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Research article

Generalized conformable variational calculus and optimal control problems with variable terminal conditions

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Abstract: This paper provides generalized transversality conditions for the problems of variational calculus and optimal control, constructed by the conformable derivative. The generalized term is used to emphasize the problems with performance indexes containing the conformable derivative and defined by the classical integral and to distinguish them from the problems with performance indexes defined by the conformable integral. Special cases of the generalized transversality conditions both for variational calculus and optimal control are exhibited and supported by illustrative examples.

Keywords: conformable derivative; fractional order; optimal control; variational calculus; generalized Euler-Lagrange equation; generalized transversality condition
Mathematics Subject Classification: 34H05, 49K15

1. Introduction

The conformable derivative has been proposed as an alternative to fractional derivatives by Khalil et al. [1] who aim to solve fractional order differential equations in analytical ways. Despite most of fractional derivative definition defined by the convolution integral with singular (i.e. Riemann–Liouville or Caputo [2]) or nonsingular (i.e. Atangana-Baleanu [3]) kernels, the conformable derivative of a real function f of order $0 < \alpha \leq 1$ was suggested via the following limits:

$$\frac{d^\alpha f(t)}{dt^\alpha} = f^{(\alpha)}(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}, t>0 \tag{1.1}$$

$$\frac{d^\alpha f(t)}{dt^\alpha}\Big|_{t=0} = f^{(\alpha)}(0) = \lim_{t\to 0} f^{(\alpha)}(t). \tag{1.2}$$

Since the conformable derivative has the limit based definition, it is a local operator like the classical derivative. Therefore, it has not the properties of memory and hereditary which are valuable for usual fractional derivatives. However, the conformable derivative supplies many rules of the classical derivative, the most important one of them is Leibniz rule which cannot be provided by usual fractional derivatives [4]. But, these mentioned points have led to a controversy in fractional scientific groups, that the conformable or any other local derivatives named with "fractional derivative", such as Katugampola, Kolwankar-Gangal, M-derivative, cannot be classified as fractional derivative [4–6]. A short time ago, Teodoro et al. have reviewed the definitions linking with the fractional derivatives and they have classified local and nonlocal operators into four classes to clarify the properties of each operator in detail [7]. According to the their classification, they have concluded that the local derivatives are not fractional derivatives. On the other hand, the conformable derivative has been recently covered into the deformed derivative concept which also covers q-derivative, fractal derivative and Haussdorf derivative, and deals with complex dynamical systems with the mathematics of local operators [8, 9]. Therefore, it is still significant to explore the conformable or any other local derivatives in the view of their mathematical and physical properties [10]. This idea is also supported by the latest experimental work on anomalous diffusion by Zhou et al. [11] who show the conformable diffusion model is better to fit the experimental data than normal diffusion. Moreover, they indicate that the best fitting order of the conformable derivative is obtained by the short-term experimental data which means the model has the benefit of predicting the long-term subdiffusion process effectively. Another work that exhibits the effectiveness of the conformable derivative in the advection-diffusion process has been presented by Avcı et al. [12]. First of all, they propose a local description of matter flux by introducing parameters of matter diffusion and flow velocity via a power-law time scaling proportional with t^{α} where α is the magnitude of anomalous transport corresponds to order of the conformable derivative, and then obtained the conformable advection-diffusion equation. Furthermore, they demonstrate as a biological application that the proposed conformable model coincides to anomalous diffusion of proteins owing to molecular crowding.

Due to simplicity and potential applications of the conformable derivative, mathematical properties of the operator have been studied by a considerable number of scientists. Abdeljawad defined left, right and sequential conformable derivatives, also gave chain rule, partial integration, conformable Taylor expansion and conformable Laplace transform [13]. The conformable Laplace transform was studied with its deeply details by Silva et al. [14]. The conformable partial derivative was defined by Atangana et al. [15]. Furthermore, complex conformable derivative was suggested and the basic properties of the operator were investigated by [16, 17]. Analytical solutions of the conformable differential equations were investigated by Hammad and Khalil [18], Anderson and Ulness [19], Ünal and Gökdoğan [20], and so on [21–23]. Although the conformable differential equations can be solved analytically, the numerical solutions were also proposed as expected [24–26].

Recently, as a result of successful applications of the conformable derivative into science and engineering problems, analysis of the conformable derivative in system theory, optimization and control areas has been needed. Evirgen [27] modeled a conformable gradient based dynamical system to solve constraint optimization problems. Conformable variational calculus problem (CVCP) was introduced by Chung [28] who obtained the conformable Euler-Lagrange equations and one-dimensional conformable Newtonian mechanics. Optimal boundary control of a heat equation defined by the conformable derivative was proposed by İskender Eroğlu and Yapişkan [29]. The necessary optimality conditions for conformable optimal control problem (COCP) were given by Lazo and Torres [30] via Hamiltonian formalism. General transversality conditions for CVCP and COCPs were explored and the special cases of the general transversality conditions were proposed by us [31]. Optimal control of a conformable diffusion equation was also given as an application problem in [31]. At the same time, Chiranjeevi and Biswas [32] proposed closed form solution of COCP with fixed or variable final conditions. For the case of fixed final time and variable final state COCP, the proposed transversality conditions in [31] and [32] overlap.

The above works except [30] considered the problems with performance index defined by the conformable integral. However, the performance index defined with the classical integral, classified as generalized problem by [30], is seen to be more often used in application problems. Therefore, the current work focuses on the generalized conformable variational calculus problem (GCVCP) and the generalized optimal control problem (GCOCP) with variable terminal point, and aims to present the transversality conditions of the problems which mentioned in this text as generalized transversality conditions. Organization of the paper is planned as follows. The needed definitions and features of conformable derivative are collected in the consecutive section. After that, the generalized transversality conditions are achieved for problems of variational calculus in Section 3 and optimal control in Section 4. The special cases of generalized transversality conditions are also proposed in their related sections and depicted with application problems supported with graphics that are plotted using MATLAB program. Finally, all results obtained throughout the paper are summarized in conclusions.

2. Basic definitions and tools

In this chapter, we give only the needed definitions and theorems to construct the other parts of the paper.

Definition 2.1. *([13]) The left conformable derivative of a real valued function f on a closed interval $[a, b]$ with the order of $0 < \alpha \leq 1$ is defined as*

$$\frac{d_a^{\alpha}}{dt_a^{\alpha}} f(t) = f_a^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{2.1}$$

If the limit exists on the open interval (a, b), then

$$f_a^{(\alpha)}(a) = \lim_{t \to a^+} f_a^{(\alpha)}(t)$$

and

$$f_a^{(\alpha)}(b) = \lim_{t \to b^-} f_a^{(\alpha)}(t).$$

It is obvious that the conformable derivative coincides with the classical derivative for $\alpha = 1$, with another words, the conformable derivative is conformable for $\alpha \to 1$. Furthermore, the conformable derivative is equivalent to $f_a^{(\alpha)}(t) = (t-a)^{1-\alpha} f'(t)$ for differentiable functions. By using this property, one can easily claim that the conformable derivative is not conformable for $\alpha \to 0$

$$\lim_{\alpha \to 0} f_a^{(\alpha)}(t) = t f'(t) \neq f(t)$$

which is also observed for most of local and nonlocal derivatives. Due to the similarities between the conformable and the classical derivative definitions, the conformable derivative has the advantageous properties such as Leibniz, quotient and chain rules (see [1, 13]).

Definition 2.2. *([13]) Let $f \in C^n\{[a,b],\mathbb{R}\}$ for $n \in \mathbb{N}^+$ and $0<\alpha\le 1$ is the order of conformable derivative. The left sequential conformable derivative of order n is defined by*

$$\left(\frac{d_a^\alpha}{dt_a^\alpha}\right)^n f(t) = \underbrace{\frac{d_a^\alpha}{dt_a^\alpha}\frac{d_a^\alpha}{dt_a^\alpha}\cdots\frac{d_a^\alpha}{dt_a^\alpha}}_{n-times} f(t).$$

The following definition is adopted from [15].

Definition 2.3. *For a real function f with m variables in which all $x_i \in [a_i,b_i]$, $i=1,2,\ldots,m$, the left conformable partial derivative of order $0<\alpha\le 1$ with respect to x_i is defined by*

$$\frac{\partial_{a_i}^\alpha}{\partial_{a_i} x_i^\alpha} f(x_1,x_2,\ldots,x_m) = \lim_{\varepsilon\to 0}\frac{f\left(x_1,\ldots,x_{i-1},x_i+\varepsilon(x_i-a_i)^{1-\alpha},\ldots,x_m\right)-f(x_1,\ldots,x_m)}{\varepsilon}. \tag{2.2}$$

Definition 2.4. *([13]) The left conformable integral of order $0<\alpha\le 1$ for a real function f defined on a closed interval $[a,b]$ is defined by*

$$I_a^\alpha f(t) = \int_a^t f(x)\, d_a^\alpha x = \int_a^t f(x)(x-a)^{\alpha-1}\, dx. \tag{2.3}$$

Theorem 2.1. *([13]) Let $f,g : [a,b] \to \mathbb{R}$ be functions such that fg is differentiable. Then the conformable partial integration formula is given by*

$$\int_a^b f(t)\, g^{(\alpha)}(t)\, d_a^\alpha t = f(t)\, g(t)\Big|_a^b - \int_a^b g(t)\, f_a^{(\alpha)}(t)\, d_a^\alpha t. \tag{2.4}$$

3. Generalized conformable variational problems

The conformable variational calculus is an expansion of the variational calculus that aims to minimize (or maximize) a performance index containing at least one conformable derivative term defined via the conformable or classical integral [31]. CVCP with classical performance index is classified as GCVCP by Lazo and Torres [30] who investigate the generalized conformable Euler-Lagrange equation of GCVCPs that contains both the conformable and classical derivatives. In this section, we aim to present the generalized transversality condition for GCVCPs. Therefore, we consider the following GCVCP

$$J[x] = \int_{t_0}^{t_e} L\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right) dt, \tag{3.1}$$

where $0<\alpha\le 1$ is the order of conformable derivative, $L \in C^\alpha\left\{[t_0,t_e]\times\mathbb{R}^2,\mathbb{R}\right\}$ is the Lagrangian and $x(t) \in C^\alpha\{[t_0,t_e],\mathbb{R}\}$ is an unknown function. Since the problem (3.1) does not contain the classical derivative term, it is a special case of the problem introduced in [30], so we first need to obtain the generalized conformable Euler-Lagrange equation of GCVCP (3.1). For this purpose, we temporarily assume that the problem (3.1) has fixed endpoint conditions also named separately as initial and terminal conditions presented as

$$x(t_0) = x_0 \text{ and } x(t_e) = x_e, \text{ where } x_0, x_e \in \mathbb{R}. \tag{3.2}$$

Before going further, we give the basic concepts needed to solve the GCVCPs.

Definition 3.1. *([31]) For the problem* (3.1) *the functions* $x(t) \in C^{\alpha}\{[t_0, t_e], \mathbb{R}\}$ *satisfying the endpoint conditions* (3.2) *are called admissible functions.*

Definition 3.2. *([31]) Assume* $x^*(t)$ *is a minimum and* $x(t)$ *is an admissible function of the problem* (3.1)*. For an arbitrary function* $\eta \in C^{\alpha}\{[t_0, t_e], \mathbb{R}\}$ *provided* $\eta(t_0) = \eta(t_e) = 0$*, the weak variation is defined by*

$$x(t) = x^*(t) + \varepsilon\eta(t), \tag{3.3}$$

where ε *is a positive small quantity independent from* t*,* x^* *and* η*.*

The following lemma is the extended version of the fundamental lemma of variational calculus to the conformable variational calculus.

Lemma 3.1. *([30]) Assume* $0 < \alpha \leq 1$ *and* $\mu, \eta \in C\{[t_0, t_e], \mathbb{R}\}$ *are arbitrary functions. If the following equation holds*

$$\int_{t_0}^{t_e} \mu(t)\,\eta(t)\, d_{t_0}^{\alpha} t = 0 \tag{3.4}$$

for any continuous η *satisfying* $\eta(t_0) = \eta(t_e) = 0$*, then*

$$\mu(t) = 0. \tag{3.5}$$

for all $t \in [t_0, t_e]$.

The following theorem gives the necessary optimality condition for the GCVCP given by (3.1) − (3.2). It is worth to note that the theorem will be proven via the variation method which differs from the proof given by [30] via Gateaux derivative.

Theorem 3.1. *(Generalized Conformable Euler–Lagrange Equation) If* $x(t)$ *is a minimum (or maximum) of the GCVCP defined by* (3.1) − (3.2)*, then* $x(t)$ *provides the generalized conformable Euler-Lagrange equation given as follows*

$$(t - t_0)^{1-\alpha} \frac{\partial L}{\partial x} - \frac{d_{t_0}^{\alpha}}{dt_{t_0}^{\alpha}} \left((t - t_0)^{1-\alpha} \frac{\partial L}{\partial x_{t_0}^{(\alpha)}} \right) = 0. \tag{3.6}$$

Proof. Let $x^*(t)$ be the extremum of the performance index J and $\eta(t)$ be an arbitrary function from the class of $C^{\alpha}\{[t_0, t_e], \mathbb{R}\}$ that satisfies $\eta(t_0) = \eta(t_e) = 0$. Then the weak variations can be written as

$$x(t) = x^*(t) + \varepsilon\eta(t), \tag{3.7}$$

$$x_{t_0}^{(\alpha)}(t) = x_{t_0}^{*(\alpha)}(t) + \varepsilon\eta_{t_0}^{(\alpha)}(t), \tag{3.8}$$

for $|\varepsilon| \ll 1$. To find the extremum $x^*(t)$, the variation of the performance index should be examined according to the weak variations of (3.7) and (3.8) as

$$\Delta J = \int_{t_0}^{t_e} L\left(t, x^*(t) + \varepsilon\eta(t), x_{t_0}^{*(\alpha)}(t) + \varepsilon\eta_{t_0}^{(\alpha)}(t)\right) dt - \int_{t_0}^{t_e} L\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right) dt.$$

After that, Taylor expansion of the function L near the point $\left(x^*, x_{t_0}^{*(\alpha)}\right)$ for $t \in [t_0, t_e]$ via the pair of variables $\left(\varepsilon\eta, \varepsilon\eta_{t_0}^{(\alpha)}\right)$ is replaced in ΔJ:

$$\begin{aligned} \Delta J = & \int_{t_0}^{t_e} \left\{ L\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right) + \tfrac{\partial L}{\partial x}\varepsilon\eta(t) + \tfrac{\partial L}{\partial x_{t_0}^{(\alpha)}}\varepsilon\eta_{t_0}^{(\alpha)}(t) \right. \\ & \left. + \tfrac{1}{2}\left(\tfrac{\partial^2 L}{\partial x^2}(\varepsilon\eta(t))^2 + 2\tfrac{\partial^2 L}{\partial x \partial x_{t_0}^{(\alpha)}}\varepsilon\eta(t)\varepsilon\eta_{t_0}^{(\alpha)}(t) + \tfrac{\partial^2 L}{\partial x_{t_0}^{(\alpha)2}}\left(\varepsilon\eta_{t_0}^{(\alpha)}(t)\right)^2\right)\right\} dt \\ & - \int_{t_0}^{t_e} L\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right) dt + O\left(\varepsilon^3\right). \end{aligned} \tag{3.9}$$

Equation (3.9) can be written

$$\Delta J = \varepsilon\delta J + \varepsilon^2\delta^2 J + O\left(\varepsilon^3\right),$$

in which

$$\delta J = \int_{t_0}^{t_e} \left(\frac{\partial L}{\partial x}\eta(t) + \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\eta_{t_0}^{(\alpha)}(t)\right) dt \tag{3.10}$$

is the first and

$$\delta^2 J = \int_{t_0}^{t_e} \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^2}\eta(t)^2 + 2\frac{\partial^2 L}{\partial x \partial x_{t_0}^{(\alpha)}}\eta(t)\eta_{t_0}^{(\alpha)}(t) + \frac{\partial^2 L}{\partial x_{t_0}^{(\alpha)^2}}\left(\eta_{t_0}^{(\alpha)}(t)\right)^2\right) dt \tag{3.11}$$

is the second variations. Suppose $x^*(t)$ is the minimum of J, then

$$\Delta J = \varepsilon\delta J + \varepsilon^2\delta^2 J + O\left(\varepsilon^3\right) \geq 0 \tag{3.12}$$

must be satisfied for all admissible $\eta(t)$. If both sides of the inequality (3.12) are divided by ε, since ε may be positive or negative, the following inequalities arise

$$\begin{aligned} \Delta J = \delta J + \varepsilon\delta^2 J + O\left(\varepsilon^2\right) \leq 0, \quad & \text{for } \varepsilon > 0, \\ \Delta J = \delta J + \varepsilon\delta^2 J + O\left(\varepsilon^2\right) \geq 0, \quad & \text{for } \varepsilon < 0. \end{aligned}$$

When the limit of the inequalities are taken for $\varepsilon \to 0$, it is held both $\delta J \leq 0$ and $\delta J \geq 0$ which give the first variation must be zero for all admissible $\eta(t)$. Therefore, the necessary condition for the minimum is expressed as follows:

$$\delta J = \int_{t_0}^{t_e} \left(\frac{\partial L}{\partial x}\eta(t) + \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\eta_{t_0}^{(\alpha)}(t)\right) dt = 0. \tag{3.13}$$

Note that, the same relation is also obtained if $x^*(t)$ is assumed to be a maximum of J. To achieve a more useful result, we use the conformable partial integration formula and remove the conformable derivative term $\eta_{t_0}^{(\alpha)}(t)$ from the equation (3.13). Therefore, we define the following transformation

$$\tilde{L}\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right) = (t - t_0)^{1-\alpha} L\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right) \tag{3.14}$$

which transforms the classical integral of the first variation to a conformable integral as below

$$\delta J = \int_{t_0}^{t_e} \left(\frac{\partial \tilde{L}}{\partial x} \eta(t) + \frac{\partial \tilde{L}}{\partial x_{t_0}^{(\alpha)}} \eta_{t_0}^{(\alpha)}(t) \right) d_{t_0}^{\alpha} t = 0. \tag{3.15}$$

Applying the conformable partial integration formula (2.4) to (3.15) gives

$$\int_{t_0}^{t_e} \frac{\partial \tilde{L}}{\partial x_{t_0}^{(\alpha)}} \eta_{t_0}^{(\alpha)}(t)\, d_{t_0}^{\alpha} t = -\int_{t_0}^{t_e} \frac{d_{t_0}^{\alpha}}{dt_{t_0}^{\alpha}} \left(\frac{\partial \tilde{L}}{\partial x_{t_0}^{(\alpha)}} \right) \eta(t)\, d_{t_0}^{\alpha} t \tag{3.16}$$

since $\eta(t_0) = \eta(t_e) = 0$. Substituting (3.16) into (3.15) and arranging the first variation according to the arbitrary function $\eta(t)$ leads to

$$\delta J = \int_{t_0}^{t_e} \left(\frac{\partial \tilde{L}}{\partial x} \eta(t) - \frac{d_{t_0}^{\alpha}}{dt_{t_0}^{\alpha}} \left(\frac{\partial \tilde{L}}{\partial x_{t_0}^{(\alpha)}} \right) \right) \eta(t)\, d_{t_0}^{\alpha} t = 0.$$

By using the Lemma (3.1), the generalized conformable Euler-Lagrange equation is finally acquired from the first variation as

$$(t - t_0)^{1-\alpha} \frac{\partial L}{\partial x} - \frac{d_{t_0}^{\alpha}}{dt_{t_0}^{\alpha}} \left((t - t_0)^{1-\alpha} \frac{\partial L}{\partial x_{t_0}^{(\alpha)}} \right) = 0,$$

that completes the proof. □

It is known that the problems encountered in physics and engineering may not include the sufficient number of endpoint conditions. However, the prior properties of the variational calculus is that the methodology always supply the right number of conditions named as transversality conditions. Therefore, variational calculus problems whose one or both endpoint conditions are missing, can be solved with Euler-Lagrange equation and transversality conditions. Here, we propose the generalized transversality condition for a GCVCP in which the performance index has fixed initial condition $x(t_0) = x_0$ and has variable terminal condition $x(\tau) = x_\tau$. For generality, the variable terminal condition is assumed to lie on a given curve. It is useful to indicate that the initial condition can be also assumed to be free, but we prefer to skip this assumption for brevity of the proof procedure which can be easily extended to the variable initial condition. So, we consider the GCVCP defined by the performance index

$$J[x] = \int_{t_0}^{\tau} L\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right) dt \tag{3.17}$$

with fixed initial condition

$$x(t_0) = x_0 \text{ is fixed,} \tag{3.18}$$

and variable terminal condition

$$x(\tau) = x_\tau \text{ is free which lies on a differentiable curve } \nu(t), \tag{3.19}$$

where $0 < \alpha \le 1$ is the order of conformable derivative, $L \in C^\alpha \left\{[t_0, \tau] \times \mathbb{R}^2, \mathbb{R}\right\}$ is the Lagrangian and $x = x(t) \in C^\alpha \{[t_0, \tau], \mathbb{R}\}$ is an unknown function.

Theorem 3.2. *(Generalized Transversality Condition for GCVCP) If a GCVCP is defined by* (3.17)-(3.19)*, then the generalized transversality condition is*

$$L\left(\tau, x(\tau), x_{t_0}^{(\alpha)}(\tau)\right)\Delta\tau + (\tau - t_0)^{1-\alpha} \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}|_\tau \varepsilon\eta(\tau) = 0, \tag{3.20}$$

where $\eta \in C^\alpha \{[t_0, \tau], \mathbb{R}\})$ and $\Delta\tau$ are arbitrary functions. Additionally, it is also assumed that $\Delta\tau$ is from the remainder class of $O(\varepsilon)$.

Proof. Assume $x^*(t)$ be an extremum of J and intersects with the target curve $\nu(t)$ at $t = \tau^*$. Consider the following weak variations for $|\varepsilon| \ll 1$

$$\begin{aligned} x(t) &= x^*(t) + \varepsilon\eta(t), \\ x_{t_0}^{(\alpha)}(t) &= x_{t_0}^{*(\alpha)}(t) + \varepsilon\eta_{t_0}^{(\alpha)}(t), \\ \tau &= \tau^* + \varepsilon\Delta\tau, \end{aligned} \tag{3.21}$$

in which $\eta(t_0) = 0$ and $\eta(\tau) \neq 0$. The variation of the performance index is calculated via these weak variations as

$$\Delta J = \int_{t_0}^{\tau^*+\varepsilon\Delta\tau} L\left(t, x^*(t) + \varepsilon\eta(t), x_{t_0}^{*(\alpha)}(t) + \varepsilon\eta_{t_0}^{(\alpha)}(t)\right)dt - \int_{t_0}^{\tau^*} L\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right)dt \tag{3.22}$$

and organized in the following form

$$\begin{aligned} \Delta J = & \int_{\tau^*}^{\tau^*+\varepsilon\Delta\tau} L\left(t, x^*(t) + \varepsilon\eta(t), x_{t_0}^{*(\alpha)}(t) + \varepsilon\eta_{t_0}^{(\alpha)}(t)\right)dt \\ & + \int_{t_0}^{\tau^*} \left(L\left(t, x(t), x_{t_0}^{(\alpha)}(t)\right) - L\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right)\right)dt. \end{aligned} \tag{3.23}$$

For $t \in [t_0, \tau]$, expanding the function L to a Taylor series near the point $\left(x^*, x_{t_0}^{*(\alpha)}\right)$ via the pair of variables $\left(\varepsilon\eta, \varepsilon\eta_{t_0}^{(\alpha)}\right)$ and substituting the expansion in ΔJ gives

$$\begin{aligned} \Delta J = & \int_{\tau^*}^{\tau^*+\varepsilon\Delta\tau} \left(L\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right) + \tfrac{\partial L}{\partial x}\varepsilon\eta(t) + \tfrac{\partial L}{\partial x_{t_0}^{(\alpha)}}\varepsilon\eta_{t_0}^{(\alpha)}(t)\right)dt \\ & + \int_{t_0}^{\tau^*} \left(L\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right) + \tfrac{\partial L}{\partial x}\varepsilon\eta(t) + \tfrac{\partial L}{\partial x_{t_0}^{(\alpha)}}\varepsilon\eta_{t_0}^{(\alpha)}(t)\right. \\ & \left. - L\left(t, x^*(t), x_{t_0}^{*(\alpha)}(t)\right)\right)dt + O\left(\varepsilon^2\right). \end{aligned} \tag{3.24}$$

When the first integral is calculated via a numerical method, the following equation is held

$$\begin{aligned}\Delta J = & \left(L\left(\tau^*, x^*(\tau^*), x_{t_0}^{*(\alpha)}(\tau^*)\right) + \tfrac{\partial L}{\partial x}\Big|_{\tau^*} \varepsilon\eta(\tau^*) + \tfrac{\partial L}{\partial x_{t_0}^{(\alpha)}}\Big|_{\tau^*} \varepsilon\eta_{t_0}^{(\alpha)}(\tau^*)\right)\varepsilon\Delta\tau \\ & + \int_{t_0}^{\tau^*}\left(\tfrac{\partial L}{\partial x}\varepsilon\eta(t) + \tfrac{\partial L}{\partial x_{t_0}^{(\alpha)}}\varepsilon\eta_{t_0}^{(\alpha)}(t)\right)dt + O\left(\varepsilon^2\right).\end{aligned} \tag{3.25}$$

Again, we can separate the variation to the first and second variations marked respectively with δJ and $\delta^2 J$. Thus, the first variation is equal to zero due to the requirement to be an extremum:

$$\delta J = L\left(\tau^*, x^*(\tau^*), x_{t_0}^{*(\alpha)}(\tau^*)\right)\Delta\tau + \int_{t_0}^{\tau^*}\left(\frac{\partial L}{\partial x}\eta(t) + \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\eta_{t_0}^{(\alpha)}(t)\right)dt = 0. \tag{3.26}$$

If the transformation (3.14) is applied to (3.26), the first variation is expressed via the following conformable integral

$$\delta J = L\left(\tau^*, x^*(\tau^*), x_{t_0}^{*(\alpha)}(\tau^*)\right)\Delta\tau + \int_{t_0}^{\tau^*}\left(\frac{\partial \tilde{L}}{\partial x}\eta(t) + \frac{\partial \tilde{L}}{\partial x_{t_0}^{(\alpha)}}\eta_{t_0}^{(\alpha)}(t)\right)d_{t_0}^{\alpha}t = 0,$$

then the conformable integration by part for formula (2.4) is used to remove $\eta_{t_0}^{(\alpha)}$ term as below

$$\begin{aligned}\delta J = & \ L\left(\tau^*, x^*(\tau^*), x_{t_0}^{*(\alpha)}(\tau^*)\right)\Delta\tau + \tfrac{\partial \tilde{L}}{\partial x_{t_0}^{(\alpha)}}\Big|_{\tau^*}\eta(\tau^*) \\ & + \int_{t_0}^{\tau^*}\eta(t)\left(\tfrac{\partial \tilde{L}}{\partial x} - \tfrac{d_{t_0}^{\alpha}}{dt_{t_0}^{\alpha}}\left(\tfrac{\partial \tilde{L}}{\partial x_{t_0}^{(\alpha)}}\right)\right)d_{t_0}^{\alpha}t = 0.\end{aligned} \tag{3.27}$$

Since the term inside of the above integral is generalized conformable Euler-Lagrange equation, the generalized transversality condition of GCVCP is achieved as

$$L\left(\tau^*, x(\tau^*), x_{t_0}^{(\alpha)}(\tau^*)\right)\Delta\tau + (\tau^* - t_0)^{1-\alpha}\frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\Big|_{\tau^*}\eta(\tau^*) = 0. \tag{3.28}$$

□

The generalized transversality condition (3.28) depends on the unknown arbitrary functions η and $\Delta\tau$. To identify these functions and get more clear generalized transversality conditions, we examine the special cases considering for the terminal condition (3.19) by the below corollary.

Corollary 3.1. ***(Specialized Transversality Conditions for GCVCP)*** *Taylor series expansion of the weak variation $x(t) = x^*(t) + \varepsilon\eta(t)$ at the point $t = \tau = \tau^* + \varepsilon\Delta\tau$ according to $\varepsilon\Delta\tau$ near the point τ^* as*

$$x(t) = x^*(\tau^* + \varepsilon\Delta\tau) + \varepsilon\eta(\tau^* + \varepsilon\Delta\tau) = x^*(\tau^*) + \dot{x}^*(\tau^*)\varepsilon\Delta\tau + \varepsilon\eta(\tau^*) + O\left(\varepsilon^2\right). \tag{3.29}$$

Intersection of the curve $x(t)$ and the target curve $v(t)$ at the point $t = \tau$ leads also the requirement of Taylor expansion for $v(\tau^* + \varepsilon\Delta\tau)$ via $\varepsilon\Delta\tau$ about the point τ^* as

$$v(\tau^* + \varepsilon\Delta\tau) = v(\tau^*) + \dot{v}(\tau^*)\varepsilon\Delta\tau + O\left(\varepsilon^2\right). \tag{3.30}$$

Equating (3.29) *with* (3.30) *and ignoring the remainder terms give the perturbation function as*

$$\eta(\tau^*) = (\dot{\nu}(\tau^*) - \dot{x}^*(\tau^*))\,\Delta\tau. \tag{3.31}$$

Therefore, substituting the equation (3.31) *into the equation* (3.28) *leads to the following transversality condition:*

$$L\left(\tau^*, x(\tau^*), x_{t_0}^{(\alpha)}(\tau^*)\right) + (\tau^* - t_0)^{1-\alpha} \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\Big|_{\tau^*} (\dot{\nu}(\tau^*) - \dot{x}^*(\tau^*)) = 0. \tag{3.32}$$

This equation can be more simplified if either τ or x_τ is specified while the other value is completely free. Therefore, we examine special forms of (3.32) *by the following situations.*

- **Terminal Curve:** If τ is free and x_τ belongs to a differentiable target curve $\nu(t)$, then the transversality condition is

$$L\left(\tau, x(\tau), x_{t_0}^{(\alpha)}(\tau)\right) + (\tau - t_0)^{1-\alpha} \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\Big|_{\tau} (\dot{\nu}(\tau) - \dot{x}(\tau)) = 0. \tag{3.33}$$

- **Vertical Terminal Line:** If τ is fixed and x_τ is free which coincides to a straight line perpendicular to the x-axis, then the transversality condition (3.32) can be rewritten by assuming $\nu(\tau) \neq 0$ as

$$\frac{1}{\dot{\nu}(\tau)}\left(L\left(\tau, x(\tau), x_{t_0}^{(\alpha)}(\tau)\right) - (\tau - t_0)^{1-\alpha}\dot{x}(\tau)\right) + (\tau - t_0)^{1-\alpha} \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\Big|_{\tau} = 0.$$

  For infinite $\dot{\nu}(\tau)$, the transversality condition is reduced to

$$(\tau - t_0)^{1-\alpha} \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\Big|_{\tau} = 0. \tag{3.34}$$

  Since this condition naturally arise in variational formulation, it is also referred as **"natural boundary condition"**.
- **Horizontal Terminal Line:** If τ is free and x_τ is fixed which coincides to a straight line parallel to x-axis, then for $\dot{\nu}(\tau) = 0$ the transversality condition is obtained as

$$L\left(\tau, x(\tau), x_{t_0}^{(\alpha)}(\tau)\right) - (\tau - t_0)^{1-\alpha}\dot{x}(\tau) \frac{\partial L}{\partial x_{t_0}^{(\alpha)}}\Big|_{\tau} = 0. \tag{3.35}$$

Example 1. *Find the minimum of*

$$J[x] = \int_0^\tau \left(1 - x_0^{(\alpha)}(t)\right)^2 dt \tag{3.36}$$

for each of the following cases:

- $x(0) = 1,\ x(\tau) = 2$
- $x(0) = 1,\ \tau = 2.$
- $x(0) = 1$ *and* $x(\tau)$ *lies on the differentiable curve* $\nu(t) = 2 + (t^\alpha - 1)^2$.

Solution 1. *The generalized conformable Euler–Lagrange equation of the problem is*

$$x_0^{(\alpha)}(t)\, x_0^{(\alpha)}(t) + (1-\alpha)\, t^{-\alpha} x_0^{(\alpha)}(t) = (1-\alpha)\, t^{-\alpha}.$$

Using the method for conformable linear differential equations with variable coefficients given by [18], the solution is achieved as

$$x(t) = c_1 + c_2 \frac{t^{2\alpha-1}}{2\alpha-1}.$$

To get the minimum solution of the problem, we need to determine the unknown coefficients. From the initial condition $x(0) = 1$, *the first coefficient is achieved as* $c_1 = 1$ *which leads to*

$$x(t) = 1 + c_2 \frac{t^{2\alpha-1}}{2\alpha-1}.$$

The other coefficient is specified via transversality conditions, so that we examine the each cases separately

- ***First Case:*** *Since* τ *is free and* $x(\tau) = 2$, *we obtain*

  $$c_2 \tau^{2\alpha-1} = 2\alpha - 1. \tag{3.37}$$

  Writing the functions L and x into the transversality condition (3.35) *gives*

  $$\left(1 - x_0^{(\alpha)}(\tau)\right)^2 - 2c_2\tau^{\alpha-1}\left(1 - x_0^{(\alpha)}(\tau)\right) = 0. \tag{3.38}$$

  If the equations (3.37) *and* (3.38) *are solved, then the minimum solution is identified. To illustrate the minimum solution, we choose* $\alpha = 0.7$ *and find the parameters as* $\tau = 0.2701$ *and* $c_2 = 0.6752$ *via Symbolic Toolbox of MATLAB. According to these parameters, the minimum solution is plotted in Figure 1.*

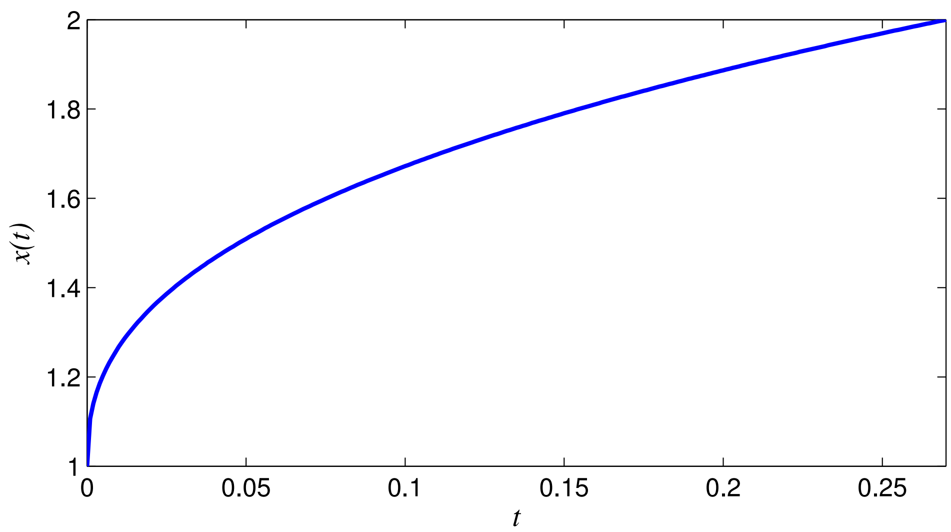

Figure 1. Solution of horizontal terminal line problem for $\alpha = 0.7$.

- ***Second Case:*** *Since* $\tau = 2$ *and* $x(\tau)$ *is free, the transversality condition* (3.34) *gives*

$$2\tau^{\alpha-1}\left(1 - c_2\tau^{\alpha-1}\right) = 0.$$

For $\alpha = 0.7$, *the unknown coefficient is determined as* $c_2 = 2.4623$ *and then the minimum function is plotted via these parameters in Figure 2.*

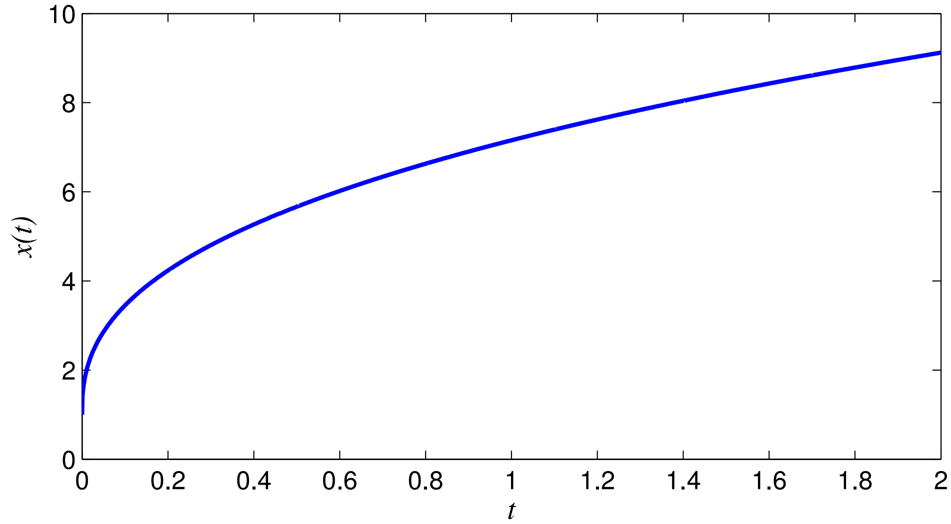

Figure 2. Solution of vertical terminal line problem for $\alpha = 0.7$.

- ***Third Case:*** *Since* τ *and* x_τ *are free but* x_τ *lies on the curve* $v(t) = 2 + (t^\alpha - 1)^2$, *the first equation needed to solve is obtained by equating* $x(\tau) = v(\tau)$ *as*

$$c_2\frac{\tau^{2\alpha-1}}{2\alpha - 1} - \left(\tau^{\alpha-1}\right)^2 - 1 = 0. \tag{3.39}$$

The second equation arising from the transversality condition (3.33) *is*

$$\left(1 - c_2\tau^{\alpha-1}\right)\left(1 - c_2\tau^{\alpha-1} - \left(4\alpha\left(\tau^\alpha - 1\right) - 2c_2\tau^{\alpha-1}\right)\right) = 0. \tag{3.40}$$

Solving the equations (3.39) − (3.40) *gives the minimum solution which is illustrated in Figure 3 for the chosen value of* $\alpha = 0.7$ *corresponds to* $\tau = 0.3715$ *and* $c_2 = 0.7430$.

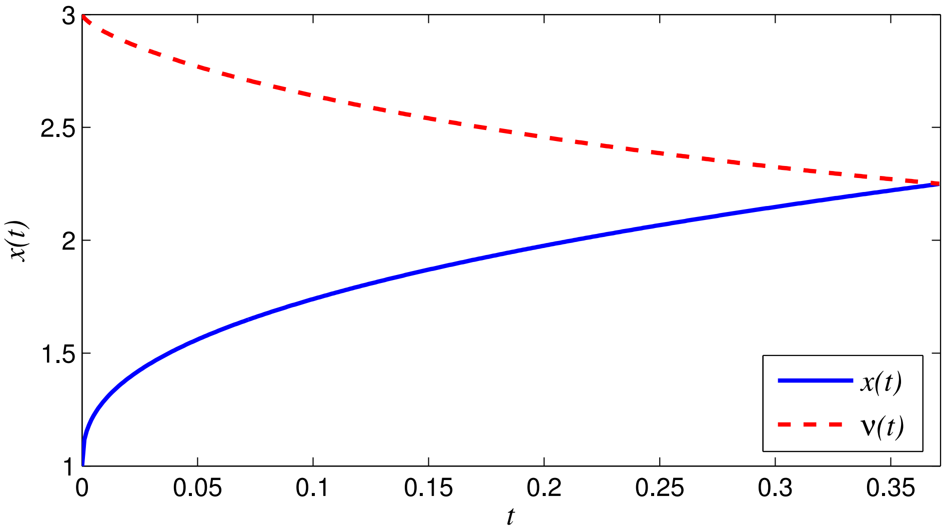

Figure 3. Solution of terminal curve problem for $\alpha = 0.7$.

4. Generalized conformable optimal control problem

Conformable optimal control aims to find the state and control functions which minimize (or maximize) a given performance index defined with classical or conformable integrals subjected to a conformable dynamic constraints, [31]. In this section, we consider COCPs with classical performance index which we call generalized conformable optimal control problems (GCOCPs) and analyze the generalized transversality conditions for GCOCPs by utilizing the generalized transversality conditions for GCVCPs. For this purpose, firstly we need to know the necessary optimality conditions for GCOCPs. When these conditions are examined, it is concluded that the GCOCPs have the same optimality conditions with COCPs which are proposed by [30] via Hamiltonian formalism. To remind these conditions, we consider the following COCP which aims to minimize the performance index

$$J[x,u] = \int_{t_0}^{t_e} L(t, x(t), u(t))\, dt$$

with fixed initial $x(t_0)$ and fixed terminal $x(t_e)$ conditions and subjected to a conformable dynamic constraint

$$x_{t_0}^{(\alpha)}(t) = f(t, x(t), u(t))$$

in which $0 < \alpha \leq 1$ is the order of conformable derivative, L, f are the functions from the class of $C^{\alpha}\left\{[t_0, t_e] \times \mathbb{R}^2, \mathbb{R}\right\}$, x is the state from the class of $C^{\alpha}\{[t_0, t_e], \mathbb{R}\}$ and u is the control function. The Hamiltonian formulation for GCOCP is

$$H(t, x, u, \lambda) = -L(t, x, u) + \lambda(t) f(t, x, u) \tag{4.1}$$

where $\lambda \in C^{\alpha}\{[t_0, t_e], \mathbb{R}\}$ is a Lagrange multiplier. Then the conformable Euler–Lagrange equations also known as necessary conditions of optimality are as follows

$$\begin{cases} x_{t_0}^{(\alpha)}(t) = \frac{\partial H}{\partial \lambda}(t, x, u, \lambda), \\ \lambda_{t_0}^{(\alpha)}(t) = -\frac{\partial H}{\partial x}(t, x, u, \lambda), \\ \frac{\partial H}{\partial u}(t, x, u, \lambda) = 0. \end{cases} \tag{4.2}$$

As to be for GCVCPs, one or both endpoint conditions may be missing for GCOCPs. Here, we again assume that the initial condition is fixed while the terminal condition is variable via the same reason explained in the previous section. Therefore, we consider the most general form of GCOCPs that aims to minimize the performance index

$$J[x, u] = \int_{t_0}^{\tau} L(t, x(t), u(t))\, dt \tag{4.3}$$

with fixed initial condition

$$x(t_0) = x_0 \tag{4.4}$$

and variable terminal condition

$$x_\tau = x(\tau) \text{ lies on a differentiable curve } \nu(t) \tag{4.5}$$

which is subjected to a conformable dynamic constraint

$$x_{t_0}^{(\alpha)}(t) = f(t, x(t), u(t)), \tag{4.6}$$

in which $0 < \alpha \leq 1$ is the order or the conformable derivative, L, f are the functions from the class of $C^{\alpha}\left\{\left[t_0, t_f\right] \times \mathbb{R}^2, \mathbb{R}\right\}$, x is the state from the class of $C^{\alpha}\left\{\left[t_0, t_f\right], \mathbb{R}\right\}$ and u is the control function. The transversality conditions of the problem is given by the below theorem.

Theorem 4.1. *(Transversality Conditions for GCOCP) If a GCOCP is defined by* (4.3) − (4.6), *then the generalized conformable transversality condition is*

$$\left[-H(\tau, x(\tau), u(\tau)) + \lambda(\tau)\, x_{t_0}^{(\alpha)}(\tau)\right] \Delta\tau + (\tau - t_0)^{1-\alpha} \lambda(\tau)\, \eta(\tau) = 0, \tag{4.7}$$

where $\eta \in C^{\alpha}\{[t_0, \tau], \mathbb{R}\}$ and $\Delta\tau$ are arbitrary functions. Additionally, it is also assumed that $\Delta\tau$ is from the remainder class of $O(\varepsilon)$.

Proof. Let $x^*(t)$ and $u^*(t)$ are the optimum functions in which $x^*(t)$ intersects with the target curve $\nu(t)$ at $t = \tau^*$. To identify optimum functions, we consider arbitrary functions $\eta(t)$, $\zeta(t)$, $\Lambda(t)$, $\Delta\tau$ that have α−order conformable derivative on $[t_0, \tau]$, and define the weak variations at $t \in [t_0, \tau]$ for $|\varepsilon| \ll 1$ as

$$\begin{aligned} x(t) &= x^*(t) + \varepsilon\eta(t), \\ x_{t_0}^{(\alpha)}(t) &= x_{t_0}^{*(\alpha)}(t) + \varepsilon\eta_{t_0}^{(\alpha)}(t), \\ u(t) &= u^*(t) + \varepsilon\zeta(t), \\ \lambda(t) &= \lambda^*(t) + \varepsilon\Lambda(t), \end{aligned} \tag{4.8}$$

$$\tau = \tau^* + \varepsilon\Delta\tau,$$

Note that, $\eta(t_0) = \zeta(t_0) = \Lambda(t_0) = 0$ and $\eta(\tau) \neq 0, \zeta(\tau) \neq 0, \Lambda(\tau) \neq 0$ since τ is unspecified. To use the Lagrange multiplier technique we define

$$\Phi[x,u,\lambda] = \int_{t_0}^{\tau} \lambda(t)\left(x_{t_0}^{(\alpha)}(t) - f(t,x(t),u(t))\right)dt$$

and then give the following functional

$$I[x,u,\lambda] = J[x,u] + \Phi[x,u,\lambda] \tag{4.9}$$

which obviously coincides to the performance index J since Φ is identically zero. To construct a clear proof, we examine the variations of the functionals J and Φ separately as in the following

$$\Delta J = \int_{\tau^*}^{\tau^*+\varepsilon\Delta\tau} L(t,x(t),u(t))\,dt + \int_{t_0}^{\tau^*} \{L(t,x(t),u(t)) - L(t,x^*(t),u^*(t))\}\,dt \tag{4.10}$$

$$\begin{aligned}\Delta\Phi = & \int_{\tau^*}^{\tau^*+\varepsilon\Delta\tau} \lambda(t)\left(x_{t_0}^{(\alpha)}(t) - f(t,x(t),u(t))\right)dt \\ & + \int_{t_0}^{\tau^*} \left\{\lambda(t)\left(x_{t_0}^{(\alpha)}(t) - f(t,x(t),u(t))\right) - \lambda^*(t)\left(x_{t_0}^{*(\alpha)}(t) - f(t,x^*(t),u^*(t))\right)\right\}dt.\end{aligned} \tag{4.11}$$

When the functions L and f are expanded to the Taylor series near the point (x^*, u^*) for $t \in [t_0, \tau]$ via the pair of variables $(\varepsilon\eta, \varepsilon\zeta)$, we get

$$\begin{aligned}\Delta J = & \int_{\tau^*}^{\tau^*+\varepsilon\Delta\tau} \left\{L(t,x^*(t),u^*(t)) + \tfrac{\partial L}{\partial x}\varepsilon\eta(t) + \tfrac{\partial L}{\partial u}\varepsilon\zeta(t)\right\}dt \\ & + \int_{t_0}^{\tau^*} \left(\tfrac{\partial L}{\partial x}\varepsilon\eta(t) + \tfrac{\partial L}{\partial u}\varepsilon\zeta(t)\right)dt + O\left(\varepsilon^2\right),\end{aligned} \tag{4.12}$$

$$\begin{aligned}\Delta\Phi = & \int_{\tau^*}^{\tau^*+\varepsilon\Delta\tau} (\lambda^*(t) + \varepsilon\Lambda(t))\left\{x_{t_0}^{*(\alpha)}(t) + \varepsilon\eta_{t_0}^{(\alpha)}(t)\right. \\ & \left. - f(t,x^*(t),u^*(t)) - \tfrac{\partial f}{\partial x}\varepsilon\eta(t) - \tfrac{\partial f}{\partial u}\varepsilon\zeta(t)\right\}dt \\ & + \int_{t_0}^{\tau^*} (\lambda^*(t) + \varepsilon\Lambda(t))\left\{x_{t_0}^{*(\alpha)}(t) + \varepsilon\eta_{t_0}^{(\alpha)}(t)\right. \\ & - f(t,x^*(t),u^*(t)) - \tfrac{\partial f}{\partial x}\varepsilon\eta(t) - \tfrac{\partial f}{\partial u}\varepsilon\zeta(t) \\ & \left. - \lambda^*(t)\left(x_{t_0}^{*(\alpha)}(t) - f(t,x^*(t),u^*(t))\right)\right\}dt + O\left(\varepsilon^2\right).\end{aligned} \tag{4.13}$$

Then, calculating the first integrals in the above variations via a numerical method gives the first variations respectively as

$$\delta J = L(\tau^*, x^*(\tau^*), u^*(\tau^*))\,\Delta\tau + \int_{t_0}^{\tau^*} \left(\frac{\partial L}{\partial x}\eta(t) + \frac{\partial L}{\partial u}\zeta(t)\right)dt = 0, \tag{4.14}$$

$$\begin{aligned}\delta\Phi = {} & \lambda^*(\tau^*)\left(x_{t_0}^{*(\alpha)}(\tau^*) - f(\tau^*, x^*(\tau^*), u^*(\tau^*))\right)\Delta\tau \\ & + \int_{t_0}^{\tau^*} \lambda^*(t)\left(\eta_{t_0}^{(\alpha)}(t) - \tfrac{\partial f}{\partial x}\eta(t) - \tfrac{\partial f}{\partial u}\zeta(t)\right)dt \\ & + \int_{t_0}^{\tau^*} \Lambda(t)\left(x_{t_0}^{*(\alpha)}(t) - f(t, x^*(t), u^*(t))\right)dt = 0.\end{aligned} \tag{4.15}$$

To use the conformable partial integration formula (2.4) for the term $\int_{t_0}^{\tau^*} \lambda^*(t)\,\eta_{t_0}^{(\alpha)}(t)\,dt$ in (4.15), we define the following transformation

$$\widetilde{\lambda}^*(t) = (t - t_0)^{1-\alpha}\,\lambda^*(t)$$

which converts the classical integral to the conformable one. If the conformable integral is solved by the formula (2.4), then the integral is written be aware of $\eta(t_0) = 0$ as

$$\int_{t_0}^{\tau^*} \lambda^*(t)\,\eta_{t_0}^{(\alpha)}(t)\,dt = \widetilde{\lambda}^*(\tau^*)\,\eta(\tau^*) - \int_{t_0}^{\tau^*} \lambda_{t_0}^{*(\alpha)}(t)\,\eta(t)\,d_{t_0}^{\alpha}t. \tag{4.16}$$

Substituting (4.16) into (4.15) leads to

$$\begin{aligned}\delta\Phi = {} & \lambda^*(\tau^*)\left(x_{t_0}^{*(\alpha)}(\tau^*) - f(\tau^*, x^*(\tau^*), u^*(\tau^*))\right)\Delta\tau + \widetilde{\lambda}^*(\tau^*)\,\eta(\tau^*) \\ & - \int_{t_0}^{\tau^*} \left\{\eta(t)\left(\widetilde{\lambda}^*(t)\,\tfrac{\partial f}{\partial x} + \widetilde{\lambda}_{t_0}^{*(\alpha)}\right) + \zeta(t)\,\widetilde{\lambda}^*(t)\,\tfrac{\partial f}{\partial u}\right\} d_{t_0}^{\alpha}t \\ & + \int_{t_0}^{\tau^*} \Lambda(t)\left(x_{t_0}^{*(\alpha)}(t) - f(t, x^*(t), u^*(t))\right)(t - t_0)^{1-\alpha}\,d_{t_0}^{\alpha}t = 0.\end{aligned}$$

From (4.9), the first variation δI corresponds to the sum of δJ and $\delta\Phi$, equation (4.9) must be written in sense of conformable integral which gives the first variation of I is achieved via Hamiltonian formalism as

$$\begin{aligned}\delta I = {} & \left(-H(\tau^*, x^*(\tau^*), u^*(\tau^*)) + \lambda^*(\tau)\,x_{t_0}^{*(\alpha)}(\tau^*)\right)\Delta\tau + \widetilde{\lambda}^*(\tau^*)\,\eta(\tau^*) \\ & - \int_{t_0}^{\tau^*} \eta(t)\left(\widetilde{\lambda}_{t_0}^{*(\alpha)}(t) + (t - t_0)^{1-\alpha}\,\tfrac{\partial H}{\partial x}\right) d_{t_0}^{\alpha}t - \int_{t_0}^{\tau^*} \zeta(t)\,(t - t_0)^{1-\alpha}\,\tfrac{\partial H}{\partial u}\,d_{t_0}^{\alpha}t \\ & + \int_{t_0}^{\tau^*} \Lambda(t)\left(x_{t_0}^{*(\alpha)}(t) - f(t, x^*(t), u^*(t))\right)(t - t_0)^{1-\alpha}\,d_{t_0}^{\alpha}t = 0.\end{aligned} \tag{4.17}$$

Because the necessary optimality conditions are equal to zero, the above integrals vanish, so the transversality condition of GCOCP is acquired as

$$\left(-H(\tau^*, x^*(\tau^*), u^*(\tau^*)) + \lambda^*(\tau^*)\,x_{t_0}^{*(\alpha)}(\tau^*)\right)\Delta\tau + (\tau^* - t_0)^{1-\alpha}\,\lambda^*(\tau^*)\,\eta(\tau^*) = 0. \tag{4.18}$$

□

More clear formula for the generalized transversality condition of GCOCP is investigated by the following corollary which removes the unknown arbitrary functions of η, $\Delta\tau$ from (4.18) and specializes the transversality conditions for particular cases.

Corollary 4.1. *(* ***Specialized Transversality Conditions for GCOCP****) Since the arbitrary function $\eta(\tau^*)$ and the value $\Delta\tau$ arise in* (4.18) *are unknown, the relation* (3.31) *given by Corollary 3.1 is valid for $\eta(\tau^*)$ and $\Delta\tau$. Substituting* (3.31) *into* (4.18)*, and arranging* (4.18) *according to $\Delta\tau$ gives*

$$\left(-H(\tau^*, x^*(\tau^*), u^*(\tau^*)) + \lambda^*(\tau^*)\, x_{t_0}^{*(\alpha)}(\tau^*) + (\tau^* - t_0)^{1-\alpha}\lambda^*(\tau^*)\left(\dot{v}(\tau^*) - \dot{x}^*(\tau^*)\right)\right)\Delta\tau = 0. \tag{4.19}$$

Since $\Delta\tau$ is arbitrary, the generalized conformable transversality condition is finally achieved as

$$-H(\tau^*, x^*(\tau^*), u^*(\tau^*)) + \lambda^*(\tau^*)\, x_{t_0}^{*(\alpha)}(\tau^*) + (\tau^* - t_0)^{1-\alpha}\lambda^*(\tau^*)\left(\dot{v}(\tau^*) - \dot{x}^*(\tau^*)\right) = 0 \tag{4.20}$$

- ***Terminal Curve:*** *The differentiable target curve $v(t)$ defines the behavior of the terminal point such that $x_\tau = v(\tau)$. Therefore, the generalized transversality condition in the most general form must be*
$$-H(\tau, x(\tau), u(\tau)) + \lambda(\tau)\, x_{t_0}^{(\alpha)}(\tau) + (\tau - t_0)^{1-\alpha}\lambda(\tau)\left(\dot{v}(\tau) - \dot{x}(\tau)\right) = 0 \tag{4.21}$$
- ***Vertical Terminal Line:*** *For fixed τ and variable x_τ the target curve corresponds to a straight line perpendicular to the x-axis whose slope at τ is infinite. Therefore, the transversality condition for infinite $\dot{v}(\tau)$ is obtained as*
$$(\tau - t_0)^{1-\alpha}\lambda(\tau) = 0. \tag{4.22}$$
- ***Horizontal Terminal Line:*** *If τ is free and x_τ is fixed, then $\dot{v}(\tau) = 0$. Thus, the transversality conditions is to be*
$$-H(\tau, x(\tau), u(\tau)) + \lambda(\tau)\, x_{t_0}^{(\alpha)}(\tau) - (\tau - t_0)^{1-\alpha}\lambda(\tau)\,\dot{x}(\tau) = 0. \tag{4.23}$$

The following example is adopted from the example given in [33] by replacing the Riemann-Liouville fractional derivative with conformable derivative.

Example 2. *Find the pair of state $x(t)$ and control $u(t)$ functions which minimizes the quadratic performance index*

$$J[x,u] = \frac{1}{2}\int_0^\tau \left(x^2(t) + u^2(t)\right)dt, \tag{4.24}$$

subjected to dynamic constraint

$$x_0^{(\alpha)}(t) = -x(t) + u(t), \tag{4.25}$$

with the following endpoint conditions

$$x(0) = 10,\ x(\tau) = 1. \tag{4.26}$$

Solution 2. *The necessary conditions for optimality* (4.2) *via the Hamiltonian function of the problem*

$$H(t,x,u,\lambda) = -\frac{1}{2}\left(x^2(t) + u^2(t)\right) + \lambda(t)\left(-x(t) + u(t)\right).$$

is calculated as

$$\begin{aligned} x_0^{(\alpha)}(t) &= -x(t) + u(t), \\ \lambda_0^{(\alpha)}(t) &= x(t) + \lambda(t), \end{aligned} \tag{4.27}$$

$$u(t) = \lambda(t).$$

These equations are solved by the analytical method proposed in [19] and the state function is obtained as

$$x(t) = c_1 e^{\sqrt{2}\frac{t^\alpha}{\alpha}} + c_2 e^{-\sqrt{2}\frac{t^\alpha}{\alpha}}. \tag{4.28}$$

The fixed initial condition $x(0) = 10$ *and fixed terminal condition* $x(\tau) = 1$ *for free* τ *give*

$$\begin{aligned} c_1 + c_2 &= 10, \\ x(\tau) &= c_1 e^{\sqrt{2}\frac{\tau^\alpha}{\alpha}} + (10 - c_1) e^{-\sqrt{2}\frac{\tau^\alpha}{\alpha}}, \end{aligned}$$

respectively. From (4.27)*, the control function* $u(t)$ *is obtained as*

$$u(t) = \left(1 + \sqrt{2}\right)(10 - c_1) e^{\sqrt{2}\frac{t^\alpha}{\alpha}} + \left(1 - \sqrt{2}\right) c_1 e^{-\sqrt{2}\frac{t^\alpha}{\alpha}}. \tag{4.29}$$

To identify the unknown coefficient c_1*, the generalized transversality condition* (4.23) *is used which gives*

$$x^2(\tau) + u^2(\tau) - 2u(\tau)\tau^{1-\alpha}\dot{x}(\tau) = 0. \tag{4.30}$$

Substituting (4.28)*,* (4.29) *into* (4.30) *and solving* (4.30) *present the optimal control. To illustrate the optimal control law, the solution is achieved by Symbolic Toolbox of MATLAB for chosen value of* $\alpha = 0.7$ *resulted as* $\tau = 0.1307$*,* $c_1 = -5.0903$ *and* $c_2 = 15.0903$*. Thus, the state and control functions are plotted in Figure* 4*.*

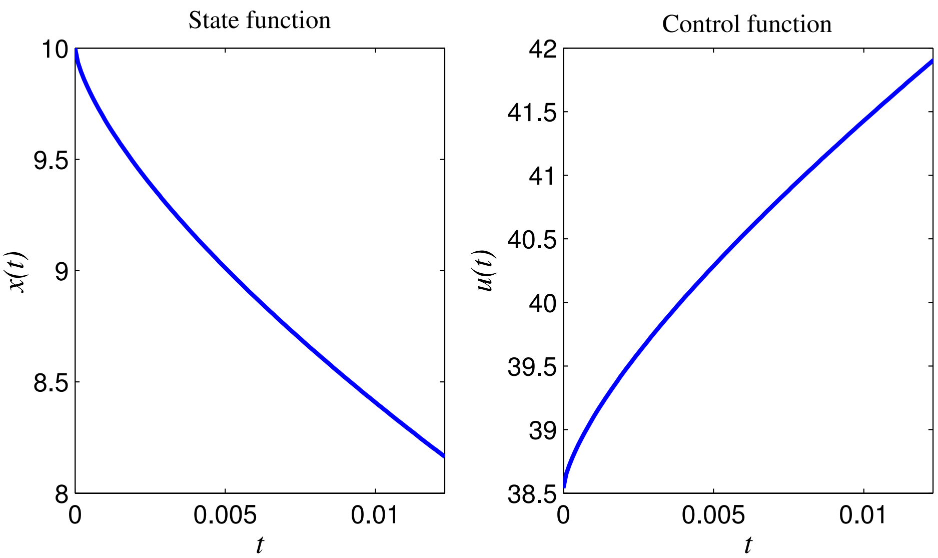

Figure 4. Optimal state and control functions of horizontal terminal line problem for $\alpha = 0.7$.

Now, we consider a GCOCP in the case of the terminal curve.

Example 3. *Consider the terminal curve problem*

$$J[u] = \int_0^\tau \sqrt{(1 + u^2(t))}dt, \tag{4.31}$$

subjected to dynamic constraint

$$x_0^{(\alpha)}(t) = u(t), \tag{4.32}$$

with the following endpoint conditions

$$x(0) = 0,\ x(\tau) = 10 - \tau^2 \tag{4.33}$$

and find the optimal control law which minimize the GCOCP.

Solution 3. *The necessary conditions for the optimality via the Hamiltonian function* $H(t, x, u, \lambda) = \sqrt{(1 + u^2(t))} - \lambda(t)u(t)$ *are determined as*

$$\begin{aligned} -\lambda_0^{(\alpha)}(t) &= 0, \\ x_0^{(\alpha)}(t) &= u(t), \\ \lambda(t) &= \frac{u(t)}{\sqrt{(1 + u^2(t))}}. \end{aligned}$$

These equations can be solved by the analytical method proposed in [19] which gives the optimal state function as

$$x(t) = \frac{t^\alpha}{\alpha} u(t) + c_1.$$

The initial condition gives $c_1 = 0$ *and so the terminal curve leads to*

$$10 - \tau^2 = \frac{\tau^\alpha}{\alpha} u(\tau).$$

This equation with the generalized transversality condition (4.21) *calculated as*

$$\tau^{1-\alpha}\left(-2\tau - u(\tau)\tau^{\alpha-1}\right)\frac{u(\tau)}{\sqrt{(1 + u^2(\tau))}} + \sqrt{(1 + u^2(\tau))} = 0$$

are solved together in which the solution presents the optimal control law. For chosen value of $\alpha = 0.7$*, the solutions are achieved as* $\tau = 0.01230$ *and* $u(t) = 152.1364$ *via MATLAB. Figure* 5 *depicts the optimal state and control functions, respectively.*

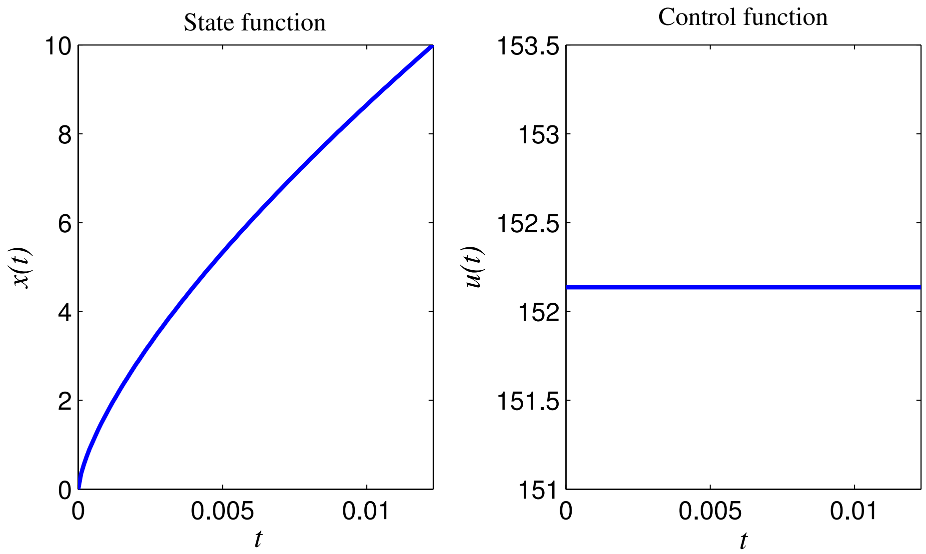

Figure 5. Optimal state and control functions of terminal curve problem for $\alpha = 0.7$.

5. Conclusions

The paper focuses on a pretty new area which is called conformable optimal control involved also conformable variational calculus as expected. Although optimal control problems with local or nonlocal operators have been investigated via performance index defined by the classical or own integral operators, classical performance indexes are more often used in application problems. Therefore, the conformable variational calculus and optimal control problems defined by the classical performance indexes are taken under consideration. The generalized Euler-Lagrange equations and the transversality conditions of the problems are obtained. Moreover, generalized transversality conditions are specialized to the terminal curve, fixed time horizon and fixed endpoint problems. Related examples for each problem have been solved and illustrated for a specific value of the order of conformable derivative.

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Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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