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Research article

Higher order energy functionals and the Chen-Maeta conjecture

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Abstract: The study of higher order energy functionals was first proposed by Eells and Sampson in 1965 and, later, by Eells and Lemaire in 1983. These functionals provide a natural generalization of the classical energy functional. More precisely, Eells and Sampson suggested the investigation of the so-called $ES-r$-energy functionals $E_r^{ES}(\varphi) = (1/2)\int_M |(d^* + d)^r(\varphi)|^2\, dV$, where $r \geq 2$ and $\varphi : M \to N$ is a map between two Riemannian manifolds. The initial part of this paper is a short overview on basic definitions, properties, recent developments and open problems concerning the functionals $E_r^{ES}(\varphi)$ and other, equally interesting, higher order energy functionals $E_r(\varphi)$ which were introduced and studied in various papers by Maeta and other authors. If a critical point φ of $E_r^{ES}(\varphi)$ (respectively, $E_r(\varphi)$) is an *isometric immersion*, then we say that its image is an $ES-r$-harmonic (respectively, r-harmonic) submanifold of N. We observe that *minimal* submanifolds are trivially both $ES-r$-harmonic and r-harmonic. Therefore, it is natural to say that an $ES-r$-harmonic (r-harmonic) submanifold is *proper* if it is not minimal. In the special case that the ambient space N is the Euclidean space $\mathbb{R}^n$ the notions of $ES-r$-harmonic and r-harmonic submanifolds coincide. The Chen-Maeta conjecture is still open: it states that, for all $r \geq 2$, any proper, r-harmonic submanifold of $\mathbb{R}^n$ is minimal. In the second part of this paper we shall focus on the study of $G = \mathrm{SO}(p+1) \times \mathrm{SO}(q+1)$-invariant submanifolds of $\mathbb{R}^n$, $n = p + q + 2$. In particular, we shall obtain an explicit description of the relevant Euler-Lagrange equations in the case that $r = 3$ and we shall discuss difficulties and possible developments towards the proof of the Chen-Maeta conjecture for 3-harmonic G-invariant hypersurfaces.

Keywords: polyharmonic maps or submanifolds; equivariant differential geometry; Chen conjecture; Maeta conjecture
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1. Introduction

Harmonic maps are the critical points of the *energy functional*

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 \, dV \,, \tag{1.1}$$

where $\varphi : M \to N$ is a smooth map between two Riemannian manifolds (M^m, g) and (N^n, h) (note that, if M is not compact, we have to consider compactly supported variations). In particular, φ is harmonic if it is a solution of the Euler-Lagrange system of equations associated to (1.1), i.e.,

$$- d^* d\varphi = \operatorname{trace} \nabla d\varphi = 0 \,. \tag{1.2}$$

The left member of (1.2) is a vector field along the map φ or, equivalently, a section of the pull-back bundle $\varphi^{-1}TN$: it is called *tension field* and denoted $\tau(\varphi)$. In addition, we recall that, if φ is an *isometric immersion*, then φ is a harmonic map if and only if the immersion φ defines a minimal submanifold of N (see [11, 12] for background). Now, let us denote ∇^M, ∇^N and ∇^φ the induced connections on the bundles TM, TN and $\varphi^{-1}TN$ respectively. The *rough Laplacian* on sections of $\varphi^{-1}TN$, denoted $\overline{\Delta}$, is defined by

$$\overline{\Delta} = d^* d = - \sum_{i=1}^{m} \left(\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla^M_{e_i} e_i} \right) ,$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field tangent to M. In recent years, the following r-order versions of the energy functional where intensively studied by Maeta and other researchers (see [4, 5, 26–29, 37–39, 45]). If $r = 2s$, $s \geq 1$:

$$\begin{aligned} E_{2s}(\varphi) &= \frac{1}{2} \int_M \langle \underbrace{(d^*d) \ldots (d^*d)}_{s \text{ times}} \varphi, \underbrace{(d^*d) \ldots (d^*d)}_{s \text{ times}} \varphi \rangle_{_N} \, dV \\ &= \frac{1}{2} \int_M \langle \overline{\Delta}^{s-1} \tau(\varphi), \overline{\Delta}^{s-1} \tau(\varphi) \rangle_{_N} \, dV \,. \end{aligned} \tag{1.3}$$

In the case that $r = 2s + 1$:

$$\begin{aligned} E_{2s+1}(\varphi) &= \frac{1}{2} \int_M \langle \, d \underbrace{(d^*d) \ldots (d^*d)}_{s \text{ times}} \varphi, \, d \underbrace{(d^*d) \ldots (d^*d)}_{s \text{ times}} \varphi \rangle_{_N} \, dV \\ &= \frac{1}{2} \int_M \sum_{j=1}^{m} \langle \nabla^\varphi_{e_j} \overline{\Delta}^{s-1} \tau(\varphi), \nabla^\varphi_{e_j} \overline{\Delta}^{s-1} \tau(\varphi) \rangle_{_N} \, dV \,. \end{aligned} \tag{1.4}$$

We say that a map φ is *r-harmonic* if, for all variations φ_t,

$$\frac{d}{dt} E_r(\varphi_t) \Big|_{t=0} = 0 \,.$$

In the case that $r = 2$, the functional (1.3) is called *bienergy* and its critical points are the so-called *biharmonic maps*. At present, a very ample literature on biharmonic maps is available and we refer to [9, 23, 35, 36, 40, 41] for an introduction to this topic. When $r \geq 3$, the Euler-Lagrange equations for

$E_r(\varphi)$ were obtained by Wang [45] and Maeta [26]. The expressions for their second variation were derived in [27], where it was also shown that a biharmonic map is not always r-harmonic ($r \geq 3$) and, more generally, that an s-harmonic map is not always r-harmonic ($2 \leq s < r$). On the other hand, any harmonic map is trivially r-harmonic for all $r \geq 2$. Therefore, we say that an r-harmonic map is *proper* if it is not harmonic (similarly, an r-harmonic submanifold, i.e., an r-harmonic isometric immersion, is *proper* if it is not minimal). As a general fact, when the ambient space has nonpositive sectional curvature there are several results which assert that, under suitable conditions, an r-harmonic submanifold is minimal (see [9], [26], [29] and [39], for instance), but the Chen conjecture that an arbitrary biharmonic submanifold of $\mathbb{R}^n$ must be minimal is still open (see [10] for recent results in this direction). More generally, the Maeta conjecture (see [26]) that any r-harmonic submanifold of the Euclidean space is minimal is open. More precisely, Maeta showed that the conjecture holds for curves, but very little is known when $\dim M > 1$: one of the goals of this paper is to study the Maeta conjecture in the case of $G = \mathrm{SO}(p+1) \times \mathrm{SO}(q+1)$-invariant submanifolds of $\mathbb{R}^n$, $n = p+q+2$. This study will be carried out in Section 3. By contrast, things drastically change when the ambient space is positively curved. More precisely, let us denote by $\mathbb{S}^n(R)$ the Euclidean sphere of radius R and write $\mathbb{S}^n$ for $\mathbb{S}^n(1)$. Then we have the following fundamental examples of proper r-harmonic submanifolds into spheres (see [23] for the biharmonic case, [28] for $r = 3$ and [37] for $r \geq 4$):

Theorem 1.1. *Assume that $r \geq 2, n \geq 2$. Then a small hypersphere $i : \mathbb{S}^{n-1}(R) \hookrightarrow \mathbb{S}^n$ is a proper r-harmonic submanifold of $\mathbb{S}^n$ if and only if the radius R is equal to $1/\sqrt{r}$.*

Theorem 1.2. *Let $r \geq 2$, $p, q \geq 1$ and assume that the radii R_1, R_2 verify $R_1^2 + R_2^2 = 1$. Then a generalized Clifford torus $i : \mathbb{S}^p(R_1) \times \mathbb{S}^q(R_2) \hookrightarrow \mathbb{S}^{p+q+1}$ is:*

(a) minimal if and only if

$$R_1^2 = \frac{p}{p+q} \quad \text{and} \quad R_2^2 = \frac{q}{p+q} \ ; \tag{1.5}$$

(b) a proper r-harmonic submanifold of $\mathbb{S}^{p+q+1}$ if and only if (1.5) *does not hold and either*

$$r = 2 \ , \quad p \neq q \quad \text{and} \quad R_1^2 = R_2^2 = \frac{1}{2}$$

or $r \geq 3$ and $t = R_1^2$ is a root of the following polynomial:

$$P(t) = r(p+q)\, t^3 + [q - p - r(q+2p)]\, t^2 + (2p + rp)\, t - p \ . \tag{1.6}$$

Remark 1.3. For a discussion on the existence of positive roots of the polynomial $P(t)$ in (1.6), which provide proper r-harmonic submanifolds, we refer to [37].

The setting for r-harmonicity which we have outlined so far represents, both from the geometric and the analytic point of view, a convenient approach to the study of higher order versions of the classical energy functional. However, we point out that actually the first idea of studying higher order versions of the energy functional was formulated in a different way. More precisely, in 1965 Eells and Sampson (see [14], and also [11]) proposed the following functionals, which we denote $E_r^{ES}(\varphi)$ to remember these two leading mathematicians:

$$E_r^{ES}(\varphi) = \frac{1}{2} \int_M |(d^* + d)^r(\varphi)|^2 \, dV \ . \tag{1.7}$$

To avoid confusion, it is important to fix the terminology: as we said above, a map φ is r-harmonic if it is a critical point of the functional $E_r(\varphi)$ defined in (1.3), (1.4). Instead, we say that a map φ is $ES-r$-harmonic if it is a critical point of the functional $E_r^{ES}(\varphi)$ defined in (1.7). The functionals $E_r^{ES}(\varphi)$ and $E_r(\varphi)$ coincide in the following cases:

(i) $r = 2, 3$;
(ii) $\dim M = 1$;
(iii) the sectional curvature tensor of N vanishes.

By contrast with the case of $E_r(\varphi)$, the explicit derivation of the Euler-Lagrange equation for the Eells-Sampson functionals $E_r^{ES}(\varphi)$ seems, in general, a very complicated task. These difficulties are explained in detail in the recent paper [6], where the Euler-Lagrange equation of the functional $E_4^{ES}(\varphi)$ was computed. To end this introduction, let us briefly point out some of the technical reasons which make the study of the functionals $E_r^{ES}(\varphi)$ rather different from that of their companions $E_r(\varphi)$. As we said, the two types of functionals coincide when $r = 2$ (the case of biharmonic maps) and $r = 3$: this is a consequence of the fact that d^* vanishes on 0-forms and $d^2\varphi = 0$, as computed in [11]. The first fundamental difference, as it was already observed in [29], arises when $r = 4$ because $d^2\tau(\varphi)$ is not necessarily zero unless N is flat or $\dim M = 1$. So, in general, we have

$$E_4^{ES}(\varphi) = \frac{1}{2}\int_M \left(|d^2\tau(\varphi)|^2 + |d^* d\tau(\varphi)|^2\right) dV = \frac{1}{2}\int_M |d^2\tau(\varphi)|^2\, dV + E_4(\varphi)\ . \tag{1.8}$$

This description of $E_4^{ES}(\varphi)$ appeared in [29], but the Euler-Lagrange equation associated to the first term on the right-side of (1.8) was computed only very recently in [6]. When $r \geq 5$ things become even more complicated. For instance, we know that the integrand of $E_5(\varphi)$ is the squared norm of a 1-form, but we cannot write $E_5^{ES}(\varphi)$ as the sum of $E_5(\varphi)$ and a functional which involves only differential forms of degree $p \neq 1$. The reason for this is the fact that, in general, the 1-form $dd^*d\tau(\varphi)$ (whose squared norm is the integrand of $E_5(\varphi)$) may mix up with $d^*d^2\tau(\varphi)$. Difficulties of this type boost as r increases.

The present paper is organised as follows: in Section 2, we describe some recent advances concerning the Eells-Sampson functionals $E_r^{ES}(\varphi)$ and propose a list of open problems on this topic. In Section 3, we study the Maeta conjecture for 3-harmonic $G = \mathrm{SO}(p+1) \times \mathrm{SO}(q+1)$-invariant submanifolds of $\mathbb{R}^n$, $n = p + q + 2$.

2. Recent results on the functionals $E_r^{ES}(\varphi)$ and open problems

In this section, we describe some recent results on the Eells-Sampson functionals $E_r^{ES}(\varphi)$ and conclude by proposing some open problems in this context. We start with the Euler-Lagrange equation for $E_4^{ES}(\varphi)$. First, we shall describe the equations in the general case. Next, we shall illustrate some relevant simplifications which occur when the target is a space form. We recall that, when $r = 4$, the Eells-Sampson functional is

$$E_4^{ES}(\varphi) = \frac{1}{2}\int_M |(d^* + d)(d\tau(\varphi))|^2\, dV = \frac{1}{2}\int_M |d^* d\tau(\varphi) + d^2\tau(\varphi)|^2\, dV.$$

Note that $d^* d\tau(\varphi) \in C(\varphi^{-1}TN) = A^0(\varphi^{-1}TN)$ and $d^2\tau(\varphi) \in C(\Lambda^2 T^*M \otimes \varphi^{-1}TN) = A^2(\varphi^{-1}TN)$. In order to simplify the formal sum in $E_4^{ES}(\varphi)$ we observe that

$$|d^* d\tau(\varphi) + d^2\tau(\varphi)|^2 = |d^* d\tau(\varphi)|^2 + |d^2\tau(\varphi)|^2 = |\overline{\Delta}\tau(\varphi)|^2 + |d^2\tau(\varphi)|^2.$$

The curvature term here acquires the form

$$|d^2\tau(\varphi)|^2 = |R^\varphi \wedge \tau(\varphi)|^2 = \frac{1}{2}\sum_{i,j} |R^N(d\varphi(X_i), d\varphi(X_j))\tau(\varphi)|^2,$$

where $\{X_i\}$ denotes a geodesic frame field around a point $p \in M$ and we shall perform the calculations at p. In the sequel, we shall omit to write the symbol $\sum$ when it is clear from the context. Therefore, we have

$$\begin{aligned} E_4^{ES}(\varphi) &= \frac{1}{2}\int_M |\overline{\Delta}\tau(\varphi)|^2 \, dV + \frac{1}{4}\int_M |R^N(d\varphi(X_i), d\varphi(X_j))\tau(\varphi)|^2 \, dV \\ &= E_4(\varphi) + \frac{1}{4}\int_M |R^N(d\varphi(X_i), d\varphi(X_j))\tau(\varphi)|^2 \, dV. \end{aligned}$$

It was first noted in [29], equation (2.8), that the four energy of Eells and Sampson contains a curvature contribution. Next, we wish to describe the Euler-Lagrange equation for $E_4^{ES}(\varphi)$. To this end we set

$$\widehat{E}_4(\varphi) = \frac{1}{2}\int_M |d^2\tau(\varphi)|^2 \, dV = \frac{1}{4}\int_M |R^N(d\varphi(X_i), d\varphi(X_j))\tau(\varphi)|^2 \, dV,$$

so that

$$E_4^{ES}(\varphi) = E_4(\varphi) + \widehat{E}_4(\varphi).$$

The first variation of $E_4(\varphi)$ was computed by Maeta in [26], while the first variation of $\widehat{E}_4(\varphi)$ was obtained in [6]. Combining these results we have:

Theorem 2.1. *Consider a smooth map $\varphi\colon (M^m, g) \to (N^n, h)$. Then the following formula holds*

$$\frac{d}{dt} E_4^{ES}(\varphi_t)\Big|_{t=0} = -\int_M \langle \tau_4^{ES}(\varphi), V\rangle \, dV,$$

where $\tau_4^{ES}(\varphi)$ is given by the following expression

$$\tau_4^{ES}(\varphi) = \tau_4(\varphi) + \hat{\tau}_4(\varphi)\,, \tag{2.1}$$

where

$$\begin{aligned} \tau_4(\varphi) =& \overline{\Delta}^3\tau(\varphi) + R^N(d\varphi(X_i), \overline{\Delta}^2\tau(\varphi))d\varphi(X_i) - R^N(\nabla^\varphi_{X_i}\overline{\Delta}\tau(\varphi), \tau(\varphi))d\varphi(X_i) \\ &+ R^N(\overline{\Delta}\tau(\varphi), \nabla^\varphi_{X_i}\tau(\varphi))d\varphi(X_i)\,, \\ \hat{\tau}_4(\varphi) =& -\frac{1}{2}(2\xi_1 + 2d^*\Omega_1 + \overline{\Delta}\Omega_0 + \operatorname{trace} R^N(d\varphi(\cdot), \Omega_0)d\varphi(\cdot))\,, \end{aligned}$$

and we have used the following abbreviations

$$\begin{array}{lcll} \Omega_0 &=& R^N(d\varphi(X_i), d\varphi(X_j))(R^N(d\varphi(X_i), d\varphi(X_j))\tau(\varphi)), & \Omega_0 \in C(\varphi^{-1}TN); \\ \Omega_1(X) &=& R^N(R^N(d\varphi(X), d\varphi(X_j))\tau(\varphi), \tau(\varphi))d\varphi(X_j), & \Omega_1 \in A^1(\varphi^{-1}TN); \\ \xi_1 &=& -(\nabla^N R^N)(d\varphi(X_j), R^N(d\varphi(X_i), d\varphi(X_j))\tau(\varphi), \tau(\varphi), d\varphi(X_i)), & \xi_1 \in C(\varphi^{-1}TN). \end{array}$$

Remark 2.2. We point out that the Euler-Lagrange equation $\tau_4^{ES}(\varphi) = 0$ is a semi-linear elliptic system of order 8. The leading terms are given by $\tau_4(\varphi)$, while $\hat{\tau}_4(\varphi)$ is a differential operator of order 4.

2.1. The case of space form target

In the case that the target manifold (N^n, h) is a real space form $N^n(\epsilon)$ with constant curvature ϵ the first variational formula of $\widehat{E}_4(\varphi)$ simplifies and we have:

$$\begin{aligned}\frac{d}{dt}\widehat{E}_4(\varphi_t)\Big|_{t=0} = \frac{1}{2}\int_M \langle & 2R^N(\nabla^{\varphi}_{X_i} V, d\varphi(X_j))\tau(\varphi) \\ & + R^N(d\varphi(X_i), d\varphi(X_j))(-\overline{\Delta} V - \operatorname{trace} R^N(d\varphi(\cdot), V)d\varphi(\cdot)), \\ & R^N(d\varphi(X_i), d\varphi(X_j))\tau(\varphi)\rangle \, dV. \end{aligned} \tag{2.2}$$

We have to compute all the terms on the right hand side of (2.2). Recall that

$$\langle R^N(\nabla^{\varphi}_{X_i} V, d\varphi(X_j))\tau(\varphi), R^N(d\varphi(X_i), d\varphi(X_j))\tau(\varphi)\rangle = \operatorname{div} Y + \langle d^*\Omega_1, V\rangle,$$

where $\Omega_1 \in A^1(\varphi^{-1}TN)$ is defined as

$$\Omega_1(X) = R^N(R^N(d\varphi(X), d\varphi(X_j))\tau(\varphi), \tau(\varphi))d\varphi(X_j)$$

and $Y = \langle \Omega_1(X_k), V\rangle X_k$ is a well-defined, global vector field on M. Next, for our purposes, it turns out to be useful to define the following vector field:

$$Z = \langle \tau(\varphi), d\varphi(X_k)\rangle X_k = -(\operatorname{div} S)^{\sharp},$$

where S is the stress-energy tensor field associated to φ. Clearly, we have

$$\operatorname{div} Z = |\tau(\varphi)|^2 + \langle d\varphi, \nabla^{\varphi}\tau(\varphi)\rangle.$$

We can now state the main result in the context of maps into a space form:

Theorem 2.3. *[6] In the case that $(N^n, h) = N^n(\epsilon)$ the terms in the expression of $\tau_4^{ES}(\varphi)$ given by* (2.1) *simplify as follows:*

$$\begin{aligned}\xi_1 &= 0,\\ \Omega_0 &= 2\epsilon^2(\operatorname{trace}\langle d\varphi(\cdot), d\varphi(Z)\rangle d\varphi(\cdot) - |d\varphi|^2 d\varphi(Z)),\\ \Omega_1 &= \epsilon^2(|Z|^2 d\varphi(\cdot) - Z^{\flat}\otimes d\varphi(Z) - \langle d\varphi(Z), d\varphi(\cdot)\rangle\tau(\varphi) + |d\varphi|^2 Z^{\flat}\otimes\tau(\varphi)).\end{aligned}$$

2.2. Some geometric applications

We observe that if $R^{\varphi}(X,Y)\tau(\varphi) = 0$ for any $X, Y \in C(TM)$, then φ is an absolute minimum for $\widehat{E}_4(\varphi)$ and so it is a critical point for $\widehat{E}_4(\varphi)$. As an application, we have (see [6]):

Proposition 2.4. *Let $\varphi\colon (M^m, g) \to (N^n, h)$ be a smooth map. Assume that $R^{\varphi}(X,Y)\tau(\varphi) = 0$ for any $X, Y \in C(TM)$. Then φ is a critical point of E_4^{ES} if and only if it is a critical point of E_4.*

Corollary 2.5. *Let $\varphi\colon (M^m, g) \to N^n(\epsilon)$ be a smooth map. Assume that $\tau(\varphi)$ is orthogonal to the image of the map. Then φ is ES-4-harmonic if and only if it is* 4*-harmonic. In particular, if $\varphi\colon M^m \to N^n(\epsilon)$ is an isometric immersion, then it is ES-4-harmonic if and only if it is* 4*-harmonic.*

2.3. Second variation

Let us consider a smooth map $\varphi\colon (M^m, g) \to (N^n, h)$ and, for simplicity, assume that M is compact. We consider a two-parameter smooth variation of φ, that is a smooth map

$$\Phi\colon \mathbb{R}\times\mathbb{R}\times M \to N, \qquad (t,s,p)\mapsto \Phi(t,s,p)=\varphi_{t,s}(p)$$

such that $\varphi_{0,0}(p)=\varphi(p)$ for any $p\in M$. To any given two-parameter variation of φ we associate the corresponding variation vector fields, i.e., the sections $V, W \in C(\varphi^{-1}TN)$ which are defined by

$$V(p)=\frac{d}{dt}\varphi_{t,0}(p)\Big|_{t=0}\in T_{\varphi(p)}N,$$
$$W(p)=\frac{d}{ds}\varphi_{0,s}(p)\Big|_{s=0}\in T_{\varphi(p)}N.$$

We will now compute

$$\frac{\partial^2}{\partial t\partial s}\widehat{E}_4(\varphi_{t,s})\Big|_{(t,s)=(0,0)}$$

starting with

$$\frac{\partial}{\partial s}\widehat{E}_4(\varphi_{t,s})\Big|_{(t,s)=(t,0)}=\frac{1}{2}\int_M \langle\nabla^{\Phi}_{\frac{\partial}{\partial s}(t,0,p)}R^{\Phi}(X_i,X_j)\tilde{\tau},R^{\varphi_{t,0}}(X_i,X_j)\tau(\varphi_{t,0})\rangle\,dV,$$

where $\tilde{\tau}\in C(\Phi^{-1}TN)$ is defined by

$$\tilde{\tau}(t,s,p)=\tau(\varphi_{t,s})_p\in T_{\varphi_{t,s}(p)}N.$$

Then we find

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}\widehat{E}_4(\varphi_{t,s})\Big|_{(t,s)=(0,0)}=&\frac{1}{2}\int_M \big(\langle\nabla^{\Phi}_{\frac{\partial}{\partial t}(0,0,p)}\nabla^{\Phi}_{\frac{\partial}{\partial s}}R^{\Phi}(X_i,X_j)\tilde{\tau},R^{\varphi}(X_i,X_j)\tau(\varphi)\rangle\\&+\langle\nabla^{\Phi}_{\frac{\partial}{\partial s}(0,0,p)}R^{\Phi}(X_i,X_j)\tilde{\tau},\nabla^{\Phi}_{\frac{\partial}{\partial t}(0,0,p)}R^{\Phi}(X_i,X_j)\tilde{\tau}\rangle\big)\,dV.\end{aligned}\tag{2.3}$$

Even if $R^{\varphi}(X,Y)\tau(\varphi)=0$ for any $X,Y\in C(TM)$, so that φ is a critical point of $\widehat{E}_4$, the Hessian of $\widehat{E}_4$ can be different from zero. Indeed, in this case we have

$$\frac{\partial^2}{\partial t\partial s}\widehat{E}_4(\varphi_{t,s})\Big|_{(t,s)=(0,0)}=\frac{1}{2}\int_M \langle\nabla^{\Phi}_{\frac{\partial}{\partial s}(0,0,p)}R^{\Phi}(X_i,X_j)\tilde{\tau},\nabla^{\Phi}_{\frac{\partial}{\partial t}(0,0,p)}R^{\Phi}(X_i,X_j)\tilde{\tau}\rangle\,dV$$

and this term will not vanish in general. We can conclude that, if $R^{\varphi}(X,Y)\tau(\varphi)=0$ and φ is a critical point for both E_4^{ES} and E_4, then the stability of φ may depend on which of the two functionals we are actually considering. Since, in this case, φ is an absolute minimum point for $\widehat{E}_4$, its index computed with respect to E_4^{ES} could be smaller than the one computed using E_4. However, in the case of a one-dimensional domain, there is no difference.

The previous discussions suggest that, in general, the notions of r-harmonicity and $ES-r$-harmonicity display significant differences. However, we have the following surprising result:

Theorem 2.6. *[6] Theorems 1.1 and 1.2 hold with the word r-harmonic replaced by $ES - r$-harmonic.*

The proof of this result requires essentially two ingredients. One is the explicit computation of the terms involving d^2. The other key tool is Proposition 2.8 below, which says that we can apply a rather general theorem of Palais which ensures the validity of the so-called principle of symmetric criticality. This result of Palais can be found in [42], p.22. However, since the paper [42] is written using a rather obsolete notation, we rewrite it here in a form which is suitable for our purposes. In order to do this, let us assume that G is a Lie group which acts on both M and N. Then G acts on $C^\infty(M,N)$ by $(g\varphi)(x) = g\varphi(g^{-1}x)$, $x \in M$. We say that a map φ is *G-equivariant* (shortly, equivariant) if $g\varphi = \varphi$ for all $g \in G$. Now, let $E : C^\infty(M,N) \to \mathbb{R}$ be a smooth function. Then we say that E is *G-invariant* if, for all $\varphi \in C^\infty(M,N)$, $E(g\varphi) = E(\varphi)$ for all $g \in G$. Now we can recall the Palais result in this context:

Theorem 2.7. *[42] Let M, N be two Riemannian manifolds and assume that G is a compact Lie group which acts on both M and N. Let $E : C^\infty(M,N) \to \mathbb{R}$ be a smooth, G-invariant function. If φ is G-equivariant, then φ is a critical point of E if and only if it is stationary with respect to G-equivariant variations, i.e., variations φ_t through G-equivariant maps.*

Palais observed in [42] that, if G is a group of *isometries* of both M and N, then the volume functional and the energy functional are both G-invariant and so the principle of symmetric criticality stated in Theorem 2.7 applies in both cases: the first, beautiful instances of this type can be found in the paper [21] for minimal submanifolds and in [44] for harmonic maps. It is also easy to show that the same is true for the bienergy functional: this is a special case in a more general setting of a reduction theory for biharmonic maps developed in [33, 34]. Now we shall extend this to the Eells-Sampson functionals $E_r^{ES}(\varphi)$, $r \geq 3$. In particular, using Theorem 2.7, we could prove the following basic result:

Proposition 2.8. *[6] Let M, N be two Riemannian manifolds and assume that G is a compact Lie group which acts by isometries on both M and N. If φ is a G-equivariant map, then φ is a critical point of $E_r^{ES}(\varphi)$ if and only if it is stationary with respect to G-equivariant variations.*

Remark 2.9. The conclusion of Proposition 2.8 is true also for the r-energy functional $E_r(\varphi)$.

Remark 2.10. Theorems 1.1, 1.2 and 2.6 were first proved for $r = 2$ (see [8] and [23]) and $r = 3$ (see [28]). The proofs given in [8, 23, 28] do not use the methods of equivariant differential geometry, but they are based on geometric constraints which the second fundamental form of a biharmonic, or 3-harmonic, immersion into $\mathbb{S}^n$ must satisfy. By contrast, we point out that the use of the principle of symmetric criticality of Proposition 2.8 enabled us to prove the existence of G-equivariant critical points even if the explicit general expression of the $ES - r$-tension field is not available. For this reason, this seems to be a very convenient approach to the study of the Eells-Sampson functionals $E_r^{ES}(\varphi)$.

2.4. Condition (C)

To our knowledge, no work in the literature clarifies and proves in which contexts the Condition (C) of Palais-Smale holds for the $ES - r$-energy (r-energy) functionals. A general belief (see [13, 15, 25]) is that, if $2r > \dim M$ and the curvature of the target is non-positive, then the $ES - r$-energy (r-energy) functionals may satisfy Condition (C). But, for each of these functionals, a further difficulty in the search of proper critical points is the fact that the minimum point in a given homotopy class can very well be reached by a harmonic map. By contrast, when the target has positive curvature, there is little hope that these higher order energy functionals satisfy Condition (C). The following result displays a homotopy class where the $ES-4$-energy functional does not reach the infimum.

Theorem 2.11. *[6] Let $\mathbb{T}^2$ denote the flat 2-torus. Then*

$$\operatorname{Inf}\left\{E_4^{ES}(\varphi)\colon \varphi\in C^{\infty}\left(\mathbb{T}^2,\mathbb{S}^2\right),\ \varphi \text{ has degree one}\right\}=0\,. \tag{2.4}$$

Moreover, the functional $E_4^{ES}(\varphi)$ does not admit a minimum in the homotopy class of maps $\varphi:\mathbb{T}^2\to\mathbb{S}^2$ of degree one.

The conclusion (2.4) in Proposition 2.11 was obtained by Lemaire (see [25]) in the case of the bienergy. It was pointed out in [6] that the same conclusion holds for $E_3(\varphi)$, $E_4(\varphi)$ and, more generally, for $r\geq 5$. We end this section with some open questions in this context.

Problem 2.12. Compute the Euler-Lagrange equation for $E_r^{ES}(\varphi)$, $r\geq 5$.

Problem 2.13. Simplify the expression (2.3) of the second variation operator associated to $\widehat{E}_4(\varphi)$, and apply it to some specific instance.

Problem 2.14. When $\dim M=2$, the energy functional is invariant with respect to conformal transformations of the domain. When $m=4$, the relationship between the bienergy and conformal changes has been thoroughly studied (see [1, 2, 32], for instance) and has produced several interesting examples. In particular, we know that the inverse stereographic projection from $\mathbb{R}^n$ to $\mathbb{S}^n\backslash\{\text{Pole}\}$ is a proper conformal biharmonic diffeomorphism. When $m=2r$, the functionals $E_r^{ES}(\varphi)$ and $E_r(\varphi)$ are invariant under homothetic changes of the metric on the domain. It would be interesting to explore the relationship between $E_r^{ES}(\varphi)$ (or $E_r(\varphi)$) and conformal transformations when $\dim M=2r$.

Problem 2.15. Establish a version of the unique continuation principle for the functionals $E_r^{ES}(\varphi)$ and $E_r(\varphi)$, $r\geq 3$ (see [7, 43] for the cases $r=1,2$).

Problem 2.16. Develop a theory of interior regularity for the critical points of the functionals $E_r^{ES}(\varphi)$ and $E_r(\varphi)$, $r\geq 3$. In this context, we cite [22, 24] for the case $r=2$. As for $r\geq 3$, we mention the papers [16] for maps from domains in $\mathbb{R}^n$, and [17] for maps into spheres.

3. $\mathrm{SO}(p+1)\times\mathrm{SO}(q+1)$-invariant 3-harmonic submanifolds of the Euclidean space

In this section, we study $\mathrm{SO}(p+1)\times\mathrm{SO}(q+1)$-invariant 3-harmonic submanifolds of the Euclidean space. The original motivation to develop a theory of G-invariant submanifolds (see [21]) was the search of examples of solutions of geometrically relevant partial differential equations or systems of equations. In this order of ideas, we first cite the counterexamples for classical (see [3]) and spherical (see [19]) Bernstein problems. In the same spirit, Hsiang [20] also proved the existence of non-Euclidean CMC immersions of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$, $n\geq 3$. By contrast, in the context of biharmonic immersions, the study of G-invariant submanifolds has contributed to substantiate the validity of the Chen conjecture (see [18], [30], [31]). In particular, we proved:

Theorem 3.1. *[31] Let G be a cohomogeneity two group of isometries acting on $\mathbb{R}^n$ ($n\geq 3$). Then any $G-$invariant biharmonic hypersurface in $\mathbb{R}^n$ is minimal.*

In the case of 3-harmonic immersions, aside from the above mentioned case of curves (see [26]), very little is known. Here we shall develop some new material which may prove useful to study the Maeta conjecture in the case of $G = \mathrm{SO}(p+1)\times\mathrm{SO}(q+1)$-invariant submanifolds of $\mathbb{R}^n$, $n = p+q+2$. In particular, as a first result, we shall extend the method of [33] to obtain the relevant system of ordinary differential equations for a 3-harmonic $\mathrm{SO}(p+1)\times\mathrm{SO}(q+1)$-invariant hypersurface of $\mathbb{R}^n$ (our analysis will include, as a special instance, the case of rotation hypersurfaces in $\mathbb{R}^n$, $n \geq 3$). We shall illustrate the difficulties which arise when we want to apply the standard existence and uniqueness theorems for ordinary differential equations. In particular, we shall see that not even *local* existence of non-minimal 3-harmonic G-invariant submanifolds can be deduced using these methods.

Assuming the canonical splitting $\mathbb{R}^n = \mathbb{R}^{p+1}\times\mathbb{R}^{q+1}$, we can suppose that an $SO(p+1)\times SO(q+1)$-invariant immersion into $\mathbb{R}^n$ is described as follows:

$$\begin{array}{cccccccccc} \varphi_\gamma & : & M = & \mathbb{S}^p & \times & \mathbb{S}^q & \times & I & \to & \mathbb{R}^{p+1}\times\mathbb{R}^{q+1} = \mathbb{R}^n \\ & & & (w & , & z & , & s) & \longmapsto & (x(s)\,w,\, y(s)\,z)\ , \end{array} \tag{3.1}$$

where $I \subset \mathbb{R}$ is an open interval and $x(s)$, $y(s)$ are smooth positive functions on I. We shall also assume that

$$\dot{x}^2 + \dot{y}^2 = 1\ , \tag{3.2}$$

so that the induced metric on the domain in (3.1) is given by:

$$g = x^2(s)\, g_{\mathbb{S}^p} + y^2(s)\, g_{\mathbb{S}^q} + ds^2\ , \tag{3.3}$$

where $g_{\mathbb{S}^p}$ and $g_{\mathbb{S}^q}$ denote the Euclidean metrics of the unit spheres $\mathbb{S}^p$ and $\mathbb{S}^q$ respectively. We shall work in the framework of equivariant differential geometry (see [19]). In particular, in this case the orbit space coincides with the flat Euclidean first quadrant

$$Q = \mathbb{R}^n/G = \left\{(x,\, y) \in \mathbb{R}^2\ :\ x,\, y \geq 0\right\}$$

and the orbit Volume function is $V(s) = x^p(s)\, y^q(s)$. We note that regular (i.e., corresponding to a point (x, y) with both x, $y > 0$) orbits are of the type $\mathbb{S}^p \times \mathbb{S}^q$. The orbit associated to the origin is a single point, while the other points on the x-axis (respectively, the y-axis) correspond to $\mathbb{S}^p$ (respectively, $\mathbb{S}^q$).

Definition 3.2. The curve $\gamma(s) = (x(s), y(s))$ in the orbit space Q, where $x(s)$, $y(s)$ are the functions which appear in (3.1), is called the *profile curve* associated to the equivariant immersion.

The general, basic principle of equivariant differential geometry (see [42]) can be roughly stated by saying that a G-invariant submanifold is a critical point of a G-invariant functional provided that the profile curve γ satisfies a suitable system of ordinary differential equations. Our basic result in this context is the following:

Theorem 3.3. *Consider an* $\mathrm{SO}(p+1)\times\mathrm{SO}(q+1)$*-invariant submanifold of* $\mathbb{R}^n$*,* $n = p+q+2$*, as in* (3.1)*, and assume that* (3.2) *holds. Then the* 3*-tension field of* φ_γ *is the horizontal lift, with respect to the canonical projection* $\pi : \mathbb{R}^n \to Q$*, of*

$$\tau_{3,x}\,\frac{\partial}{\partial x} + \tau_{3,y}\,\frac{\partial}{\partial y}\,, \quad \text{where}$$

$$
\begin{aligned}
\tau_{3,x} \quad = \quad & \frac{-1}{2x^5y^5}\Big[-(-4+p)py^5(-p^2-5p\dot{x}^2+(2p^2+3p-12)\dot{x}^6+\dot{x}^4(12+2p-p^2+p(-3+2p)\dot{y}^2)) \\
&+2pxy^4\Big(5pq\dot{x}\dot{y}-(-12+p+2p^2)q\dot{x}^5\dot{y}+5py\ddot{x}+(84-37p-8p^2+2p^3)y\dot{x}^4\ddot{x}+y\dot{x}^2(-30+p+p^2 \\
&+p(12-9p+p^2)\dot{y}^2)\ddot{x}+\dot{x}^3\dot{y}((-6-3p+p^2)q-2(-2+p)pq\dot{y}^2+p(8-7p+p^2)y\ddot{y})\Big) \\
&+px^2y^3\Big(3y^2(2-p+(-2+p)p\dot{y}^2\ddot{x}^2-2(2+p)q\dot{x}^4((-3+p+q)\dot{y}^2-(-3+p)y\ddot{y})+2y\dot{x}^3((-38+p+5p^2)q\dot{y}\ddot{x} \\
&+(-34+11p+3p^2)yx^{(3)})+2y\dot{x}(2(-3+p)pq\dot{y}^3\ddot{x}+\dot{y}\ddot{x}(-(-2+p)q+p(-12+7p)y\ddot{y})-2(-2+p)yx^{(3)} \\
&+2(-2+p)py\dot{y}^2x^{(3)})+\dot{x}^2(-2pq(-3+p+q)\dot{y}^4+q\dot{y}^2((-2+p)(p+q)+4p(-5+2p)y\ddot{y}) \\
&+y^2(6(-26+9p+2p^2)\ddot{x}^2+p(-4+3p)\ddot{y}^2)+2p(-5+3p)y^2\dot{y}y^{(3)})\Big) \\
&+2px^3y^2\Big(q\dot{x}^3(-2(-3+p(-1+q)+q)\dot{y}^3+(-9+2q+p(-3+2q))y\dot{y}\ddot{y}+(3+p)y^2y^{(3)}) \\
&+y\dot{x}^2(q(-17+6q+p(-5+4q))\dot{y}^2\ddot{x}+(17+3p)qy\dot{y}x^{(3)}+y((17+5p)q\ddot{x}\ddot{y}-(-11+p)yx^{(4)})) \\
&+y(p(-3+q)q\dot{y}^4\ddot{x}+2pqy\dot{y}^3x^{(3)}+y\dot{y}(-2(q-2py\ddot{y})x^{(3)}+5py\ddot{x}y^{(3)}) \\
&+\dot{y}^2(q\ddot{x}(2+p-q+10py\ddot{y})+py^2x^{(4)})-y(2q\ddot{x}\ddot{y}+y(2(-6+p)\ddot{x}^3-2p\ddot{x}\ddot{y}^2+x^{(4)}))) \\
&+\dot{x}(-2p(-2+q)q\dot{y}^5+q\dot{y}^3((1+p)(-2+q)+p(-13+5q)y\ddot{y})+8pqy^2\dot{y}^2y^{(3)}+y^2(-qy^{(3)} \\
&+y((47-7p)\ddot{x}x^{(3)}+3p\ddot{y}y^{(3)}))+y\dot{y}((3+2p-q)q\ddot{y}+qy(3(9+p)\ddot{x}^2+10p\ddot{y}^2)+3py^2y^{(4)}))\Big) \\
&+px^4y\Big(q^3+q(-12+11q-2q^2)\dot{y}^6+qy^2(1+(6+q)\dot{x}^2)\ddot{y}^2+q\dot{y}^4(12-6q+q^2+(12+q-2q^2)\dot{x}^2 \\
&+4(10-8q+q^2)y\ddot{y})+2qy\dot{y}^3((-12+q+q^2)\dot{x}\ddot{x}+(-9+5q)yy^{(3)})+q\dot{y}^2(-5q+2y(3(-2+q) \\
&+(-12-q+q^2)\dot{x}^2)\ddot{y}+y^2(-3(-4+q)\ddot{x}^2+(-34+20q)\ddot{y}^2-2(-7+q)\dot{x}x^{(3)})+8y^3y^{(4)}) \\
&-2qy^3(6\ddot{x}^2\ddot{y}-2\ddot{y}^3+7\dot{x}\ddot{y}x^{(3)}+6\dot{x}\ddot{x}y^{(3)}+\dot{x}^2y^{(4)})+y^4(-13(x^{(3)})^2+(y^{(3)})^2-18\ddot{x}x^{(4)}+2\ddot{y}y^{(4)}-6\dot{x}x^{(5)}) \\
&+2y^2\dot{y}(qy^{(3)}+4q\dot{x}^2y^{(3)}+y(-13q\ddot{x}x^{(3)}+12q\ddot{y}y^{(3)})-q\dot{x}((-18+q)\ddot{x}\ddot{y}+5yx^{(4)})+y^2y^{(5)})\Big) \\
&-2x^5\Big(q\dot{x}(3(8-6q+q^2)\dot{y}^5+(-60+39q-5q^2)y\dot{y}^3\ddot{y}+(20-11q+q^2)y^2\dot{y}^2y^{(3)}+y^2\dot{y}((30-15q+q^2)\ddot{y}^2 \\
&+(-5+2q)yy^{(4)})+y^3((-10+3q)\ddot{y}y^{(3)}+yy^{(5)}))+y(-3q(8-6q+q^2)\dot{y}^4\ddot{x}+q(14-9q+q^2)y\dot{y}^3x^{(3)} \\
&+qy\dot{y}^2((48-29q+3q^2)\ddot{x}\ddot{y}+3(-2+q)yx^{(4)})+qy^2\dot{y}(3(-7+3q)\ddot{y}x^{(3)}+(-16+7q)\ddot{x}y^{(3)} \\
&+3yx^{(5)})+y^2(4q\ddot{x}((-3+q)\ddot{y}^2+yy^{(4)})+y(7qx^{(3)}y^{(3)}+6q\ddot{y}x^{(4)}+yx^{(6)}))))\Big)\Big] \\
\tau_{3,y} \quad = \quad & \frac{1}{2x^5y^5}\Big[\Big(q\left(2q^3-5q^2-24q+48\right)\dot{y}^6+q\left(-q^3+6q^2+\left(2q^2-11q+12\right)\dot{x}^2q+4q-2\left(2q^3-8q^2-37q+84\right)y\ddot{y}-48\right)\dot{y}^4 \\
&-2qy\Big(q\left(q^2-7q+8\right)\dot{x}\ddot{x}+\left(3q^2+11q-34\right)yy^{(3)}\Big)\dot{y}^3-q\Big(-2(q-11)y^{(4)}y^3+\Big(q(3q-4)\ddot{x}^2+2\Big(3\left(2q^2+9q-26\right)\ddot{y}^2 \\
&+q(3q-5)\dot{x}x^{(3)}\Big)\Big)y^2+2\left(q^2+\left(q^2-9q+12\right)\dot{x}^2q+q-30\right)\ddot{y}y+5(q-4)q\Big)\dot{y}^2-2qy^2\Big(-3y^{(5)}y^2+\Big(3q\ddot{x}x^{(3)} \\
&+(47-7q)\ddot{y}y^{(3)}\Big)y+2(q-2)q\dot{x}^2y^{(3)}-2(q-2)y^{(3)}+q\dot{x}\left((7q-12)\ddot{x}\ddot{y}+3yx^{(4)}\right)\Big)\dot{y}-(q-4)q^3+3qy^2\Big(-(q-2)q\dot{x}^2 \\
&+q-2\Big)\ddot{y}^2-10q^2y\ddot{y}-2qy^3\left(-2(q-6)\ddot{y}^3+2q\ddot{x}^2\ddot{y}+5q\dot{x}x^{(3)}\ddot{y}+4q\dot{x}\ddot{x}y^{(3)}+\left(q\dot{x}^2-1\right)y^{(4)}\right)-qy^4\Big(\left(x^{(3)}\right)^2-13\left(y^{(3)}\right)^2 \\
&+2\left(\ddot{x}x^{(4)}-9\ddot{y}y^{(4)}+\dot{x}x^{(5)}\right)\Big)+2y^5y^{(6)}\Big)x^5+2py\Big(2q^2\left((q-2)\dot{y}^3-(q-3)y\dot{y}\ddot{y}-y^2y^{(3)}\right)\dot{x}^3-2qy\Big(q(2q-5)\ddot{x}\ddot{y}^2 \\
&+4qyx^{(3)}\dot{y}+y\left(5q\ddot{x}\ddot{y}+2yx^{(4)}\right)\Big)\dot{x}^2+\Big(q\left(2q^2+q-12\right)\dot{y}^5-q\left(q^2-3q+\left(5q^2+q-38\right)y\ddot{y}-6\right)\dot{y}^3-q(3q+17)y^2y^{(3)}\dot{y}^2 \\
&-q\left(-5y^{(4)}y^3+\left(10q\ddot{x}^2+3(q+9)\ddot{y}^2\right)y^2-(q-2)\ddot{y}y+5q\right)\dot{y}+y^2\left(3y^{(5)}y^2+\left(13q\ddot{y}y^{(3)}-12q\ddot{x}x^{(3)}\right)y+2qy^{(3)}\right)\Big)\dot{x} \\
&+y\Big(q\left(-q^2+q+6\right)\ddot{x}\dot{y}^4-q(q+3)yx^{(3)}\dot{y}^3+qy\left(yx^{(4)}-(5q+17)\ddot{x}\ddot{y}\right)\dot{y}^2+y\left((6y\ddot{y}q+q)\,x^{(3)}+y\left(7q\ddot{x}y^{(3)}+yx^{(5)}\right)\right)\dot{y} \\
&+y\left(\left(7x^{(3)}y^{(3)}+4\ddot{y}x^{(4)}+6\ddot{x}y^{(4)}\right)y^2-2q\ddot{x}\left(\ddot{x}^2-3\ddot{y}^2\right)y+2q\ddot{x}\ddot{y}\right)\Big)\Big)x^4-py^2\Big(-2q^2\left((p+q-3)\dot{y}^2-(p-3)y\ddot{y}\right)\dot{x}^4 \\
&+2qy\left((5p-13)q\dot{y}\ddot{x}+(5p-9)yx^{(3)}\right)\dot{x}^3+\Big(-2q(q+2)(p+q-3)\dot{y}^4+q\left((q-2)(p+q)+2(-5q+p(4q+6)-17)y\ddot{y}\right)\dot{y}^2 \\
&-2(p-7)qy^2y^{(3)}\dot{y}+y\left(-6(p-2)y^{(4)}y^2+q\left((20p-34)\ddot{x}^2-3(p-4)\ddot{y}^2\right)y+2q(-p+q+2)\ddot{y}\right)\Big)\dot{x}^2 \\
&+2y\Big(q(2p(q+1)-3(q+3))\ddot{x}\dot{y}^3+4qyx^{(3)}\dot{y}^2+\left((5-2p)x^{(4)}y^2+q\ddot{x}\,(-p+2q-(p-18)y\ddot{y}+3)\right)\dot{y} \\
&+y\left((q+(16-7p)y\ddot{y})\,x^{(3)}+3(7-3p)y\ddot{x}y^{(3)}\right)\Big)\dot{x}+y^2\ddot{x}\left(\ddot{x}\left((p+6)q\dot{y}^2+q-8(p-3)y\ddot{y}\right)+2(10-3p)y\dot{y}x^{(3)}\right)\Big)x^3 \\
&+2py^3\dot{x}\Big(2(p-2)q^2\dot{y}\dot{x}^4-2\left(p^2-8p+10\right)qy\ddot{x}\dot{x}^3+\Big(2q(qp+p-q-3)\dot{y}^3-q\left((p-2)(q+1)+\left(p^2+p-12\right)y\ddot{y}\right)\dot{y} \\
&+\left(p^2-9p+14\right)y^2y^{(3)}\Big)\dot{x}^2+y\Big(\ddot{x}\left(\left(-p^2+p+12\right)q\dot{y}^2-3(p-2)q+\left(3p^2-29p+48\right)y\ddot{y}\right)+\left(p^2-11p+20\right)y\dot{y}x^{(3)}\Big)\dot{x} \\
&+\left(p^2-15p+30\right)y^2\dot{y}\ddot{x}^2\Big)x^2-py^4\Big(\left(-2p^2+11p-12\right)q\dot{x}^6+\Big(\left(-2p^2+p+12\right)q\dot{y}^2+\left(p^2-6p+12\right)q \\
&+6\left(p^2-6p+8\right)y\ddot{y}\Big)\dot{x}^4+2\left(5p^2-39p+60\right)y\dot{y}\ddot{x}\dot{x}^3-5pq\dot{x}^2+p^2q\Big)x+6p\left(p^2-6p+8\right)y^5\dot{x}^5\dot{y}\Big]
\end{aligned}
\tag{3.4}
$$

In particular, a map φ_γ as in (3.1)*, which satisfies* (3.2)*, defines a* 3*-harmonic submanifold if and only if its profile curve $\gamma(s) = (x(s),\ y(s))$ satisfies the ODE's system*

$$\tau_{3,x} = 0,\ \ \tau_{3,y} = 0\,. \tag{3.5}$$

The case of rotation hypersurfaces can be deduced by Theorem 3.3. Indeed, considering the special case of surfaces into $\mathbb{R}^3$ for simplicity, we have:

Corollary 3.4. *Let*

$$\begin{array}{cccccccc} \varphi_{\mathrm{rot}} & : & M = & \mathbb{S}^1 & \times & I & \to & \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3 \\ & & & (w & , & s\,) & \longmapsto & (x(s)\,w,\, y(s)) \,, \end{array}$$

where $I \subset \mathbb{R}$ is an open interval, $x(s)$, $y(s)$ are smooth functions on I and $x(s) > 0$ on I. Assume that (3.2) *holds. Then φ_{rot} defines a 3-harmonic rotation surface in $\mathbb{R}^3$ if and only if* (3.5) *holds, where:*

$$\begin{aligned} \tau_{3,x} \;=\;& \Big[- x^4 \left(y^{(3)}\right)^2 - 2x^4 y^{(5)} \dot{y} - 2x^4 y^{(4)} \ddot{y} - 8x^3 x^{(3)} \dot{y}\ddot{y} + 3x^2 \ddot{x}^2 \left(\dot{y}^2 - 1\right) + 21\dot{x}^6 + 2x^5 x^{(6)} \\ & -2x^3 x^{(4)} \dot{y}^2 + 13x^4 \left(x^{(3)}\right)^2 + 2x^3 x^{(4)} - 20x^3 \ddot{x}^3 + 4x\dot{x}^3 \left(10xx^{(3)} - \dot{y}\ddot{y}\right) \\ & +\dot{x}^4 \left(-82x\ddot{x} + 3\dot{y}^2 - 39\right) + 2x\ddot{x} \left(-2x^2 \ddot{y}^2 - 5x^2 y^{(3)} \dot{y} + 9x^{(4)} x^3 - 5\right) \\ & +2x^2 \dot{x}\Big(3x \left(xx^{(5)} - y^{(3)} \ddot{y}\right) + 2x^{(3)} \dot{y}^2 + \dot{y} \left(5\ddot{x}\ddot{y} - 3xy^{(4)}\right) - 2x^{(3)} \left(20x\ddot{x} + 1\right) \Big) \\ & +\dot{x}^2 \left(x^2 \ddot{y}^2 - 8x\ddot{x} \left(\dot{y}^2 - 7\right) + 4x^2 y^{(3)} \dot{y} + 90x^2 \ddot{x}^2 - 20x^{(4)} x^3 + 15\right) + 3\Big] \frac{1}{2x^5} \\ \tau_{3,y} \;=\;& \Big[- 9x\dot{x}^4 \ddot{y} + 9\dot{x}^5 \dot{y} + 2x\dot{x}^3 \left(3xy^{(3)} - 13\ddot{x}\dot{y}\right) + x^2 \dot{x}^2 \left(10x^{(3)} \dot{y} + 22\ddot{x}\ddot{y} - 3xy^{(4)}\right) \\ & +x^3 \left(-8\ddot{x}^2 \ddot{y} + \ddot{x} \left(6xy^{(4)} - 7x^{(3)} \dot{y}\right) + x \left(x^{(5)} \dot{y} + 4x^{(4)} \ddot{y} + 7x^{(3)} y^{(3)} + xy^{(6)}\right)\right) \\ & +x^2 \dot{x}\Big((16\ddot{x}^2 - 3xx^{(4)})\dot{y} + 3x \left(-3x^{(3)} \ddot{y} - 4y^{(3)} \ddot{x} + xy^{(5)}\right) \Big)\Big] \frac{1}{x^5} \end{aligned} \tag{3.6}$$

Remark 3.5. Inspection of (3.4) and (3.6) shows that system (3.5) is always of the type

$$\begin{cases} x^{(6)} = F(x, \dot{x}, \ddot{x}, x^{(3)}, x^{(4)}, x^{(5)}, y, \dot{y}, \ddot{y}, y^{(3)}, y^{(4)}, y^{(5)}) \\ y^{(6)} = H(x, \dot{x}, \ddot{x}, x^{(3)}, x^{(4)}, x^{(5)}, y, \dot{y}, \ddot{y}, y^{(3)}, y^{(4)}, y^{(5)}) \end{cases} \tag{3.7}$$

for some suitable smooth functions F and H. It follows easily that the standard local existence and uniqueness theorem for ordinary differential equations can be applied and ensures the existence of local solutions of (3.5). Moreover, the initial conditions at some $s = s_0$ can easily be chosen in such a way that their associated tension field does not vanish, so that they would not correspond to harmonic maps. The problem is that, so doing, nothing guarantees that (3.2) holds and, consequently, these maps may well not be isometric immersions. The same applies to the maps of Corollary 3.4.

Proof of Theorem 3.3. We adapt to the 3-energy the method introduced in [33] (see [33], alternative proof of Proposition 4.3). The ideas behind this proof are conceptually not difficult. However, as a look at (3.4) may suggest, the involved computations are huge. Therefore, it is wise to carry them out by means of a suitable software (we used *Wolfram Mathematica*®). Here we describe the theoretical steps. First, we fix on the domain a metric

$$g_0 = x_0^2(s)\, g_{\mathbb{S}^p} + y_0^2(s)\, g_{\mathbb{S}^q} + ds^2 \ ,$$

where $x_0(s)$, $y_0(s)$ are smooth positive functions which verify (3.2). Also, we set $V_0(s) = x_0^p(s)\, y_0^q(s)$. Now, the first idea is to find the system of equations which defines when a map of the type (3.1) (without assuming (3.2)) is 3-harmonic with respect to the metric g_0. Then, setting $x_0(s) = x(s)$, $y_0(s) = y(s)$ into this system of equations, we shall obtain (3.5). Now, the energy of a map of the type (3.1) with respect to g_0 is given by

$$E(\varphi_\gamma) = \omega_0 \int_I L(s,\, x,\, \dot{x},\, y,\, \dot{y})\, ds \,,$$

where $\omega_0 = \mathrm{Vol}(\mathbb{S}^p)\,\mathrm{Vol}(\mathbb{S}^q)$ and the Lagrangian L is the product of the energy density and the volume term as follows:

$$L = e(\varphi_\gamma)\; V_0(s)\,, \quad \text{where}$$

$$e(\varphi_\gamma) \;=\; \frac{1}{2}\left[\dot{x}^2(s)+\dot{y}^2(s)\;+\;p\,\frac{x^2(s)}{x_0^2(s)}\;+\;q\,\frac{y^2(s)}{y_0^2(s)}\right].$$

Now, as in [33], the tension field $\tau(\varphi_\gamma)$ is the horizontal lift of

$$\tau_x\,\frac{\partial}{\partial x}\;+\;\tau_y\,\frac{\partial}{\partial y}\,, \quad \text{where}$$

$$\begin{aligned}
\tau_x &= \frac{1}{V_0}\left[\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}\right]\\
\tau_y &= \frac{1}{V_0}\left[\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}\right]
\end{aligned}$$

We proceed in a similar fashion for the 3-energy. Indeed, computing in local coordinates with respect to g_0, we obtain:

$$E_3(\varphi_\gamma) = \omega_0 \int_I L_3\left(s,x,\dot{x},\ddot{x},x^{(3)},y,\dot{y},\ddot{y},y^{(3)}\right) ds\,, \quad \text{where}$$

$$L_3 = \frac{1}{2}\left|d\tau(\varphi_\gamma)\right|^2\; V_0(s) = \frac{1}{2}\left[\dot{\tau}_x^2+\dot{\tau}_y^2+\;p\,\frac{\tau_x^2}{x_0^2}\;+\;q\,\frac{\tau_y^2}{y_0^2}\right] V_0(s)\,.$$

We are in the framework of Proposition 2.8: it follows that the 3-tension field $\tau_3(\varphi_\gamma)$ with respect to g_0 is a G-equivariant section. Therefore, arguing as in [33], we deduce that $\tau_3(\varphi_\gamma)$ is the horizontal lift of

$$\tau_{3,x}\,\frac{\partial}{\partial x}\;+\;\tau_{3,y}\,\frac{\partial}{\partial y}\,, \quad \text{where}$$

$$\begin{aligned}
\tau_{3,x} &= \frac{1}{V_0}\left[\frac{d^3}{ds^3}\left(\frac{\partial L}{\partial x^{(3)}}\right)-\frac{d^2}{ds^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)+\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}\right] \qquad (3.8)\\
\tau_{3,y} &= \frac{1}{V_0}\left[\frac{d^3}{ds^3}\left(\frac{\partial L}{\partial y^{(3)}}\right)-\frac{d^2}{ds^2}\left(\frac{\partial L}{\partial \ddot{y}}\right)+\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}\right].
\end{aligned}$$

Finally, we compute (3.8) explicitly with the software *Mathematica*. Then, setting $x_0(s) = x(s)$, $y_0(s) = y(s)$ and simplifying with *Mathematica* we obtain (3.4). □

Proof of Corollary 3.4. As explained in [30], the calculation in this case can be performed simply by setting $p = 1, q = 0$ in (3.4) and dropping the requirement $y(s) > 0$. Then simplification using the software *Mathematica* leads us to (3.6). □

Remark 3.6. In Remark 3.5 we pointed out that the standard local existence theory for ODE's does not guarantee that local solutions are isometric immersions. It is natural to try to understand whether this problem depends on the method or it represents a real difficulty. To answer this question, we dropped the assumption $\dot{x_0}^2+\dot{y_0}^2 = 1$ and started our process with a fixed metric now given by

$$g_0 = x_0^2(s)\,g_{\mathbb{S}^p}+y_0^2(s)\,g_{\mathbb{S}^q}+\left(\dot{x_0}^2(s)+\dot{y_0}^2(s)\right)ds^2\,.$$

After the suitable adjustments in the computations, we found that a map of the type (3.1), which does not necessarily satisfies (3.2), is a 3-harmonic isometric immersion provided that the profile curve $\gamma(s) = (x(s), y(s))$ satisfies a system of the following type:

$$\begin{cases} a_{11}\, x^{(6)} + a_{12}\, y^{(6)} = F_1(x, \dot{x}, \ddot{x}, x^{(3)}, x^{(4)}, x^{(5)}, y, \dot{y}, \ddot{y}, y^{(3)}, y^{(4)}, y^{(5)}) \\ a_{21}\, x^{(6)} + a_{22}\, y^{(6)} = F_2(x, \dot{x}, \ddot{x}, x^{(3)}, x^{(4)}, x^{(5)}, y, \dot{y}, \ddot{y}, y^{(3)}, y^{(4)}, y^{(5)}), \end{cases} \tag{3.9}$$

where F_1, F_2 are smooth functions and the coefficient matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -\frac{\dot{y}^2}{(\dot{x}^2+\dot{y}^2)^4} & \frac{\dot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^4} \\ \frac{\dot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^4} & -\frac{\dot{x}^2}{(\dot{x}^2+\dot{y}^2)^4} \end{bmatrix}.$$

Since $\det A = 0$, system (3.9) cannot be written in the form (3.7) and therefore the standard local existence theorem for ODE's cannot be directly applied. This fact substantiates the claim that it is not easy to overcome the difficulties illustrated in Remark 3.5.

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Conflict of interest

The author declares that there is no conflicts of interest in this paper.

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