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Research article

Existence results for hybrid fractional differential equations with three-point boundary conditions

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Abstract: We investigate the existence and uniqueness of solutions of problems of three point boundary values of hybrid fractional differential equations with a fractional derivative of Caputo of order $\alpha \in [1, 2]$, the results are obtained drawing on the standard fixed point theorems. The results are illustrated by a some examples.

Keywords: fractional differential equations; boundary value problems; hybrid; existence results; fixed point theorem
Mathematics Subject Classification: 34A08, 34A12, 34B15

1. Introduction

The domain of fractional calculus is interested with the generalization of the classical integer order differentiation and integration to an arbitrary order. Fractional calculus has found important applications in different fields of science, especially in problems related to biology, chemistry, mathematical physics, economics, control theory, blood flow phenomena and aerodynamics, etc (see, for example, [1–15]. For some recent developments on the existence and uniqueness of solutions for differential equations nvolving the fractional derivatives, for more information, we advise reading these papers [16–24] and the references therein. In this literature, we show some contributions of researchers to the finding of the existence and uniqueness of the solution for the different fractional differential equations. Bai [16] studied the existence and uniqueness of positive solutions for the following three-point fractional boundary value problem:

$$\begin{cases} {}^cD^q_{0^+}x(t) = f(t, x(t)), \quad t \in (0, 1), \; q \in (1, 2], \\ x(0) = 0, \\ x(1) = \beta x(\eta), \quad \eta \in (0, 1), \end{cases} \tag{1.1}$$

where D^q denotes the Riemann-Liouville fractional derivative, and $0 < \beta\eta^{q-1} < 1$.

Ahmad et al. in [17] discussed the existence and uniqueness of solutions for the following boundary value problem of fractional order differential equations with three-point integral boundary conditions:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in (0,1), \ q \in (1,2], \\ x(0) = 0, \\ x(1) = \alpha \int_0^\eta x(s)ds, \quad \eta \in (0,1), \end{cases} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, and $\alpha \in \mathbb{R}$, $\frac{2}{\eta} \neq \alpha$.

In [18], the authors discussed the existence and uniqueness of solutions for the following nonlinear fractional differential equations with three-point fractional integral boundary conditions:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in (0,1), \ q \in (1,2], \\ x(0) = 0, \\ x(1) = \alpha I^p x(\eta), \quad \eta \in (0,1), \end{cases} \tag{1.3}$$

where ${}^cD$ denotes the Caputo fractional derivative of order q, I^p is the Riemann-Liouville fractional integral of order p and $\alpha \in \mathbb{R}$, $\alpha \neq \frac{\Gamma(p+2)}{\eta^{p+1}}$.

In [20] existence and uniqueness results are obtained for the following boundary value problem of fractional order differential equations with three-point fractional integral boundary conditions:

$$\begin{cases} {}^cD^\alpha(x(t)) = f(t, x(t), {}^cD^\beta x(t)), \quad t \in [0,1], \ \alpha \in (1,2] \text{ and } \beta \in (0,1), \\ x(0) = 0, \\ bx(1) = c - aI^\gamma x(\eta), \quad \eta \in (0,1), \end{cases} \tag{1.4}$$

where ${}^cD$ denotes the Caputo fractional derivative of order q, I^γ the Riemann-Liouville fractional integral of order γ, f is a given continuous function, and a, b, c are real constants with $a\eta^{1+\gamma} \neq -b\Gamma(\beta + 2)$. We are concerned to study the hybrid fractional differential equations (also called the quadratic perturbations of nonlinear differential equations) because they have been extensively studied and have achieved a great deal of interest. First time, Dhage and Lakshmikantham in [26] proposed hybrid differential equations and showed some essential results on this kind of differential equations. In this class of the equations, the perturbations of the original differential equations are involved in many ways, for hybrid fractional differential equations, we refer to [25–35] and references therein.

In this paper, we discuss existence and uniqueness results for hybrid fractional differential equations with three-point boundary hybrid conditions, these results are determined, by applying fixed point theorems such as Banach's fixed point theorem and Leray-Schauder Nonlinear Alternative. Our assumed problem will more complicated and general than the problems considered before and aforementioned above, we study the existence and uniqueness of solutions for the hybrid fractional differential equations given by with boundary hybrid conditions where $t \in [0,1]$, γ and $q_i \in (0,1], i = 1, ..., m$, with $m \in \mathbb{N}$, $\eta \in (0,1)$, and $\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b \neq 0$.

$${}^cD^\alpha_{0^+}(x(t) - \sum_{i=1}^m I^{q_i}_{0^+} h_i(t, x(t), {}^cD^\beta_{0^+} x(t), I^q_{0^+} x(t))) = f(t, x(t), {}^cD^\beta_{0^+} x(t), I^q_{0^+} x(t)) \quad \alpha \in (1,2], \beta \text{ and } q \in (0,1), \tag{1.5}$$

$$\begin{cases} [x(t) - \sum_{i=1}^{m} I_{0^+}^{q_i} h_i(t, x(t), {}^c D_{0^+}^{\beta} x(t), I_{0^+}^{q} x(t))]_{t=0} = 0, \\ a I_{0^+}^{\gamma} [x(t) - \sum_{i=1}^{m} I_{0^+}^{q_i} h_i(t, x(t), {}^c D_{0^+}^{\beta} x(t), I_{0^+}^{q} x(t))]_{t=\eta} + b[x(t) - \sum_{i=1}^{m} I_{0^+}^{q_i} h_i(t, x(t), {}^c D_{0^+}^{\beta} x(t), I_{0^+}^{q} x(t))]_{t=1} = c, \end{cases} \tag{1.6}$$

${}^c D_{0^+}^{\alpha}$ denotes the Caputo fractional derivative of order α and $I_{0^+}^{q}$ denotes Riemann-Liouville fractional integral of order q, and a, b, c are real constants with $f, h_i \in C([0, 1] \times \mathbb{R}^3, \mathbb{R})$.

This article is structured as follows. In part 2, we introduce notations, definitions, and lemmas. Next, in part 3, we prove the existence results for problems 1.5 and 1.6 using the fixed point theorem. Finally, we illustrate the results with examples.

2. Preliminaries

In this section, we present the notation, definitions to be used throughout this article.

Definition 2.1. *[25] The Riemann-Liouville fractional integral of order q for 0 continuous function $f : [0, +\infty) \longrightarrow \mathbb{R}$ is defined as*

$$I_{0^+}^{q} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{\alpha - 1} f(s) ds,$$

where Γ is the Euler gamma function.

Definition 2.2. *[34] Let $\alpha > 0$ and $n = [\alpha] + 1$. If $f \in C^n([0, 1])$, then the Caputo fractional derivative of order α defined by*

$${}^c D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n - \alpha - 1} f^{(n)}(s) ds,$$

exists almost everywhere on $[0, 1]$, $[\alpha]$ is the integer part of α.

Lemma 2.3. *[4] Let $\alpha > \beta > 0$ and $f \in L^1([0, 1])$. Then for all $t \in [0, 1]$, we have:*

$I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t),$

${}^c D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t),$

${}^c D_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha-\beta} f(t).$

Lemma 2.4. *Let $\alpha > 0$. Then the differential equation*

$$({}^c D_{0^+}^{\alpha} f)(t) = 0,$$

has a solution

$$f(t) = \sum_{j=0}^{m-1} c_i t^i, \quad c_i \in \mathbb{R}, \ j = 0, ..., m - 1,$$

where m–$1 < \alpha < m$.

Theorem 2.5. *[19] Let X be a Banach space, let B be a closed, convex subset of X, let U be an open subset of B and $0 \in U$. Suppose that*
$P : U \longrightarrow B$ is a continuous and compact map. Then either
(a) P has a fixed point in U, or
(b) there exist an $x \in \partial U$ (the boundary of U) and $\lambda \in (0,1)$ with $x = \lambda P(x)$.

3. Main results

In this section, we show the existence results for the boundary value problems on the interval $[0,1]$.

Lemma 3.1. *Let $A(t)$ be continuous function on $[0,1]$. Then the solution of the boundary value problem*

$$\begin{cases} {}^cD^{\alpha}_{0^+}(x(t) - \sum\limits_{i=1}^{m} I^{q_i}_{0^+}h_i(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^{q}_{0^+}x(t))) = A(t), \quad \alpha \in (1,2], \beta \, and \, q \in (0,1), \\ [x(t) - \sum\limits_{i=1}^{m} I^{q_i}_{0^+}h_i(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^{q}_{0^+}x(t))]_{t=0} = 0, \\ aI^{\gamma}_{0^+}[x(t) - \sum\limits_{i=1}^{m} I^{q_i}_{0^+}h_i(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^{q}_{0^+}x(t))]_{t=\eta} + b[x(t) - \sum\limits_{i=1}^{m} I^{q_i}_{0^+}h_i(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^{q}_{0^+}x(t))]_{t=1} = c, \end{cases} \tag{3.1}$$

where $t \in [0,1]$, γ, η and $q_i \in (0,1)$, $i = 1, ..., m$, $m \in \mathbb{N}$,
is given by

$$x(t) = I^{\alpha}_{0^+}A(t) + \frac{t(c - bI^{\alpha}_{0^+}A(1) - aI^{\gamma+\alpha}_{0^+}A(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} + \sum_{i=1}^{m} I^{q_i}_{0^+}h_i(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^{q}_{0^+}x(t)). \tag{3.2}$$

Proof. For $1 < \alpha \leq 2$ and some constants $c_0, c_1 \in \mathbb{R}$, the general solution of the equation

$${}^cD^{\alpha}_{0^+}(x(t) - \sum_{i=1}^{m} I^{q_i}_{0^+}h_i(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^{q}_{0^+}x(t))) = A(t),$$

can be written as

$$x(t) = I^{\alpha}_{0^+}A(t) + c_0 + c_1 t + \sum_{i=1}^{m} I^{q_i}_{0^+}h_i(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^{q}_{0^+}x(t)), \tag{3.3}$$

applying the boundary conditions, we find that

$$c_0 = 0,$$

and

$$c_1 = \frac{c - bI^{\alpha}_{0^+}A(1) - aI^{\alpha+\gamma}_{0^+}A(\eta)}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}. \tag{3.4}$$

Substituting the values of c_0, c_1, we obtain the result, this completes the proof. □

Now we list the following hypotheses
(H_1) The functions $f,\ h_i : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ are continuous.
(H_2) There exist positive functions $\phi_i,\ i=1,...m$ with bounds $\|\phi_i\|_{\frac{1}{\tau_i}}$, such that

$$|h_i(t,x_1,y_1,z_1)-h_i(t,x_2,y_2,z_2)| \le \phi_i(t)(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|),$$

for $t\in[0,1]$, $(x_k,y_k,z_k)\in\mathbb{R}^3$, $k=1,\ 2$ and $\phi_i(t)\in L^{\frac{1}{\tau_i}}([0,1],\mathbb{R}^+)$ and $\tau_i\in(0,1-\alpha),\ i=1,..,m.$
(H_3) There exist positive function ψ with bounds $\|\psi\|_{\frac{1}{\mu}}$, such that

$$|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)| \le \psi(t)(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|),$$

for $t\in[0,1]$, $(x_k,y_k,z_k)\in\mathbb{R}^3$, $k=1,\ 2$ and $\psi(t)\in L^{\frac{1}{\mu}}([0,1],\mathbb{R}^+)$ and $\mu\in(0,1-\alpha)$.
(H_4) If $\Delta+\Lambda+\Theta<1$, where Δ, Λ and Θ are given by

$$\Delta = \frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}\left(1+\frac{|b|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}\right)+\frac{|a|\|\psi\|_{\frac{1}{\mu}}\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}+\sum_{i=1}^{m}\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q_i-\tau_i})^{1-\tau_i}}{\Gamma(q_i)}.$$

$$\Lambda = \frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)}+\frac{\|\psi\|_{\frac{1}{\mu}}}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}\left(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}+\frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|}\right)$$
$$+\sum_{i=1}^{m}\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q+q_i-\tau_i})^{1-\tau_i}}{\Gamma(q+q_i)}.$$

$$\Theta = \frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\beta-\mu})^{1-\mu}}{\Gamma(\alpha-\beta)}+\frac{\|\psi\|_{\frac{1}{\mu}}}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}\left(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}+\frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|}\right)$$
$$+\sum_{i=1}^{m}\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{\beta-q_i-\tau_i})^{1-\tau_i}}{\Gamma(\beta-q_i)}.$$

Theorem 3.2. *Assume that condition (H_1-H_4) hold, then problems (1.5) and (1.6) have a unique solution defined on* $[0,1]$

Proof. Define the space

$$X=\left\{x : x,\ I^q_{0^+}x \text{ and } {}^cD^\beta_{0^+}x\in C([0,1],\mathbb{R}),\ 0<q,\beta<1\right\},$$

endowed with the norm

$$\|x\| = max_{t\in[0,1]}|x(t)|+max_{t\in[0,1]}|I^q_{0^+}x(t)|+max_{t\in[0,1]}|{}^cD^\beta_{0^+}x(t)|.$$

We put
$Fx(t) = f(t, x(t), ^c D^{\beta}_{0^+}x(t), I^q_{0^+}x(t))$.
$H_i x(t) = h_i(t, x(t), ^c D^{\beta}_{0^+}x(t), I^q_{0^+}x(t)),\ i = 1, ..., m. \qquad m \in \mathbb{N}$.
Obviously, $(X, \|x\|)$ is Banach espace. In order to obtain the existence results of problems (1.5) and (1.6), by Lemma 3.1, we define an operator $S : X \longrightarrow X$ as follows

$$Sx(t) = I^{\alpha}_{0^+}Fx(t) + \frac{t(c - b(I^{\alpha}_{0^+}F)(1) - a(I^{\gamma+\alpha}_{0^+}Fx)(\eta))}{\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} + \sum_{i=0}^{m} I^{q_i}_{0^+}H_i x(t).$$

Since f, h_i continuous, it is easy to see that

$$(I^q_{0^+}Sx)(t) = I^{\alpha+q}_{0^+}F(t) + (\frac{t^q(c - b(I^{\alpha}_{0^+}F)(1) - a(I^{\gamma+\alpha}_{0^+}Fx)(\eta))}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b)}) + \sum_{i=0}^{m} I^{q+q_i}_{0^+}H_i x(t),$$

and

$$(^c D^{\beta}_{0^+}Sx)(t) = (I^{\alpha-\beta}_{0^+}Fx)(t) + (\frac{t^{1-\beta}(c - b(I^{\alpha}_{0^+}F)(1) - a(I^{\gamma+\alpha}_{0^+}Fx)(\eta))}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b)}) + (\sum_{i=0}^{m} I^{q_i-\beta}_{0^+}H_i x)(t).$$

Let $x, y \in X$. Then for each $t \in [0, 1]$, we have

$$\begin{aligned} |(Sx)(t) - (Sy)(t)| &\leq |I^{\alpha}_{0^+}(Fx - Fy)(t)| + \frac{|b||I^{\alpha}_{0^+}(Fx - Fy)(1) + |a||I^{\gamma+\alpha}_{0^+}(Fx - Fy)(\eta)}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\ &\quad + \sum_{i=0}^{m} I^{q_i-\beta}_{0^+}(H_i x - H_i y)(t) \\ &\leq I^{\alpha}_{0^+}(\psi\|x - y\|)(t) + \frac{|b|I^{\alpha}_{0^+}(\psi_i\|x - y\|)(1) + |a|I^{\gamma+\alpha}_{0^+}(\psi_i\|x - y\|)(\eta)}{|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\ &\quad + \sum_{i=0}^{m} I^{q_i-\beta}_{0^+}\phi_i\|x - y\|, \end{aligned}$$

by the Holder inequality, we have

$$\begin{aligned} |(Sx)(t) - (Sy)(t)| &\leq \frac{\|\psi\|_{\frac{1}{\mu}}(\dfrac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}(1 + \frac{|b|}{|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|})\|x - y\| + \frac{|a|\|\psi\|_{\frac{1}{\mu}}\eta^{\alpha+\gamma-\mu}(\dfrac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\ &\times \|x - y\| + \sum_{i=1}^{m} \frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\dfrac{1-\tau_i}{q_i-\tau_i})^{1-\tau_i}}{\Gamma(q_i)}\|x - y\| \\ &= \Delta\|x - y\|, \end{aligned}$$

similary, we have

$$
\begin{aligned}
|I^q_{0^+}(Sx)(t) - I^q_{0^+}(Sy)(t)| &\leq \{\frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)} + \frac{\|\psi\|_{\frac{1}{\mu}}}{\Gamma(q+1)(|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|)}(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)} \\
&\quad + \frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)}) + \sum_{i=1}^{m}\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q+q_i-\tau_i})^{1-\tau_i}}{\Gamma(q+q_i)}\}\|x-y\| \\
&= \Lambda\|x-y\|,
\end{aligned}
$$

and

$$
\begin{aligned}
|^cD^\beta_{0^+}(Sx)(t) -^c D^\beta_{0^+}(Sy)(t)| &\leq \{\frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\beta-\mu})^{1-\mu}}{\Gamma(\alpha-\beta)} + \frac{\|\psi\|_{\frac{1}{\mu}}}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)} \\
&\quad + \frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|}) + \sum_{i=1}^{m}\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{\beta-q_i-\tau_i})^{1-\tau_i}}{\Gamma(\beta-q_i)}\}\|x-y\| \\
&= \Theta\|x-y\|.
\end{aligned}
$$

Form the inequalities above, we can deduce that

$$\|Sx(t) - Sy(t)\| \leq (\Theta + \Delta + \Lambda)\|x-y\|.$$

By the contraction principale, we know that problem 3.1 has a unique solution. □

Theorem 3.3. *assume that*
(1) *We put* $H^i_0 = \sup_{t\in[0,1]} h_i(t,0,0,0), i = 1,...,m, \qquad m \in \mathbb{N}.$
(2) *There exist tree non-decreasing functions* $\rho_1,\rho_2,\rho_3 : [0,\infty) \to [0,\infty)$ *and a function* $\psi \in L^{\frac{1}{\mu}}([0,1],\mathbb{R}^+)$ *with* $\mu \in (0,\alpha-1)$

$$|f(t,x,y,z)| \leq \psi(t)(\rho_1(|x|) + \rho_2(|y|) + \rho_3(|z|)),$$

for $t \in [0,1]$ *and* $(x,y,z) \in \mathbb{R}^3$.
(3) *There exists a constant* $Z > 0$ *such that*

$$\frac{Z}{W_1(Z) + \|\psi\|_{\frac{1}{\mu}}W_2(\rho_1(Z)+\rho_2(Z)+\rho_3(Z))} \geq 1. \tag{3.5}$$

Where

$$W_1(Z) = (\frac{|c|(\Gamma(q+1)\Gamma(2-\beta)+\Gamma(2-\beta)+\Gamma(q+1))}{\Gamma(q+1)\Gamma(2-\beta)(|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|)}) + \sum_{i=1}^{m}(H^i_0 + Z\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q_i-\tau_i})^{1-\tau_i}}{\Gamma(q_i)})$$

$$+\sum_{i=1}^{m}(H_0^i+Z\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q+q_i-\tau_i})^{1-\tau_i}}{\Gamma(q+q_i)})+\sum_{i=1}^{m}(H_0^i+Z\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q_i-\beta-\tau_i})^{1-\tau_i}}{\Gamma(q_i-\beta)}),$$

$$\begin{aligned}W_2 &= (\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)})(1+\frac{|b|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}+\frac{|b|}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}+\frac{|b|}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)})\\&+(\frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|})(1+\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(2-\beta)})+\frac{(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)}+\frac{(\frac{1-\mu}{\alpha-\beta-\mu})^{1-\mu}}{\Gamma(\alpha-\beta)}.\end{aligned}$$

Then problem (1.5) and (1.6) has at least one solution on $[0,1]$.

Proof. Define the a ball B_r as

$$B_r=\{x\in X:\|x\|\leq r\},$$

where the constant r satisfies

$$r\geqslant W_1(r)+\|\psi\|_{\frac{1}{\mu}}W_2(\rho_1(r)+\rho_2(r)+\rho_3(r)).$$

Clearly, B_r is a closed convex bounded subset of the Banach space X. By Lemma 3.1 the boundary value problems (1.5) and (1.6) are equivalent to the equation

$$Sx(t) = I_{0^+}^{\alpha}Fx(t)+\frac{t(c-bI_{0^+}^{\alpha}Fx(1)-aI_{0^+}^{\gamma+\alpha}Fx(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b}+\sum_{i=1}^{m}I_{0^+}^{q_i}H_ix(t), \tag{3.6}$$

$$|Sx(t)| \leq |I_{0^+}^{\alpha}Fx(t)|+\frac{(|c|+|b|I_{0^+}^{\alpha}|Fx(1)|+|a|I_{0^+}^{\gamma+\alpha}|Fx(\eta)|)}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}+\sum_{i=1}^{m}|I_{0^+}^{q_i}H_ix(t)|, \tag{3.7}$$

by the Holder inequality and the hypotheses, we have

$$\begin{aligned}|Sx(t)| &\leq (\frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)})(\rho_1(r)+\rho_2(r)+\rho_3(r))(1+\frac{|b|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|})+\sum_{i=1}^{m}(H_0^i+r\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q_i-\tau_i})^{1-\tau_i}}{\Gamma(q_i)})\\&+(\frac{|c|\Gamma(\alpha+\gamma)+(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}),\end{aligned}$$

similary, we have

$$|(I_{0^+}^q Sx)(t)| \le (\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}(\frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)}) + (\frac{|c|}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)})$$
$$+(\frac{|b|(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}) + \sum_{i=1}^{m}(H_0^i + r\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q+q_i-\tau_i})^{1-\tau_i}}{\Gamma(q+q_i)})$$
$$+(\frac{(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)\Gamma(q+1)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}),$$

and

$$|{}^cD_{0^+}^{\beta} Sx(t)| \le (\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}(\frac{(\frac{1-\mu}{\alpha-\beta-\mu})^{1-\mu}}{\Gamma(\alpha-\beta)}) + (\frac{|c|}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)})$$
$$+(\frac{|b|(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}) \sum_{i=1}^{m}(H_0^i + r\frac{\|\phi_i\|_{\frac{1}{\tau_i}}(\frac{1-\tau_i}{q_i-\beta-\tau_i})^{1-\tau_i}}{\Gamma(q_i-\beta)})$$
$$+(\frac{(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}).$$

That is to say, we have

$$\|Sx(t)\| \le W_1(r) + \|\psi\|_{\frac{1}{\mu}} W_2(\rho_1(r)+\rho_2(r)+\rho_3(r)). \tag{3.8}$$

Secondly, we prove that S maps bounded sets into equicontinuous sets. Let $B_{r'}$ be any bounded set of X. Notice that f and h_i are continuous, therefore, without loss of generality, we can assume that there is an M_f and M_{h_i}, $i = 1, ..., m$, such that

$$\sup_{t\in[0,1]} f(t, x(t), {}^cD_{0^+}^{\alpha}x(t), I_{0^+}^q x(t)) = M_f,$$

and

$$\sup_{t\in[0,1]} h_i(t, x(t), {}^cD_{0^+}^{\alpha}x(t), I_{0^+}^q x(t)) = M_{h_i}.$$

Now let $0 \leq t_1 \leq t_2 \leq 1$.We have the following facts:

$$
\begin{aligned}
|Sx(t_2) - Sx(t_1)| &= |I_{0^+}^{\alpha}Fx(t_2) + \frac{t_2(c - b(I_{0^+}^{\alpha}F)(1) - a(I_{0^+}^{\gamma+\alpha}Fx)(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} + \sum_{i=0}^{m} I_{0^+}^{q_i}H_i x(t_2) - I_{0^+}^{\alpha}Fx(t_1) \\
&\quad - \frac{t_1(c - b(I_{0^+}^{\alpha}F)(1) - a(I_{0^+}^{\gamma+\alpha}Fx)(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} - \sum_{i=0}^{m} I_{0^+}^{q_i}H_i x(t_1)| \\
&\leq \int_0^{t_1} \frac{(t_2 - s)^{\alpha+1} - (t_1 - s)^{\alpha+1}}{\Gamma(\alpha)} f(s, x(s),^c D_{0^+}^{\alpha}x(s), I_{0^+}^{q}x(s))ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha+1}}{\Gamma(\alpha)} f(s, x(s),^c D_{0^+}^{\alpha}x(s), I_{0^+}^{q}x(s)ds| \\
&\quad +|t_2 - t_1|(\frac{(c - b(I_{0^+}^{\alpha}F)(1) - a(I_{0^+}^{\gamma+\alpha}Fx)(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}) \\
&\quad + \sum_{i=0}^{m}(\int_0^{t_1} \frac{(t_2 - s)^{\alpha+1} - (t_1 - s)^{q_i+1}}{\Gamma(q_i)} h_i(s, x(s),^c D_{0^+}^{\beta}x(s), I_{0^+}^{q}x(s))ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{q_i+1}}{\Gamma(q_i)} h_i(s, x(s),^c D_{0^+}^{\beta}x(s), I_{0^+}^{q}x(s))) \\
&\leq \frac{M_f(t_2 - t_1)^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_f(t_1^{\alpha} - t_2^{\alpha}) + (t_2 - t_1)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \sum_{i=0}^{m}(\frac{M_{h_i}(t_2 - t_1)^{q_i}}{\Gamma(q_i+1)} + \frac{M_{h_i}(t_1^{q_i} - t_2^{q_i}) + (t_2 - t_1)^{q_i}}{\Gamma(q_i+1)}) \\
&\leq \frac{2M_f(t_2 - t_1)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=0}^{m} \frac{2M_{h_i}(t_2 - t_1)^{q_i}}{\Gamma(q_i+1)},
\end{aligned}
$$

we can get

$$|Sx(t_1) - Sx(t_2)| \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.$$

Similarly, we can obtain that

$$
\begin{aligned}
|I_{0^+}^{q}Sx(t_1) - I_{0^+}^{q}Sx(t_2)| \longrightarrow 0 \,\text{as}\, t_2 \longrightarrow t_1, \\
|^c D_{0^+}^{\alpha}Sx(t_1) -^c D_{0^+}^{\alpha}Sx(t_2)| \longrightarrow 0 \,\text{as}\, t_2 \longrightarrow t_1.
\end{aligned}
$$

This implies that

$$||Sx(t_1) - Sx(t_2)|| \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.$$

Finally, we let $x = \lambda S x$ for $\lambda \in (0, 1)$. Due to (3.8) and for each $t \in [0, 1]$ we have

$$\|x\| = \|\lambda S x\| \le W_1(\|x\|) + \|\psi\|_{\frac{1}{\mu}} W_2(\rho_1(\|x\|) + \rho_2(\|x\|) + \rho_3(\|x\|)).$$

That is to say,

$$\frac{\|x\|}{W_1(\|x\|) + \|\psi\|_{\frac{1}{\mu}} W_2(\rho_1(\|x\|) + \rho_2(\|x\|) + \rho_3(\|x\|))} \le 1.$$

From (3.5), there exists $Z > 0$ such that $x \ne Z$. Define a set

$$O = \{y \in X : \|y\| \le Z\}.$$

The operator $S : \bar{O} \longrightarrow X$ is continuous and completely continuous.
By the definition of the set O there is no $x \in \partial O$ such that $x = \lambda S x$ for some $0 < \lambda < 1$.
Consequently, by Theorem 2.5, we obtain that S has a fixed point $x \in O$ which is a solution of problems (1.5) and (1.6). This is the end of the proof. □

4. Examples

In this section, we give two examples to illustrate the main results.

4.1. Example 1

Consider the following fractional differential equation

$$\begin{cases} {}^cD_{0^+}^{\frac{4}{3}}(x(t) - \sum\limits_{i=1}^{3} I_{0^+}^{q_i} H_i x(t)) = \dfrac{e^{\sin(\pi(1+t^2)}}{e^1(4+t)^2}\dfrac{|x|}{1+|x|} + \dfrac{1}{(4+\sin^2(t))^2}(|{}^cD_{0^+}^{\frac{3}{4}}x(t)| + |I_{0^+}^{\frac{7}{8}}x(t)|), \\ [x(t) - \sum\limits_{i=1}^{3} I_{0^+}^{q_i} H(t)]_{t=0} = 0, \\ \frac{1}{2} I_{0^+}^{\frac{1}{2}}[x(t) - \sum\limits_{i=1}^{3} I_{0^+}^{q_i} H_i x(t)]_{t=\frac{1}{2}} + \frac{1}{3}[x(t) - \sum\limits_{i=1}^{3} I_{0^+}^{q_i} H_i x(t)]_{t=1} = \frac{1}{4}. \end{cases} \tag{4.1}$$

We take

$$q_1 = \frac{90}{91}, \quad q_2 = \frac{9}{10}, \quad q_3 = \frac{29}{30}, \quad \mu = \frac{1}{4}$$
$$\tau_1 = \frac{6}{7}, \quad \tau_2 = \frac{2}{3}, \quad \tau_3 = \frac{1}{2}$$

$$\begin{aligned} f(t, x(t), {}^cD_{0^+}^{\frac{3}{4}}x(t), I_{0^+}^{\frac{7}{8}}x(t)) &= \frac{\exp(\sin(\pi(1+t^2))}{e^1(4+t)^2}\frac{|x|}{1+|x|} + \frac{1}{(4+\sin^2(t))^2}(|{}^cD_{0^+}^{\frac{3}{4}}x(t)| + |I_{0^+}^{\frac{7}{8}}x(t)|), \\ h_1(t, x(t), {}^cD_{0^+}^{\frac{3}{4}}x(t), I_{0^+}^{\frac{7}{8}}x(t)) &= \frac{t^2}{10}\Big(\frac{x(t) + {}^cD_{0^+}^{\frac{3}{4}}x(t)}{x(t) + {}^cD_{0^+}^{\frac{3}{4}}x(t) + 1} + |I_{0^+}^{\frac{7}{8}}x(t)|\Big) + \exp(-t^2), \\ h_2(t, x(t), {}^cD_{0^+}^{\frac{3}{4}}x(t), I_{0^+}^{\frac{7}{8}}x(t)) &= \frac{e^{\sqrt{2}t}\sin(\pi t^2)}{(\sqrt{3} + \sqrt{5}e^{\sqrt{2}t})^2}\Big(\cos\Big(\frac{x(t) + {}^cD_{0^+}^{\frac{3}{4}}x(t)}{x(t) + {}^cD_{0^+}^{\frac{3}{4}}x(t) + 1}\Big) + \frac{e^{\sqrt{2}t}\cos(\pi t)}{(\sqrt{3} + \sqrt{5}e^{\sqrt{2}t})^2}|I_{0^+}^{\frac{7}{8}}x(t)|\Big), \end{aligned}$$

$$h_3(t,x(t),{}^cD^{\frac{3}{4}}_{0^+}x(t),I^{\frac{7}{8}}_{0^+}x(t)) = \frac{\exp(-2t)}{10}(\log(x(t)+1)+\cos{}^cD^{\frac{3}{4}}_{0^+}x(t)+\sin(I^{\frac{7}{8}}_{0^+}x(t))).$$

We can show that

$$\begin{aligned}
|F(x(t))-F(y(t))| &\le \frac{1}{16}(|x-y|+|{}^cD^{\frac{3}{4}}_{0^+}x-{}^cD^{\frac{3}{4}}_{0^+}y|+|I^{\frac{7}{8}}_{0^+}x-{}^cD^{\frac{7}{8}}_{0^+}y|),\\
|H_1(x(t))-H_1(y(t))| &\le \frac{t^2}{10}(|x-y|+|{}^cD^{\frac{3}{4}}_{0^+}x-{}^cD^{\frac{3}{4}}_{0^+}y|+|I^{\frac{7}{8}}_{0^+}x-{}^cD^{\frac{7}{8}}_{0^+}y|),\\
|H_2(x(t))-H_2(y(t))| &\le \frac{\exp(\sqrt{2}t)}{(\sqrt{3}+\sqrt{5}\exp(\sqrt{2}t))^2}(|x-y|+|{}^cD^{\frac{3}{4}}_{0^+}x-{}^cD^{\frac{3}{4}}_{0^+}y|+|I^{\frac{7}{8}}_{0^+}x-I^{\frac{7}{8}}_{0^+}y|),\\
|H_3(x(t))-H_3(y(t))| &\le \frac{\exp(-2t)}{10}(|x-y|+|{}^cD^{\frac{3}{4}}_{0^+}x-{}^cD^{\frac{3}{4}}_{0^+}y|+|I^{\frac{7}{8}}_{0^+}x-I^{\frac{7}{8}}_{0^+}y|),
\end{aligned}$$

where

$$\psi=\tfrac{1}{16},\qquad \phi_1=\frac{t^2}{10},\qquad \phi_2=\frac{\exp(\sqrt{2}t)}{(\sqrt{3}+\sqrt{5}\exp(\sqrt{2}t))^2},\qquad \phi_3=\frac{\exp(-2t)}{10}.$$

Then, we have

$$\|\psi\|_{\frac{1}{\mu}}\approx 0.0423,\qquad \|\phi_1\|_{\tau_1}\approx 0.0356,\qquad \|\phi_2\|_{\tau_2}\approx 0.05114,\qquad \|\phi_3\|_{\tau_3}\approx 0.0495,$$

and

$$\Delta\approx 0.2109,\qquad \Lambda\approx 0.1528,\qquad \Theta\approx 0.2276,$$

and

$$\Delta+\Lambda+\Theta\approx 0.5914<1.$$

By Theorem 3.2, we know that proplem 4.1 has a unique solution defined on $[0,1]$.

4.2. Example 2

Consider the following fractional differential equation

$$\begin{cases}
{}^cD^{\frac{5}{3}}_{0^+}(x(t)-\sum\limits_{i=1}^{2}I^{q_i}_{0^+}H_ix(t))=\dfrac{e^{-4t}}{10}\sin(x(t)+\frac{1}{2}I^{\frac{8}{9}}_{0^+}x(t))+\dfrac{e^{-4t}}{4}D^{\frac{1}{50}}_{0^+}x(t),\\
[x(t)-\sum\limits_{i=1}^{2}I^{q_i}_{0^+}H(t)]_{t=0}=0,\\
\frac{1}{10}I^{\frac{1}{3}}_{0^+}[x(t)-\sum\limits_{i=1}^{2}I^{q_i}_{0^+}H_ix(t)]_{t=\frac{1}{4}}+\frac{1}{50}[x(t)-\sum\limits_{i=1}^{2}I^{q_i}_{0^+}H_ix(t)]_{t=1}=\dfrac{1}{2}.
\end{cases}\tag{4.2}$$

We choose

$$\mu=\frac{2}{3},\qquad q_1=\frac{100}{101},\qquad q_2=\frac{11}{12},\qquad \tau_1=\frac{1}{10},\qquad \tau_2=\frac{1}{20}.$$

$$f(t,x(t),{}^cD^{\frac{1}{50}}x(t),I^{\frac{8}{9}}_{0^+}x(t))=\frac{e^{-2t}}{10}\sin(\tfrac{1}{20}x(t)+\tfrac{1}{30}I^{\frac{8}{9}}_{0^+}x(t))+\tfrac{e^{-4t}}{60}{}^cD^{\frac{1}{50}}_{0^+}x(t),$$

$$h_1(t,x(t),{}^cD^{\frac{1}{50}}x(t),I^{\frac{8}{9}}_{0^+}x(t))=\frac{e^{-t^2}}{(5+t)^2}\frac{|x|}{1+|x|}+\frac{1}{16}I^{\frac{8}{9}}_{0^+}x+\frac{{}^cD^{\frac{1}{50}}x}{(4+\sin^2(x))^2}+\frac{t}{10},$$

$$h_2(t,x(t),^c D^{\frac{1}{50}}x(t), I_{0^+}^{\frac{8}{9}}x(t)) = \frac{t}{2}(\frac{sin(x(t)+^c D_{0^+}^{\frac{1}{50}}x(t))}{x(t)+^c D_{0^+}^{\frac{1}{50}}x(t)+1} + \sin(I_{0^+}^{\frac{8}{9}}x(t)) + \frac{1}{30}).$$

We can demonstrate that

$$|f(t,x(t),^c D^{\frac{1}{50}}x(t), I_{0^+}^{\frac{8}{9}}x(t))| \leq \psi(t)(\rho_1(|x|)+\rho_2(|^c D^{\frac{1}{50}}x|)+\rho_3(|I_{0^+}^{\frac{8}{9}}x|)),$$

$$h_1(t,x(t),^c D^{\frac{1}{50}}x(t), I_{0^+}^{\frac{8}{9}}x(t)) - h_1(t,y(t),^c D^{\frac{1}{50}}y(t), I_{0^+}^{\frac{8}{9}}y(t)) \leq \tfrac{1}{16}(|x-y| + |^c D^{\frac{1}{50}}x -^c D^{\frac{1}{50}}y| + |I_{0^+}^{\frac{8}{9}}x - I_{0^+}^{\frac{8}{9}}y|),$$

$$h_2(t,x(t),^c D^{\frac{1}{50}}x(t), I_{0^+}^{\frac{8}{9}}x(t)) - h_1(t,y,^c D^{\frac{1}{50}}y, I_{0^+}^{\frac{8}{9}}y) \leq \tfrac{t}{2}(|x-y| + |^c D^{\frac{1}{50}}x -^c D^{\frac{1}{50}}y| + |I_{0^+}^{\frac{8}{9}}x - I_{0^+}^{\frac{8}{9}}y|),$$

where

$$\psi(t) = \frac{e^{-2t}}{10}, \qquad \phi_1 = \frac{1}{16}, \qquad \phi_2 = \frac{t}{2}.$$

$$\rho_1(|x|) = \frac{1}{20}|x|, \qquad \rho_2(|^c D^{\frac{1}{50}}x|) = \frac{1}{30}|^c D^{\frac{1}{50}}x|, \qquad \rho_3(|I_{0^+}^{\frac{8}{9}}x|) = \frac{1}{60}|I_{0^+}^{\frac{8}{9}}x|.$$

Hence we have

$$\|\phi_1\|_{\tau_1} \approx 0.0625, \qquad \|\phi_2\|_{\tau_2} \approx 0.2714, \qquad H_0^1 = \frac{1}{10}, \qquad H_0^1 = \frac{1}{60}.$$

After calculation, it ensues by 3.5 that the constant Z provides the inequality $Z > 31,9308$. Since all the stipulations of theorem 3.3 are completed, the problem 4.2 has at least one solution on $[0,1]$.

5. Conclusions

In this article, we have presented some sufficient conditions include the existence of solutions for a kind of hybrid fractional differential equations inclose fractional Caputo derivative of order $1 < \alpha \leq 2$. Our results depend on a fixed point theorems such as Banach's fixed point theorem and Leray-Schauder nonlinear alternative. Our results prolong and complete those in the literature.

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Conflict of interest

All authors declare no conflicts of interest.

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