= \min(\frac{1}{F_{x_{n+1},x_n}(\phi(\frac{t}{c}))} - 1, \frac{1}{F_{x_n,x_{n-1}}(\phi(\frac{t}{c}))} - 1). \qquad (3.2)
\end{aligned}$$

We now claim that for all $t > 0$, $n \geq 1$, $c \in (0, 1)$,

$$\min(\frac{1}{F_{x_{n+1},x_n}(\phi(\frac{t}{c}))} - 1, \frac{1}{F_{x_n,x_{n-1}}(\phi(\frac{t}{c}))} - 1) = \frac{1}{F_{x_n,x_{n-1}}(\phi(\frac{t}{c}))} - 1. \qquad (3.3)$$

If possible, let for some $s > 0$,

$$\min(\frac{1}{F_{x_{n+1},x_n}(\phi(\frac{s}{c}))} - 1, \frac{1}{F_{x_n,x_{n-1}}(\phi(\frac{s}{c}))} - 1) = \frac{1}{F_{x_{n+1},x_n}(\phi(\frac{s}{c}))} - 1,$$

then we have from (3.2),

$$\frac{1}{F_{x_{n+1},x_n}(\phi(s))} - 1 \leq \frac{1}{F_{x_{n+1},x_n}(\phi(\frac{s}{c}))} - 1,$$

that is,

$$F_{x_{n+1},x_n}(\phi(s)) \geq F_{x_{n+1},x_n}(\phi(\frac{s}{c})), \qquad (3.4)$$

which is impossible as for $c \in (0,1)$ (since $\phi(\frac{s}{c}) > \phi(s)$, that is, $F_{x_{n+1},x_n}(\phi(\frac{s}{c})) \geq F_{x_{n+1},x_n}(\phi(s))$, by the monotone property of F and for $c \in (0,1)$).

Then, for all $t > 0$, $\frac{1}{F_{x_{n+1},x_n}(\phi(t))} - 1 \leq \frac{1}{F_{x_n,x_{n-1}}(\phi(\frac{t}{c}))} - 1$,
that is,

$$\begin{aligned} F_{x_{n+1},x_n}(\phi(t)) &\geq F_{x_n,x_{n-1}}(\phi(\frac{t}{c})) \\ &\geq F_{x_{n-1},x_{n-2}}(\phi(\frac{t}{c^2})) \\ &\geq \ldots\ldots\ldots\ldots\ldots\ldots \\ &\geq F_{x_1,x_0}(\phi(\frac{t}{c^n})), \end{aligned}$$

Therefore,

$$F_{x_{n+1},x_n}(\phi(t)) \geq F_{x_1,x_0}(\phi(\frac{t}{c^n})). \tag{3.5}$$

Now, taking limit as $n \to \infty$ on both sides of (3.5), for all $t > 0$, we obtain

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(\phi(t)) = 1. \tag{3.6}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence.

On the contrary, there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\epsilon) < 1 - \lambda. \tag{3.7}$$

We take $m(k)$ corresponding to $n(k)$ to be the smallest integer satisfying (3.7), so that

$$F_{x_{m(k)-1},x_{n(k)}}(\epsilon) \geq 1 - \lambda. \tag{3.8}$$

If $\epsilon_1 < \epsilon$ then we have

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_1) \leq F_{x_{m(k)},x_{n(k)}}(\epsilon).$$

So, it is feasible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $m(k) > n(k) > k$ and satisfying (3.7), (3.8) whenever ϵ is replaced by a smaller positive value. By the continuity of ϕ at 0 and strictly monotone increasing property with $\phi(0) = 0$, it is possible to find $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

Then, by the above logic, it is possible to get an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2)) < 1 - \lambda, \tag{3.9}$$

and

$$F_{x_{m(k)-1},x_{n(k)}}(\phi(\epsilon_2)) \geq 1 - \lambda. \tag{3.10}$$

Now, from (3.9), we get

$$1 - \lambda > F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2)),$$

that is,

$$\frac{1}{1-\lambda} < \frac{1}{F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2))},$$

that is,

$$\frac{1}{1-\lambda} - 1 < \frac{1}{F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2))} - 1,$$

which implies,

$$\begin{aligned}
\frac{\lambda}{1-\lambda} &< \frac{1}{F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2))} - 1, \\
&\leq \alpha(x_{m(k)-1}, x_{n(k)-1}, t)(\frac{1}{F_{fx_{m(k)-1},fx_{n(k)-1}}(\Phi(\epsilon_2))} - 1), \\
&\leq \min(\frac{1}{F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\frac{\epsilon_2}{c}))} - 1, \frac{1}{F_{x_{m(k)-1},x_{m(k)}}(\phi(\frac{\epsilon_2}{c}))} - 1, \frac{1}{F_{x_{n(k)-1},x_{n(k)}}(\phi(\frac{\epsilon_2}{c}))} - 1) \qquad (3.11)
\end{aligned}$$

(using the inequality (3.1))

Now, using the property of (iv) of the Menger space, we have

$$\begin{aligned}
F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\frac{\epsilon_2}{c})) &\geq \Delta(F_{x_{m(k)-1},x_{n(k)}}(\phi(\epsilon_2)), F_{x_{n(k)},x_{n(k)-1}}(\phi(\frac{\epsilon_2}{c})) - \phi(\epsilon_2)) \\
&\geq \Delta(1-\lambda, 1-\lambda)(\text{using (3.6) and (3.10)}) \\
&= 1-\lambda,
\end{aligned}$$

that is,

$$\frac{1}{F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\frac{\epsilon_2}{c}))} - 1 \leq \frac{1}{1-\lambda} - 1 = \frac{\lambda}{1-\lambda}. \qquad (3.12)$$

Now, using (3.6), for sufficiently large k, we have

$$F_{x_{m(k)-1},x_{m(k)}}(\phi(\frac{\epsilon_2}{c})) \geq 1-\lambda,$$

$$\frac{1}{F_{x_{m(k)-1},x_{m(k)}}(\phi(\frac{\epsilon_2}{c}))} - 1 \leq \frac{1}{1-\lambda} - 1 = \frac{\lambda}{1-\lambda}. \qquad (3.13)$$

$$F_{x_{n(k)-1},x_{n(k)}}(\phi(\frac{\epsilon_2}{c})) \geq 1-\lambda,$$

that is,

$$\frac{1}{F_{x_{n(k)-1},x_{n(k)}}(\phi(\frac{\epsilon_2}{c}))} - 1 \leq \frac{1}{1-\lambda} - 1 = \frac{\lambda}{1-\lambda}. \qquad (3.14)$$

Now using (3.12), (3.13) and (3.14) in (3.11), we have

$$\begin{aligned}\frac{\lambda}{1-\lambda} &< \min\Big(\frac{1}{F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\frac{\epsilon_2}{c}))}-1, \frac{1}{F_{x_{m(k)-1},x_{m(k)}}(\phi(\frac{\epsilon_2}{c}))}-1, \frac{1}{F_{x_{n(k)-1},x_{n(k)}}(\phi(\frac{\epsilon_2}{c}))}-1\Big)\\ &\leq \min\Big(\frac{\lambda}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{\lambda}{1-\lambda}\Big)\\ &= \frac{\lambda}{1-\lambda},\end{aligned}$$

that is,

$$\frac{\lambda}{1-\lambda} < \frac{\lambda}{1-\lambda},$$

which is a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence.

Since (X, F, Δ) be a complete Menger space, therefore $x_n \to u$ as $n \to \infty$, for some $u \in X$. Moreover, we get

$$F_{fu,u}(\epsilon) \geq \Delta(F_{fu,x_{n+1}}(\frac{\epsilon}{2}), F_{x_{n+1},u}(\frac{\epsilon}{2})). \tag{3.15}$$

Next, using the properties of function ϕ, we can find $t_2 > 0$ such that $\phi(t_2) < \frac{\epsilon}{2}$. Again $x_n \to u$ as $n \to \infty$ and hence there exists $n_0 \in N$ such that, for all $n > n_0$ (sufficiently large), we have

$$\begin{aligned}\frac{1}{F_{x_{n+1},fu}(\frac{\epsilon}{2})}-1 &\leq \frac{1}{F_{fx_n,fu}(\phi(t_2))}-1\\ &\leq \alpha(x_n,u,t)\Big(\frac{1}{F_{fx_n,fu}(\phi(t_2))}-1\Big)\\ &\leq \min\Big(\frac{1}{F_{x_n,u}(\phi(\frac{t_2}{c}))}-1, \frac{1}{F_{x_n,fx_n}(\phi(\frac{t_2}{c}))}-1, \frac{1}{F_{u,fu}(\phi(\frac{t_2}{c}))}-1\Big)\\ &= \min\Big(\frac{1}{F_{x_n,u}(\phi(\frac{t_2}{c}))}-1, \frac{1}{F_{x_n,x_{n+1}}(\phi(\frac{t_2}{c}))}-1, \frac{1}{F_{u,fu}(\phi(\frac{t_2}{c}))}-1\Big).\end{aligned}$$

Taking limit $n \to \infty$ on both sides, we have

$$\frac{1}{F_{u,fu}(\phi(t_2))}-1 \leq \min\Big(0,0,\frac{1}{F_{u,fu}(\phi(\frac{t_2}{c}))}-1\Big) = 0$$

$$\Rightarrow \frac{1}{F_{u,fu}(\phi(t_2))} \leq 1$$

$$\Rightarrow F_{u,fu}(\phi(t_2)) \geq 1.$$

Thus, $$fu = u.$$

The uniqueness of the fixed point is established next. Let x and y be two fixed point of f, that is, $fx = x$ and $fy = y$. By the virtue of ϕ there exists $s > 0$ such that $F_{x,y}(\phi(s)) > 0$. Then, by an application of (3.1), we have

$$\frac{1}{F_{fx,fy}(\phi(s))}-1 \leq \alpha(x,y,t)\Big(\frac{1}{F_{fx,fy}(\phi(s))}-1\Big)$$

$$\begin{aligned}
&\leq \min(\frac{1}{F_{x,y}(\phi(\frac{s}{c}))}-1, \frac{1}{F_{x,fx}(\phi(\frac{s}{c}))}-1, \frac{1}{F_{y,fy}(\phi(\frac{s}{c}))}-1) \\
&= \min(\frac{1}{F_{x,y}(\phi(\frac{s}{c}))}-1, \frac{1}{F_{x,x}(\phi(\frac{s}{c}))}-1, \frac{1}{F_{y,y}(\phi(\frac{s}{c}))}-1) \\
&= \min(\frac{1}{F_{x,y}(\phi(\frac{s}{c}))}-1, 0, 0) \\
&= 0,
\end{aligned}$$

which implies,

$$\frac{1}{F_{fx,fy}(\phi(s))}-1 \leq 0,$$

$$\Rightarrow F_{fx,fy}(\phi(s)) \geq 1,$$

that is,

$$F_{x,y}(\phi(s)) = 1.$$

Hence $x = y$, that is, the fixed point is unique. □

If we replace $\phi(t)$ by t in Theorem 3.1, we get the following Corollary.

Corollary 3.1. *Let (X, F, Δ) be a complete Menger space and $f : X \to X$ be a mapping satisfying the following inequality for all $x, y \in X$,*

$$\frac{1}{F_{fx,fy}(t)}-1 \leq \min(\frac{1}{F_{x,y}(\frac{t}{c})}-1, \frac{1}{F_{x,fx}(\frac{t}{c})}-1, \frac{1}{F_{y,fy}(\frac{t}{c})}-1) \tag{3.16}$$

where $t > 0$, $0 < c < 1$. Then f has a unique fixed point in X.

4. Example

Example 4.1. *Let $X = [0, 1]$, the t-norm Δ is a continuous t-norm and F be defined as*

$$F_{x,y}(t) = \frac{t}{t+|x-y|}.$$

Then (X, F, Δ) is a complete Menger space. If we define $f : X \to X$ as follows:

$$fx = \frac{x}{6} \text{ for all } x \in [0, 1],$$

then the mapping f satisfies all the conditions of Theorem 3.1, for $c = \frac{2}{3}$, where 0 is the unique fixed point of f.

Example 4.2. *Let $X = \{\alpha, \beta, \gamma\}$, the t-norm Δ is a minimum t-norm and F be defined as*

$$F_{\beta,\gamma}(t) = F_{\gamma,\alpha}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.75, & \text{if } 0 < t \leq 2, \\ 1, & \text{if } t > 2, \end{cases}$$

and

$$F_{\alpha,\beta}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0, \end{cases}$$

Then (X, F, Δ) is a complete Menger space. If we define $f : X \to X$ as follows: $f\alpha = \alpha, f\beta = \alpha, f\gamma = \beta$ then the mapping f satisfies all the conditions of Theorem 3.1 where $\phi(t) = t$, $c \in (0, 1)$ and α is the unique fixed point of f in X.

4.1. Application

Some recent references [5, 14, 23] help us to establish the following application.

We consider the following boundary value problem of second order differential equation :

$$-\frac{d^2x}{dt^2} = g(t,\ x(t)), \qquad t \in [0,1]$$

$$x(0) = x(1) = 0,$$

where $g : [0,1] \times R \longrightarrow R$ is a continuous function.

$$x'' = 0 \Rightarrow D^2 x = 0 \tag{4.1}$$

and boundary values are $x(0) = 0,\ x(1) = 0$.
The auxiliary equation is

$$D^2 = 0.$$

Therefore, the general solution is

$$x(t) = At + B.$$

Now, The Green's function $G(t,s)$ exists for the associated boundary-values problem and is given by

$$G(t,s) = \begin{cases} a_1 t + a_2, & 0 \le t < s \\ b_1 t + b_2, & s < t \le 1 \end{cases}$$

The Green's function must satisfy the following three properties:

i) $G(t,s)$ is continuous at $x = s$

i.e.,

$$b_1 s + b_2 = a_1 s + a_2 \Rightarrow s(b_1 - a_1) + b_2 - a_2 = 0 \tag{4.2}$$

ii) The determination of G has a discontinuity of magnitude $-\frac{1}{p_0(s)}$ at the point $x = s$, where $p_0(t) =$ co-efficient of the highest order derivative

i.e.,

$$\Big(\frac{\partial G}{\partial t}\Big)_{t=s+0} - \Big(\frac{\partial G}{\partial t}\Big)_{t=s-0} = -1 \Rightarrow b_1 - a_1 = -1 \tag{4.3}$$

iii) $G(t,s)$ must satisfy the boundary condition

$$G(0,s) = 0 \ \Rightarrow a_2 = 0 \tag{4.4}$$

and

$$G(1,s) = 0 \ \Rightarrow\ b_1 + b_2 = 0. \tag{4.5}$$

Therefore, $G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ -st + s, & 0 \le s \le t \le 1 \end{cases}$

Let $C(I)$ $(I = [0,1])$ be the space of all continuous functions defined on I. It is well known that such a space with the metric given by

$$d(x,y) = \|x - y\|_\infty = \max_{t \in I} |x(t) - y(t)|$$

is a complete metric space.

We have to show that the above mentioned differential equation satisfies the following inequality,

$$\alpha(x,y,t)(\frac{1}{F_{fx,fy}(\phi(t))}-1) \leq \min(\frac{1}{F_{x,y}(\phi(\frac{t}{c}))}-1, \frac{1}{F_{x,fx}(\phi(\frac{t}{c}))}-1, \frac{1}{F_{y,fy}(\phi(\frac{t}{c}))}-1)$$

taking $\alpha(x,y,z)=1$, $\phi(t)=t$,

we have

$$\frac{1}{F_{fx,fy}(\phi(t))}-1 \leq \min(\frac{1}{F_{x,y}(\phi(\frac{t}{c}))}-1, \frac{1}{F_{x,fx}(\phi(\frac{t}{c}))}-1, \frac{1}{F_{y,fy}(\phi(\frac{t}{c}))}-1).$$

Taking $F_{x,y}(t)=\frac{t}{t+d(x,y)}$,
that is,

$$\frac{1}{\frac{t}{t+d(fx,fy)}}-1 \leq \min(\frac{1}{\frac{\frac{t}{c}}{\frac{t}{c}+d(x,y)}}-1, \frac{1}{\frac{\frac{t}{c}}{\frac{t}{c}+d(x,fx)}}-1, \frac{1}{\frac{\frac{t}{c}}{\frac{t}{c}+d(y,fy)}}-1),$$

that is,

$$\frac{t+d(fx,fy)}{t}-1 \leq \min(\frac{\frac{t}{c}+d(x,y)}{\frac{t}{c}}-1, \frac{\frac{t}{c}+d(x,fx)}{\frac{t}{c}}-1, \frac{\frac{t}{c}+d(y,fy)}{\frac{t}{c}}-1),$$

that is,

$$\frac{d(fx,fy)}{t} \leq \min(\frac{\frac{t}{c}+d(x,y)-\frac{t}{c}}{\frac{t}{c}}, \frac{\frac{t}{c}+d(x,fx)-\frac{t}{c}}{\frac{t}{c}}, \frac{\frac{t}{c}+d(y,fy)-\frac{t}{c}}{\frac{t}{c}}),$$

that is,

$$\frac{d(fx,fy)}{t} \leq \min(\frac{cd(x,y)}{t}, \frac{cd(x,fx)}{t}, \frac{cd(y,fy)}{t}), \text{for } t \neq 0$$

that is,

$$d(fx,fy) \leq \min c(d(x,y), d(x,fx), d(y,fy)).$$

We have $c > 0$ such that for all $x,y \in C(I,R)$ and for all $t,s \in I$, for all $a,b \in R$, we get

$$|g(t,a)-g(t,b)| \leq c\, min\{|x(s)-y(s)|, |x(s)-fx(s)|, |y(s)-fy(s)|\}.$$

Now, It is well known that $x \in C^2(I)$ is a solution of given differential equation is equivalent to that $x \in C(I)$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s)g(s,x(s))ds, \text{for all } t \in I. \tag{4.6}$$

Define the operator $f : C(I) \to C(I)$ by

$$f(x(t)) = \int_0^1 G(t,s)g(s,x(s))ds, \text{for all } t \in I.$$

To find $x^* \in C(I)$ that is a fixed point of f.

So,

$$\begin{aligned}
|f(x(t)) - f(y(t))| &= |\int_0^1 G(t,s)[g(s,x(s)) - g(s,y(s))]ds| \\
&\le \int_0^1 G(t,s)|g(s,x(s)) - g(s,y(s))|ds \\
&\le \int_0^1 G(t,s)\, c \cdot min\{d(x,y), d(x,fx), d(y,fy))\}ds \\
&= c \cdot min\{d(x,\ y), d(x,\ fx),\ d(y,\ fy)\} \int_0^1 G(t,s)ds \\
&\le c \cdot min\{d(x,y), d(x,fx), d(y,fy))\} \times \frac{1}{8} \\
&= 0.
\end{aligned}$$

Note that for all $t \in I$,

$$\int_0^1 G(t,s)ds = -\frac{t^2}{2} + \frac{t}{2},$$

which implies that,

$$\sup_{t \in I} \int_0^1 G(t,s)ds = \frac{1}{8}.$$

Also ,

$$min\{d(x,y), d(x,fx), d(y,fy)\} = min\{d(x,y), 0, 0\} = 0.$$

implies

$$d(fx,fy) = min\{d(x,y), d(x,fx), d(y,fy)\}, \text{for all } x,y \in C([0,1],R).$$

Therefore by Theorem 3.1 with $\phi(t) = t$ for all $t \ge 0$ and $\alpha(x,y,t) = 1$ for all $x,y \in C([0,1],R)$ and $t > 0$, we conclude that the uniqueness of the operator f is $fx^* = x^* \in C([0,1],R)$, which also serves the purpose of unique solution of (4.6), our proposed integral equation.

5. Conclusion

In the course of mathematical analysis and allied stream related to it, probabilistic metric spaces has an important role. The structural theory was created primarily after 1960. Many researchers have taken their interest in this area of research. Some authors have recently demonstrated that PM spaces are also applicable in nuclear fusion. One of the references may be noted as [29]. This paper [29] outlines the application to identify regimes of containment and disruption of plasma.

Conflict of interest

The authors declare no conflict of interest.

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