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Research article

Subalgebras of type (α,β) based on m-polar fuzzy points in BCK/BCI-algebras

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Abstract: In this article, we present the idea of quasi-coincidence of an m-polar fuzzy point with an m-polar fuzzy subset. By utilizing this new idea, we further introduce the notion of m-polar (α,β)-fuzzy subalgebras in BCK/BCI-algebras which is a generalization of the idea of (α,β)-bipolar fuzzy subalgebras in BCK/BCI-algebras. Some interesting results of the BCK/BCI-algebras in terms of m-polar (α,β)-fuzzy subalgebras are given. By using m-polar $(\in,\in\vee q)$-fuzzy subalgebras, some interesting results are obtained. Conditions for an m-polar fuzzy set to be an m-polar $(q,\in\vee q)$-fuzzy subalgebra and an m-polar $(\in,\in\vee q)$-fuzzy subalgebra are provided. Characterizations of m-polar $(\in,\in\vee q)$-fuzzy subalgebras in BCK/BCI-algebras by using level cut subsets are explored.

Keywords: BCK/BCI-algebra; m-polar fuzzy subalgebra; m-polar (α,β)-fuzzy subalgebra; m-polar $(\in,\in\vee q)$-fuzzy subalgebra
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1. Introduction

BCK-algebras, one of the oldest branches of general algebras, first appeared in mathematical sciences in 1966 by Imai and Iséki [1], which are applied to several areas, such as topology, group theory, semigroups, graphs and functional analysis, etc. Such algebras generalize Boolean D-poset (MV-algebras) as well as Boolean rings. In the same year, as an extension of BCK-algebras, Iséki [2] introduced the idea of BCI-algebras. BCK/BCI-algebras are established from two distinct approaches: Propositional calculi and set theory. Several results and properties of BCK/BCI-algebras are discussed in the works [3, 4].

Fuzzy set theory, initially established by Zadeh [5] in 1965, was applied by several researchers to generalize some of the essential ideas of algebraic structures. Fuzzy algebraic structures play a prominent role in different domains in mathematics and other sciences such as theoretical physics, topological spaces, real analysis, coding theory, set theory, logic, information sciences and the like. In 1994, bipolar fuzzy ($\mathscr{BF}$, for short) sets were developed by Zhang [6] and is a more platform that extends the crisp (classical) sets and fuzzy sets. Hybrid models of fuzzy sets have been implemented in several algebraic structures, such as hemirings [7], BRK-algebras [8] and BCK/BCI-algebras [9–16]. In many real life problems, multi-polar information plays a fundamental role in distinct areas of the science, such as neurobiology and technology. Data sometimes comes from m components ($m \geq 2$), for example consider the following sentence "Harvard University Is a Good University". The degree of membership of this sentence may not be a real number in the standard interval $[0, 1]$. In fact, Harvard University is a good university in several components: good in ranking, location, facilities and education, etc. Any component may be a real number in the interval $[0, 1]$. If we have m components, then the degree of membership of the fuzzy sentence is an element of $[0, 1]^m$, that is, a m-tuple of real number in $[0, 1]$.

Based on these observations, Chen et al. [17] broadened the theory of $\mathscr{BF}$ sets to get the idea of m-polar fuzzy (m-$\mathscr{PF}$, for short) sets in 2014, and proved that $\mathscr{BF}$ sets and 2-$\mathscr{PF}$ sets are cryptomorphic mathematical notions. In m-$\mathscr{PF}$ sets, the grades of membership functions $\widehat{\mathcal{W}}$ are extended from the unit interval [0,1] into the cubic $[0, 1]^m$. Recently, m-$\mathscr{PF}$ set theory was applied to some algebraic structures such as lie algebras [18, 19] and groups [20]. In BCK/BCI-algebras, the notion of m-$\mathscr{PF}$ subalgebras was first initiated in 2018 by Al-Masarwah and Ahmad [21]. After that, in [22] they studied the normalizations of m-$\mathscr{PF}$ subalgebras in BCK/BCI-algebras. Various applications of m-$\mathscr{PF}$ sets and other hybrid models of fuzzy sets in the real life-issues in the field of decision making problems are studied in [23, 24].

The framework of the fuzzy subgroup, initially proposed by Rosenfeld [25] in 1971, is a fundamental concept of fuzzy algebras. The notion of "belongingness" of a fuzzy point with a fuzzy set was given by Murali [26]. Besides, the concept of "quasi-coincidence" of a fuzzy point with a fuzzy set [27], played a fundamental role to construct distinct innovative types of fuzzy subgroups. In the literature, Bhakat and Das [28] first generalized the notion of fuzzy subgroups to (α,β)-fuzzy subgroups. They proposed the idea of $(\in, \in \vee q)$-fuzzy subgroups as a special case of (α,β)-fuzzy subgroups. In this aspect, Dudek et al. [29] and Narayanan et al. [30] extended these results to near-rings and hemirings. In BCK/BCIalgebras, Xi [31] introduced the idea of fuzzy subalgebras in 1991. Jun [32] presented the study of (α,β)-fuzzy subalgebras as a generalization of fuzzy subalgebras. Further, Muhiuddin and Al-Roqi discussed more results of this concept in [33]. Jana et al. [34] presented an $(\in, \in \vee q)$-intuitionistic fuzzy subalgebra in BCI-algebras. Also, Jana et al. [35] established the idea of $(\in, \in \vee q)$-$\mathscr{BF}$ subalgebras in BCK/BCI-algebras. This concept is a fundamental and useful generalization of Lee's [36] $\mathscr{BF}$ subalgebras.

Inspired by the previous studies and by using m-$\mathscr{PF}$ sets and m-$\mathscr{PF}$ points, we present a new idea called m-polar (α,β)-fuzzy subalgebras in BCK/BCI-algebras and we establish some interesting characterization results. In particular, we introduce the concept of m-polar $(\in, \in \vee q)$-fuzzy subalgebras and we give some related theorems. We provide conditions for an m-$\mathscr{PF}$ set to be an m-polar $(q, \in \vee q)$-fuzzy subalgebra and an m-polar $(\in, \in \vee q)$-fuzzy subalgebra. We explore the characterizations of m-polar $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras by using level cut subsets. To show the novelty of this model, some contributions of different authors toward generalized m-$\mathscr{PF}$ subalgebras in BCK/BCI-algebras are analyzed in Table 1.

Table 1. Contributions toward generalized m-$\mathscr{PF}$ subalgebras.

Authors	Year	Contributions
Rosenfeld [25]	1971	Introduction of fuzzy subgroups.
Bhakat and Das [28]	1996	Generalization of fuzzy subgroups.
Xi [31]	1991	Introduction of fuzzy subalgebras.
Jun [32]	2005	Generalization of fuzzy subalgebras.
Lee [36]	2009	Introduction of $\mathscr{BF}$ subalgebras.
Jana et al. [35]	2017	Generalization of $\mathscr{BF}$ subalgebras.
Al-Masarwah and Ahmad [21]	2018	Introduction of m-$\mathscr{PF}$ subalgebras.
Al-Masarwah and Ahmad	This paper	Generalization of m-$\mathscr{PF}$ subalgebras.

2. Preliminaries

In the current section, we recall the basic concepts of BCK/BCI-algebras which will be very helpful in further study of the paper.

A structure $(\mathcal{P}, *)$ is called a BCI-algebra if $\mathcal{P}$ contains a constant 0 and satisfies the following conditions: For all $h, k, l \in \mathcal{P}$,

(I) $(h * (h * k)) * k = 0$,

(II) $((h * k) * (h * l)) * (l * k) = 0$,

(III) $h * h = 0$,

(IV) $h * k = 0$ and $k * h = 0$ imply $h = k$.

If a BCI-algebra $(\mathcal{P}, *)$ satisfies $0 * h = 0$, then $\mathcal{P}$ is said to be a BCK-algebra. In any BCK/BCI-algebra $(\mathcal{P}, *)$, the following valid: For all $h, k, l \in \mathcal{P}$,

(1) $h * 0 = h$,

(2) $(h * k) * l = (h * l) * k$,

(3) $h * k \leq h$,

(4) $(h * k) * l \leq (h * l) * (k * l)$,

(5) $h \leq k \Rightarrow h * l \leq k * l, l * k \leq l * h$,

where $h \leq k \Leftrightarrow h * k = 0$.

A subset $C \neq \phi$ of a BCK/BCI-algebra $(\mathcal{P}, *)$ is a subalgebra of $(\mathcal{P}, *)$ if $h * k \in C$, $\forall h, k \in C$.

Here we mentioned some of the related definitions and results which are directly used in our work. For details we refer the researcher to the works [3, 4, 37] for more information regarding BCK/BCI-algebras. From now on, let $\mathcal{P}$ denote a BCK/BCI-algebra unless otherwise specified.

Definition 2.1. [17] A function $\widehat{\mathcal{W}} : \mathcal{P} \to [0, 1]^m$ is defined from $\mathcal{P}(\neq \phi)$ to an m-tuple of real number in $[0, 1]$, is called an m-$\mathscr{PF}$ set. The membership degree of each element $h \in \mathcal{P}$ is denoted by

$$\widehat{\mathcal{W}}(h) = (p_1 \circ \mathcal{W}(h), p_2 \circ \mathcal{W}(h), ..., p_m \circ \mathcal{W}(h))$$

where $p_j \circ \mathcal{W} : [0,1]^m \to [0,1]$ is the j-th projection mapping. The smallest and largest values in $[0,1]^m$ are $\widehat{0} = (0,0,...,0)$ and $\widehat{1} = (1,1,...,1)$, respectively.

Definition 2.2. [21] An m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$ is called an m-$\mathscr{PF}$ subalgebra of $\mathcal{P}$ if for all $h, k \in \mathcal{P}$,

$$\widehat{\mathcal{W}}(h * k) \geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k)\},$$

i.e.,

$$p_j \circ \mathcal{W}(h * k) \geq \inf\{p_j \circ \mathcal{W}(h), p_j \circ \mathcal{W}(k)\}$$

for all $j = 1, 2, ..., m$.

3. m-polar (α,β)-fuzzy subalgebras

In the current section, we propose the concept of m-polar (α,β)-fuzzy subalgebras, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}, \alpha \neq \in \wedge q$, and study some related properties.

An m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$ having the form

$$\widehat{\mathcal{W}}(k) = \begin{cases} \widehat{\eta} = (\eta_1, \eta_2, ..., \eta_m) \in (0,1]^m, & \text{if } k = h \\ \widehat{0} = (0,0,...,0), & \text{if } k \neq h \end{cases}$$

is called an m-$\mathscr{PF}$ point with support h and value $\widehat{\eta} = (\eta_1, \eta_2, ..., \eta_m)$ and is denoted by $h_{\widehat{\eta}}$.

For an m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$, we say that

(1) $h_{\widehat{\eta}}$ is belong to $\widehat{\mathcal{W}}$, denoted by $h_{\widehat{\eta}} \in \widehat{\mathcal{W}}$, if $\widehat{\mathcal{W}}(h) \geq \widehat{\eta}$ i.e., $p_j \circ \mathcal{W}(h) \geq \eta_j$ for each $j = 1, 2, ..., m$.

(2) $h_{\widehat{\eta}}$ is quasi-coincident with $\widehat{\mathcal{W}}$, denoted by $h_{\widehat{\eta}} q \widehat{\mathcal{W}}$, if $\widehat{\mathcal{W}}(h) + \widehat{\eta} > \widehat{1}$ i.e., $p_j \circ \mathcal{W}(h) + \eta_j > 1$ for each $j = 1, 2, ..., m$.

(3) $h_{\widehat{\eta}}$ is belong to $\widehat{\mathcal{W}}$ or $h_{\widehat{\eta}}$ is quasi-coincident with $\widehat{\mathcal{W}}$, denoted by $h_{\widehat{\eta}} \in \vee q \widehat{\mathcal{W}}$, if $h_{\widehat{\eta}} \in \widehat{\mathcal{W}}$ or $h_{\widehat{\eta}} q \widehat{\mathcal{W}}$.

(4) $h_{\widehat{\eta}}$ is belong to $\widehat{\mathcal{W}}$ and $h_{\widehat{\eta}}$ is quasi-coincident with $\widehat{\mathcal{W}}$, denoted by $h_{\widehat{\eta}} \in \wedge q \widehat{\mathcal{W}}$, if $h_{\widehat{\eta}} \in \widehat{\mathcal{W}}$ and $h_{\widehat{\eta}} q \widehat{\mathcal{W}}$.

(5) $h_{\widehat{\eta}} \overline{\alpha} \widehat{\mathcal{W}}$ if $h_{\widehat{\eta}} \alpha \widehat{\mathcal{W}}$ does not hold.

If C is a nonempty subset of $\mathcal{P}$, then the m-polar characteristic function of C denoted and defined by

$$\widehat{\chi}_C(h) = \begin{cases} \widehat{1}, & \text{if } h \in C \\ \widehat{0}, & \text{if } h \notin C. \end{cases}$$

Clearly, $\widehat{\chi}_C$ is an m-$\mathscr{PF}$ subset of $\mathcal{P}$.

Definition 3.1. An m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$ is called an m-polar (α,β)-fuzzy subalgebra of $\mathcal{P}$ if it satisfies the following condition:

$$h_{\widehat{\eta}} \alpha \widehat{\mathcal{W}}, k_{\widehat{\zeta}} \alpha \widehat{\mathcal{W}} \Rightarrow (h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} \beta \widehat{\mathcal{W}}$$

for all $\widehat{\eta}, \widehat{\zeta} \in (0,1]^m$ and $h, k \in \mathcal{P}$, where $\alpha \neq \in \wedge q$.

In Definition 3.1 $\alpha \neq \in \wedge q$. To explain this, let $\widehat{\mathcal{W}}$ be an m-$\mathscr{PF}$ set of $\mathcal{P}$ such that $\widehat{\mathcal{W}}(h) \leq \widehat{0.5}$ $\forall h \in \mathcal{P}$. Let $h \in \mathcal{P}$ be such that $h_{\widehat{\eta}} \in \wedge q\widehat{\mathcal{W}}$ for $\widehat{\eta} \in (0,1]^m$. Then, $h_{\widehat{\eta}} \in \widehat{\mathcal{W}}$ and $h_{\widehat{\eta}} q\widehat{\mathcal{W}}$ i.e., $\widehat{\mathcal{W}}(h) \geq \widehat{\eta}$ and $\widehat{\mathcal{W}}(h) + \widehat{\eta} > \widehat{1}$. It implies that $\widehat{1} < \widehat{\mathcal{W}}(h) + \widehat{\eta} \leq \widehat{\mathcal{W}}(h) + \widehat{\mathcal{W}}(h) = 2\widehat{\mathcal{W}}(h)$, so $\widehat{\mathcal{W}}(h) > \widehat{0.5}$. This means that $\{h_{\widehat{\eta}} \mid h_{\widehat{\eta}} \in q\widehat{\mathcal{W}}\} = \phi$.

Example 3.1. Consider a *BCI*-algebra $\mathcal{P} = \{0, k, l, n\}$ with the operation $*$ which is given in Table 2:

Table 2. Tabular representation of the binary operation $*$.

$*$	0	k	l	n
0	0	k	l	n
k	k	0	n	l
l	l	n	0	k
n	n	l	k	0

Let $\widehat{\mathcal{W}} : \mathcal{P} \to [0,1]^3$ be a 3-$\mathscr{PF}$ set defined by:

$$\widehat{\mathcal{W}}(h) = \begin{cases} (0.8, 0.7, 0.6), & \text{if } h = 0 \\ (0.9, 0.8, 0.7), & \text{if } h = k \\ (0.5, 0.4, 0.3), & \text{if } h = l, n. \end{cases}$$

Then, $\widehat{\mathcal{W}}$ is a 3-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$.

For any m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$. Consider the set $\widehat{\mathcal{W}}_{\widehat{0}} = \{h \in \mathcal{P} \mid \widehat{\mathcal{W}}(h) > \widehat{0}\}$.

Theorem 3.2. *If $\widehat{\mathcal{W}}$ is a non-zero m-polar $(\in, \beta)$-fuzzy subalgebra of $\mathcal{P}$, then $\widehat{\mathcal{W}}_{\widehat{0}}$ is a subalgebra of $\mathcal{P}$, where $\beta \in \{\in, q\}$.*

Proof. Let $\widehat{\mathcal{W}}$ be a non-zero m-polar $(\in, \beta)$-fuzzy subalgebra of $\mathcal{P}$ and $h, k \in \mathcal{P}$. We consider the following:
(1) For $(\beta = \in)$. Let $h, k \in \widehat{\mathcal{W}}_{\widehat{0}}$. Then, $\widehat{\mathcal{W}}(h) > \widehat{0}$ and $\widehat{\mathcal{W}}(k) > \widehat{0}$. Note that $h_{\widehat{\mathcal{W}}(h)} \in \widehat{\mathcal{W}}$ and $k_{\widehat{\mathcal{W}}(k)} \in \widehat{\mathcal{W}}$. If $\widehat{\mathcal{W}}(h * k) = \widehat{0}$, then $\widehat{\mathcal{W}}(h * k) = \widehat{0} < \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k)\}$. Thus, $(h * k)_{\inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k)\}} \overline{\in} \widehat{\mathcal{W}}$, a contradiction. So $\widehat{\mathcal{W}}(h * k) > \widehat{0}$, i.e., $h * k \in \widehat{\mathcal{W}}_{\widehat{0}}$.

(2) For $(\beta = q)$. Let $h, k \in \widehat{\mathcal{W}}_{\widehat{0}}$. Then, $\widehat{\mathcal{W}}(h) > \widehat{0}$ and $\widehat{\mathcal{W}}(k) > \widehat{0}$. If $\widehat{\mathcal{W}}(h * k) = \widehat{0}$, then $\widehat{\mathcal{W}}(h * k) + \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k)\} = \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k)\} \leq \widehat{1}$. Thus, $(h * k)_{\inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k)\}} \overline{q} \widehat{\mathcal{W}}$, a contradiction. So $\widehat{\mathcal{W}}(h * k) > \widehat{0}$, i.e., $h * k \in \widehat{\mathcal{W}}_{\widehat{0}}$. Hence, in any case, we have $\widehat{\mathcal{W}}_{\widehat{0}}$ is a subalgebra of $\mathcal{P}$. □

Theorem 3.3. *If $\widehat{\mathcal{W}}$ is a non-zero m-polar $(q, \in)$-fuzzy subalgebra of $\mathcal{P}$, then $\widehat{\mathcal{W}}_{\widehat{0}}$ is a subalgebra of $\mathcal{P}$.*

Proof. Let $h, k \in \widehat{\mathcal{W}}_{\widehat{0}}$ for $h, k \in \mathcal{P}$. Then, $\widehat{\mathcal{W}}(h) > \widehat{0}$ and $\widehat{\mathcal{W}}(k) > \widehat{0}$. It follows $x_{\widehat{1}} q\widehat{\mathcal{W}}$ and $y_{\widehat{1}} q\widehat{\mathcal{W}}$. Since $\widehat{\mathcal{W}}$ is an m-polar $(q, \in)$-fuzzy subalgebra of $\mathcal{P}$, we have $(h * k)_{\widehat{1}} \in \widehat{\mathcal{W}}$. If $\widehat{\mathcal{W}}(h * k) = \widehat{0} < \widehat{1}$, then $(h * k)_{\widehat{1}} \overline{\in} \widehat{\mathcal{W}}$, a contradiction. So $\widehat{\mathcal{W}}(h * k) > \widehat{0}$, i.e., $h * k \in \widehat{\mathcal{W}}_{\widehat{0}}$ for $h, k \in \mathcal{P}$. Hence, $\widehat{\mathcal{W}}_{\widehat{0}}$ is a subalgebra of $\mathcal{P}$. □

Theorem 3.4. *If $\widehat{\mathcal{W}}$ is a non-zero m-polar (q, q)-fuzzy subalgebra of $\mathcal{P}$, then $\widehat{\mathcal{W}}_{\widehat{0}}$ is a subalgebra of $\mathcal{P}$.*

Proof. Let $h, k \in \widehat{\mathcal{W}}_{\widehat{0}}$ for $h, k \in \mathcal{P}$. Then, $\widehat{\mathcal{W}}(h) > \widehat{0}$ and $\widehat{\mathcal{W}}(k) > \widehat{0}$. Thus, $\widehat{\mathcal{W}}(h) + \widehat{1} > \widehat{1}$ and $\widehat{\mathcal{W}}(k) + \widehat{1} > \widehat{1}$. It follows $x_{\widehat{1}} q \widehat{\mathcal{W}}$ and $y_{\widehat{1}} q \widehat{\mathcal{W}}$. If $\widehat{\mathcal{W}}(h * k) = \widehat{0}$, then $\widehat{\mathcal{W}}(h * k) + \widehat{1} = \widehat{0} + \widehat{1} = \widehat{1}$, and so $(h * k)_{\widehat{1}} \overline{q} \widehat{\mathcal{W}}$. This is impossible, and hence $\widehat{\mathcal{W}}(h * k) > \widehat{0}$ i.e., $h * k \in \widehat{\mathcal{W}}_{\widehat{0}}$ for $h, k \in \mathcal{P}$. Thus, $\widehat{\mathcal{W}}_{\widehat{0}}$ is a subalgebra of $\mathcal{P}$. □

In the following theorem, we give characterizations of an m-polar $(\in, \in \vee q)$-fuzzy subalgebra.

Theorem 3.5. *For an m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$, the conditions (J) and (H) are equivalent, where*

$$(J) \quad h_{\widehat{\eta}} \in \widehat{\mathcal{W}}, k_{\widehat{\zeta}} \in \widehat{\mathcal{W}} \Rightarrow (h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} \in \vee q \widehat{\mathcal{W}},$$
$$(H) \quad \widehat{\mathcal{W}}(h * k) \geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}$$

for all $h, k \in \mathcal{P}$ and $\widehat{\eta}, \widehat{\zeta} \in (0, 1]^m$.

Proof. (J) ⇒ (H). Assume that (H) does not valid, i.e., there exist $h, k \in \mathcal{P}$ such that $\widehat{\mathcal{W}}(h * k) < \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}$. Then, $\widehat{\mathcal{W}}(h * k) < \widehat{\eta} \leq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}$ for some $\widehat{\eta} \in (0, 1]^m$. Thus, $h_{\widehat{\eta}} \in \widehat{\mathcal{W}}$ and $y_{\widehat{\eta}} \in \widehat{\mathcal{W}}$, but $(h * k)_{\widehat{\eta}} \overline{\in \vee q} \widehat{\mathcal{W}}$, a contradiction. Thus, $\widehat{\mathcal{W}}(h * k) \geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}$ for all $h, k \in \mathcal{P}$.

(H) ⇒ (J). Let $h_{\widehat{\eta}} \in \widehat{\mathcal{W}}, k_{\widehat{\zeta}} \in \widehat{\mathcal{W}}$. Then, $\widehat{\mathcal{W}}(h) \geq \widehat{\eta}$ and $\widehat{\mathcal{W}}(k) \geq \widehat{\zeta}$. If $(h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} \in \widehat{\mathcal{W}}$, then (J) holds. If $(h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} \overline{\in} \widehat{\mathcal{W}}$, then $\widehat{\mathcal{W}}(h * k) < \inf\{\widehat{\eta}, \widehat{\zeta}\}$. Since

$$\begin{aligned} \widehat{\mathcal{W}}(h * k) &\geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\} \\ &\geq \inf\{\widehat{\eta}, \widehat{\zeta}, \widehat{0.5}\}. \end{aligned}$$

It implies that $\widehat{\mathcal{W}}(h * k) \geq \widehat{0.5}$ and $\inf\{\widehat{\eta}, \widehat{\zeta}\} > \widehat{0.5}$. Hence, $\widehat{\mathcal{W}}(h * k) + \inf\{\widehat{\eta}, \widehat{\zeta}\} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$, implies $(h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} q \widehat{\mathcal{W}}$. Thus, $(h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} \in \vee q \widehat{\mathcal{W}}$. □

Corollary 3.6. *An m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$ if it satisfies the condition (H).*

Theorem 3.7. *An m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$ if and only if $\widehat{\mathcal{W}}_{\widehat{\eta}} = \{h \in \mathcal{P} \mid \widehat{\mathcal{W}}(h) \geq \widehat{\eta}\}$ is a subalgebra of $\mathcal{P}$ for all $\widehat{\eta} \in (0, 0.5]^m$.*

Proof. Let $\widehat{\mathcal{W}}$ be an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$ and let $h, k \in \widehat{\mathcal{W}}_{\widehat{\eta}}$ for $\widehat{\eta} \in (0, 0.5]^m$. Then,

$$\widehat{\mathcal{W}}(h) \geq \widehat{\eta} \text{ and } \widehat{\mathcal{W}}(k) \geq \widehat{\eta}.$$

Thus, we have

$$\begin{aligned} \widehat{\mathcal{W}}(h * k) &\geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\} \\ &\geq \inf\{\widehat{\eta}, \widehat{\eta}, \widehat{0.5}\} \\ &= \inf\{\widehat{\eta}, \widehat{0.5}\} \\ &= \widehat{\eta}, \end{aligned}$$

that is, $\widehat{\mathcal{W}}(h * k) \geq \widehat{\eta}$, which implies, $h * k \in \widehat{\mathcal{W}}_{\widehat{\eta}}$. Hence, $\widehat{\mathcal{W}}_{\widehat{\eta}}$ is a subalgebra of $\mathcal{P}$.

Conversely, assume that $\widehat{\mathcal{W}}_{\widehat{\eta}}$ is a subalgebra of $\mathcal{P}$ for all $\widehat{\eta} \in (0, 0.5]^m$. Suppose $h, k \in \mathcal{P}$ such that

$$\widehat{\mathcal{W}}(h * k) < \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}.$$

Select $\widehat{\psi} \in (0, 0.5]^m$ such that

$$\widehat{\mathcal{W}}(h * k) < \widehat{\psi} \leq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}.$$

Then, $x_{\widehat{\psi}} \in \widehat{\mathcal{W}}$, $y_{\widehat{\psi}} \in \widehat{\mathcal{W}}$, but $(h * k)_{\widehat{\psi}} \overline{\in \vee q} \widehat{\mathcal{W}}$. Which is a contradiction. Thus, $\widehat{\mathcal{W}}(h * k) \geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}$ for all $h, k \in \mathcal{P}$. Hence, $\widehat{\mathcal{W}}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$. □

In the following theorem, we give conditions for an m-$\mathscr{PF}$ set to be an m-polar $(q, \in \vee q)$-fuzzy subalgebra.

Theorem 3.8. *Let C be a subalgebra of $\mathcal{P}$ and let $\widehat{\mathcal{W}}$ be an m-$\mathscr{PF}$ subset of $\mathcal{P}$ such that*

(1) $\widehat{\mathcal{W}}(h) \geq \widehat{0.5}$, for all $h \in C$.

(2) $\widehat{\mathcal{W}}(h) = \widehat{0}$, for all $h \notin C$,

Then, $\widehat{\mathcal{W}}$ is an m-polar $(q, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$.

Proof. Assume C is a subalgebra of $\mathcal{P}$, $h, k \in \mathcal{P}$ and $\widehat{\eta}, \widehat{\zeta} \in (0, 1]^m$. If $h_{\widehat{\eta}} q \widehat{\mathcal{W}}$ and $k_{\widehat{\zeta}} q \widehat{\mathcal{W}}$, then $\widehat{\mathcal{W}}(h) + \widehat{\eta} > \widehat{1}$ and $\widehat{\mathcal{W}}(k) + \widehat{\zeta} > \widehat{1}$. Thus, $h, k \in C$ and so $h * k \in C$ because if not, then $h \notin C$ or $k \notin C$. Thus, $\widehat{\mathcal{W}}(h) = \widehat{0}$ or $\widehat{\mathcal{W}}(k) = \widehat{0}$, and so $\widehat{\eta} > \widehat{1}$ or $\widehat{\zeta} > \widehat{1}$. This is a contradiction. If $\inf\{\widehat{\eta}, \widehat{\zeta}\} \leq \widehat{0.5}$, then

$$\widehat{\mathcal{W}}(h * k) \geq \widehat{0.5} \geq \inf\{\widehat{\eta}, \widehat{\zeta}\}.$$

Hence, $(h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} \in \widehat{\mathcal{W}}$. If $\inf\{\widehat{\eta}, \widehat{\zeta}\} > \widehat{0.5}$, then

$$\widehat{\mathcal{W}}(h * k) + \inf\{\widehat{\eta}, \widehat{\zeta}\} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$$

and so $(h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} q \widehat{\mathcal{W}}$. Thus, $(h * k)_{\inf\{\widehat{\eta}, \widehat{\zeta}\}} \in \vee q \widehat{\mathcal{W}}$. Hence, $\widehat{\mathcal{W}}$ is an m-polar $(q, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$.

□

Corollary 3.9. *Let $\phi \neq C \subseteq X$ and $\widehat{\chi}_C$ be the m-polar characteristic function of C. Then, C is a subalgebra of $\mathcal{P}$ if and only if $\widehat{\chi}_C$ is an m-polar $(\alpha, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$, where $\alpha \in \{\in, q\}$.*

We consider a relation between an m-polar $(q, \in \vee q)$-fuzzy subalgebra and an m-polar $(\in, \in \vee q)$-fuzzy subalgebra.

Theorem 3.10. *Every m-polar $(q, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$.*

Proof. Let $\widehat{\mathcal{W}}$ be an m-polar $(q, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$. Let $h, k \in \mathcal{P}$ be such that $h_{\widehat{\eta}}, k_{\widehat{\zeta}} \in \widehat{\mathcal{W}}$ for $\widehat{\eta}, \widehat{\zeta} \in (0, 1]^m$. Then,

$$\widehat{\mathcal{W}}(h) \geq \widehat{\eta} \text{ and } \widehat{\mathcal{W}}(k) \geq \widehat{\zeta}.$$

Suppose that $(h*k)_{\inf\{\widehat{\eta},\widehat{\zeta}\}}\overline{\in\vee q}\widehat{\mathcal{W}}$. Then,

$$\widehat{\mathcal{W}}(h*k)<\inf\{\widehat{\eta},\widehat{\zeta}\} \tag{3.1}$$

and

$$\widehat{\mathcal{W}}(h*k)+\inf\{\widehat{\eta},\widehat{\zeta}\}\leq\widehat{1}. \tag{3.2}$$

From Eqs (3.1) and (3.2), we get

$$\widehat{\mathcal{W}}(h*k)<\widehat{0.5}. \tag{3.3}$$

Combining Eqs (3.1) and (3.3), we have

$$\widehat{\mathcal{W}}(h*k)<\inf\{\widehat{\eta},\widehat{\zeta},\widehat{0.5}\}.$$

Thus,

$$\begin{aligned}\widehat{1}-\widehat{\mathcal{W}}(h*k) &> \widehat{1}-\inf\{\widehat{\eta},\widehat{\zeta},\widehat{0.5}\}\\ &= \sup\{1-\widehat{\eta},1-\widehat{\zeta},\widehat{0.5}\}\\ &\geq \sup\{1-\widehat{\mathcal{W}}(h),1-\widehat{\mathcal{W}}(k),\widehat{0.5}\}.\end{aligned}$$

Choose $\widehat{\psi}\in(0,1]^m$ such that $\widehat{1}-\widehat{\mathcal{W}}(h*k)\geq\widehat{\psi}>\sup\{1-\widehat{\mathcal{W}}(h),1-\widehat{\mathcal{W}}(k),\widehat{0.5}\}$. It follows that $\widehat{\mathcal{W}}(h)+\widehat{\psi}>\widehat{1}$, $\widehat{\mathcal{W}}(k)+\widehat{\psi}>\widehat{1}$, and $\widehat{\mathcal{W}}(h*k)+\widehat{\psi}\leq\widehat{1}$. Thus, $x_{\widehat{\psi}}q\widehat{\mathcal{W}}$, $y_{\widehat{\psi}}q\widehat{\mathcal{W}}$, but $(h*k)_{\widehat{\psi}}\overline{\in\vee q}\widehat{\mathcal{W}}$, a contradiction. Hence, $\widehat{\mathcal{W}}$ is an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$. □

Remark 3.11. The converse of Theorem 3.10 is not true in general. For example, a 3-polar $(\in,\in\vee q)$-fuzzy subalgebra $\widehat{\mathcal{W}}$ of $\mathcal{P}$ in Example 3.1 is not a 3-polar $(q,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$, since

$$k_{(0.41,0.42,.43)}q\widehat{\mathcal{W}} \text{ and } l_{(0.77,0.78,.79)}q\widehat{\mathcal{W}},$$

but

$$(k*l)_{\inf\{(0.41,0.42,.43),(0.77,0.78,.79)\}}=n_{(0.41,0.42,.43)}\overline{\in\vee q}\widehat{\mathcal{W}}.$$

The following corollary follows from Theorem 3.8 and Theorem 3.10.

Corollary 3.12. *For a subalgebra C of $\mathcal{P}$, let $\widehat{\mathcal{W}}$ be an m-$\mathscr{PF}$ subset of $\mathcal{P}$ satisfying conditions (1), and (2) of Theorem 3.8. Then, $\widehat{\mathcal{W}}$ is an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$.*

We give a condition for an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$ to be an m-polar $(q,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$.

Theorem 3.13. *Let $\widehat{\mathcal{W}}$ be an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$ and any m-$\mathscr{PF}$ point has the value $\widehat{\eta}\in(0,0.5]^m$. Then, $\widehat{\mathcal{W}}$ is an m-polar $(q,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$.*

Proof. Let $\widehat{\mathcal{W}}$ be an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$. For $h,k\in\mathcal{P}$, let $\widehat{\eta},\widehat{\zeta}\in(0,0.5]^m$ be such that $h_{\widehat{\eta}}q\widehat{\mathcal{W}}$ and $k_{\widehat{\zeta}}q\widehat{\mathcal{W}}$. Then, $\widehat{\mathcal{W}}(h)>\widehat{1}-\widehat{\eta}\geq\widehat{\eta}$ and $\widehat{\mathcal{W}}(k)>\widehat{1}-\widehat{\zeta}\geq\widehat{\zeta}$, i.e., $h_{\widehat{\eta}}\in\widehat{\mathcal{W}}$ and $k_{\widehat{\zeta}}\in\widehat{\mathcal{W}}$. Since $\widehat{\mathcal{W}}$ is an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$, it implies that $(h*k)_{\inf\{\widehat{\eta},\widehat{\zeta}\}}\in\vee q\widehat{\mathcal{W}}$. Consequently, $\widehat{\mathcal{W}}$ is an m-polar $(q,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$. □

Theorem 3.14. *Both m-polar* $(\in,\in)$*-fuzzy subalgebra and m-polar* $(\in\vee q,\in\vee q)$*-fuzzy subalgebra of* $\mathcal{P}$ *are an m-polar* $(\in,\in\vee q)$*-fuzzy subalgebra of* $\mathcal{P}$.

Proof. Obviously, an m-polar $(\in,\in)$-fuzzy subalgebra of $\mathcal{P}$ is an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$. Now, let $\widehat{\mathcal{W}}$ be an m-polar $(\in\vee q,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$. For any $h,k\in\mathcal{P}$, let $\widehat{\eta},\widehat{\zeta}\in(0,1]^m$ be such that $h_{\widehat{\eta}}\in\widehat{\mathcal{W}}$ and $k_{\widehat{\zeta}}\in\widehat{\mathcal{W}}$. Then, $h_{\widehat{\eta}}\in\vee q\widehat{\mathcal{W}}$ and $k_{\widehat{\zeta}}\in\vee q\widehat{\mathcal{W}}$, it follows that $(h*k)_{\inf\{\widehat{\eta},\widehat{\zeta}\}}\in\vee q\widehat{\mathcal{W}}$. Thus, $\widehat{\mathcal{W}}$ is an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$. □

Remark 3.15. The converse of Theorem 3.14 is not true in general. For example, a 3-polar $(\in,\in\vee q)$-fuzzy subalgebra $\widehat{\mathcal{W}}$ of $\mathcal{P}$ in Example 3.1 is not a 3-polar $(\in\vee q,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$, since

$$k_{(0.5,0.52,.53)}\in\vee q\widehat{\mathcal{W}} \text{ and } n_{(0.8,0.82,.83)}\in\vee q\widehat{\mathcal{W}},$$

but

$$(k*n)_{\inf\{(0.50,0.52,.53),(0.80,0.82,.83)\}}=l_{(0.50,0.52,.53)}\overline{\in\vee q}\widehat{\mathcal{W}}.$$

Also, it is not a 3-polar $(\in,\in)$-fuzzy subalgebra of $\mathcal{P}$, since

$$k_{(0.62,0.63,.64)}\in\widehat{\mathcal{W}} \text{ and } k_{(0.66,0.67,.68)}\in\widehat{\mathcal{W}},$$

but

$$(k*k)_{\inf\{(0.62,0.63,.64),(0.66,0.67,.68)\}}=0_{(0.62,0.63,.64)}\overline{\in}\widehat{\mathcal{W}}.$$

We give a condition for an m-polar $(\in,\in\vee q)$-fuzzy subalgebra to be an m-polar $(\in,\in)$-fuzzy subalgebra.

Theorem 3.16. *Let* $\widehat{\mathcal{W}}$ *be an m-polar* $(\in,\in\vee q)$*-fuzzy subalgebra of* $\mathcal{P}$ *such that* $\widehat{\mathcal{W}}(h)<\widehat{0.5}$ *for all* $h\in\mathcal{P}$. *Then,* $\widehat{\mathcal{W}}$ *is an m-polar* $(\in,\in)$*-fuzzy subalgebra of* $\mathcal{P}$.

Proof. Let $h_{\widehat{\eta}}\in\widehat{\mathcal{W}}$ and $k_{\widehat{\zeta}}\in\widehat{\mathcal{W}}$ for $h,k\in\mathcal{P}$ and $\widehat{\eta},\widehat{\zeta}\in(0,1]^m$. Then, $\widehat{\mathcal{W}}(h)\geq\widehat{\eta}$ and $\widehat{\mathcal{W}}(k)\geq\widehat{\zeta}$. Since $\widehat{\mathcal{W}}$ is an m-polar $(\in,\in\vee q)$-fuzzy subalgebra of $\mathcal{P}$, by using condition (H) in Theorem 3.5, we conclude that $\widehat{\mathcal{W}}(h*k)\geq\inf\{\widehat{\mathcal{W}}(h),\widehat{\mathcal{W}}(k),\widehat{0.5}\}$. Since $\widehat{\mathcal{W}}(h)<\widehat{0.5}$ for all $h\in\mathcal{P}$, then $\widehat{\mathcal{W}}(h*k)\geq\inf\{\widehat{\mathcal{W}}(h),\widehat{\mathcal{W}}(k)\geq\inf\{\widehat{\eta},\widehat{\zeta}\}$. Therefore, $(h*k)_{\inf\{\widehat{\eta},\widehat{\zeta}\}}\in\widehat{\mathcal{W}}$. Hence, $\widehat{\mathcal{W}}$ is an m-polar $(\in,\in)$-fuzzy subalgebra of $\mathcal{P}$. □

In the following figure, we summarize and display the relations between some types of m-polar (α,β)-fuzzy subalgebras in where $\alpha,\beta\in\{\in,q,\in\vee q\}$, $\beta\notin q$.

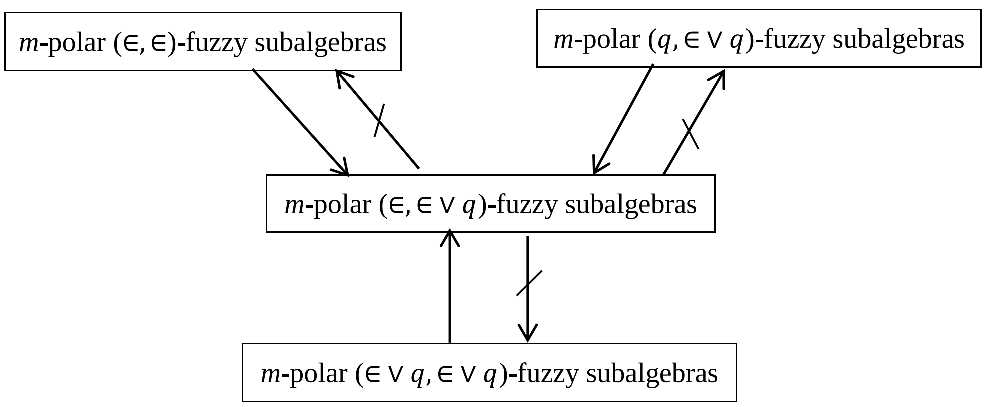

Figure 1. Some relations in this study.

Theorem 3.17. *Let $\{\widehat{\mathcal{W}_j}\}_{j\in I}$ be a family of m-polar $(\in, \in \vee q)$-fuzzy subalgebras of $\mathcal{P}$. Then, the intersection, denoted by $\cap_{j\in I}\widehat{\mathcal{W}}_j$, of $\{\widehat{\mathcal{W}_j}\}_{j\in I}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$.*

Proof. Let $\{\widehat{\mathcal{W}_j}\}_{j\in I}$ be a family of m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$ and $h, k \in \mathcal{P}$. Then, $\widehat{\mathcal{W}}_j(h * k) \geq \inf\{\widehat{\mathcal{W}}_j(h), \widehat{\mathcal{W}}_j(k), \widehat{0.5}\} \forall j \in I$. Thus,

$$\begin{aligned}
(\cap_{j\in I}\widehat{\mathcal{W}}_j)(h * k) &= \cap_{j\in I}\widehat{\mathcal{W}}_j(h * k) \\
&\geq \cap_{j\in I}(\inf\{\widehat{\mathcal{W}}_j(h), \widehat{\mathcal{W}}_j(k), \widehat{0.5}\}) \\
&= \inf\{(\cap_{j\in I}\widehat{\mathcal{W}}_j)(h), (\cap_{j\in I}\widehat{\mathcal{W}}_j)(k), \widehat{0.5}\}.
\end{aligned}$$

Therefore, $(\cap_{j\in I}\widehat{\mathcal{W}}_j)(h * k) \geq \inf\{(\cap_{j\in I}\widehat{\mathcal{W}}_j)(h), (\cap_{j\in I}\widehat{\mathcal{W}}_j)(k), \widehat{0.5}\}$. Hence, $\cap_{j\in I}\widehat{\mathcal{W}}_j$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$. □

The following example shows that the union of two m-polar $(\in, \in \vee q)$-fuzzy subalgebras of $\mathcal{P}$ may not be an m-polar $(\in, \in \vee q)$-fuzzy subalgebras of $\mathcal{P}$.

Example 3.2. Let $\mathcal{P} = \{0, k, l, n\}$ be a BCI-algebra with the operation $*$ which is given in Example 3.1, and let $\widehat{\mathcal{W}} : \mathcal{P} \to [0, 1]^m$ be an m-$\mathscr{P}\mathscr{F}$ set defined by:

$$\widehat{\mathcal{W}}(h) = \begin{cases} (0.6, ..., 0.6), & \text{if } h = 0 \\ (0.7, ..., 0.7), & \text{if } h = k \\ (0.3, ..., 0.3), & \text{if } h = l, n. \end{cases}$$

Then,

$$\widehat{\mathcal{W}}_{\widehat{\eta}} = \begin{cases} \mathcal{P}, & \text{if } \widehat{\eta} \in (0, 0.3]^m \\ \{0, k\}, & \text{if } \widehat{\eta} \in (0.3, 0.4]^m. \end{cases}$$

Since $\mathcal{P}$ and $\{0, k\}$ are subalgebras of $\mathcal{P}$, $\widehat{\mathcal{W}}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebras of $\mathcal{P}$ by Theorem 3.7. Let $\widehat{\mathcal{F}} : \mathcal{P} \to [0, 1]^m$ be an m-$\mathscr{P}\mathscr{F}$ set defined by:

$$\widehat{\mathcal{F}}(h) = \begin{cases} (0.4, ..., 0.4), & \text{if } h = 0 \\ (0.3, ..., 0.3), & \text{if } h = k, n \\ (0.5, ..., 0.5), & \text{if } h = l. \end{cases}$$

Then,

$$\widehat{\mathcal{F}}_{\widehat{\eta}} = \begin{cases} \mathcal{P}, & \text{if } \widehat{\eta} \in (0, 0.3]^m \\ \{0, l\}, & \text{if } \widehat{\eta} \in (0.3, 0.4]^m. \end{cases}$$

Since $\mathcal{P}$ and $\{0, l\}$ are subalgebras of $\mathcal{P}$, $\widehat{\mathcal{F}}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebras of $\mathcal{P}$ by Theorem 3.7. The union $\widehat{\mathcal{W}} \cup \widehat{\mathcal{F}}$ of $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{F}}$ is given by:

$$(\widehat{\mathcal{W}} \cup \widehat{\mathcal{F}})(h) = \begin{cases} (0.6, ..., 0.6), & \text{if } h = 0 \\ (0.7, ..., 0.7), & \text{if } h = k \\ (0.5, ..., 0.5), & \text{if } h = l \\ (0.3, ..., 0.3), & \text{if } h = n. \end{cases}$$

Hence,

$$(\widehat{\mathcal{W}} \cup \widehat{\mathcal{F}})_{\widehat{\eta}} = \begin{cases} \mathcal{P}, & \text{if } \widehat{\eta} \in (0, 0.3]^m \\ \{0, k, l\}, & \text{if } \widehat{\eta} \in (0.3, 0.4]^m. \end{cases}$$

Since $\{0, k, l\}$ is not a subalgebra of $\mathcal{P}$, it follows from Theorem 3.7 that $\widehat{\mathcal{W}} \cup \widehat{\mathcal{F}}$ is not an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$.

For any m-$\mathscr{P}\mathscr{F}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$ and $\widehat{\eta} \in (0, 1]^m$, we denote

$$\langle\widehat{\mathcal{W}}\rangle_{\widehat{\eta}} = \{h \in \mathcal{P} \mid h_{\widehat{\eta}} q \widehat{\mathcal{W}}\},$$

and

$$[\widehat{\mathcal{W}}]_{\widehat{\eta}} = \{h \in \mathcal{P} \mid h_{\widehat{\eta}} \in \vee q \widehat{\mathcal{W}}\}.$$

The sets $\langle\widehat{\mathcal{W}}\rangle_{\widehat{\eta}}$ and $[\widehat{\mathcal{W}}]_{\widehat{\eta}}$ are called q-level cut subset and $\in \vee q$-level cut subset of $\widehat{\mathcal{W}}$, respectively.

It is clear that

$$[\widehat{\mathcal{W}}]_{\widehat{\eta}} = \langle\widehat{\mathcal{W}}\rangle_{\widehat{\eta}} \cup \widehat{\mathcal{W}}_{\widehat{\eta}}.$$

In the following two theorems, we discuss the relation between crisp subalgebras and m-polar $(\in, \in \vee q)$-fuzzy subalgebras of $\mathcal{P}$ through level cut subsets.

Theorem 3.18. *An m-$\mathscr{P}\mathscr{F}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$ if and only if $\langle\widehat{\mathcal{W}}\rangle_{\widehat{\eta}} \neq \phi$ is a subalgebra of $\mathcal{P}$ for all $\widehat{\eta} \in (0.5, 1]^m$.*

Proof. Assume $\widehat{\mathcal{W}}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$. Let $h, k \in \langle\widehat{\mathcal{W}}\rangle_{\widehat{\eta}}$. Then,

$$h_{\widehat{\eta}} q \widehat{\mathcal{W}} \text{ and } y_{\widehat{\eta}} q \widehat{\mathcal{W}}.$$

This implies that

$$\begin{array}{ccc} \widehat{\mathcal{W}}(h) + \widehat{\eta} > \widehat{1} & , & \widehat{\mathcal{W}}(k) + \widehat{\eta} > \widehat{1} \\ \widehat{\mathcal{W}}(h) > \widehat{1} - \widehat{\eta} & , & \widehat{\mathcal{W}}(k) > \widehat{1} - \widehat{\eta}. \end{array}$$

By hypothesis

$$\begin{aligned} \widehat{\mathcal{W}}(h * k) &\geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\} \\ &> \inf\{\widehat{1} - \widehat{\eta}, \widehat{1} - \widehat{\eta}, \widehat{0.5}\} \\ &= \inf\{\widehat{1} - \widehat{\eta}, \widehat{0.5}\} \\ &= \widehat{1} - \widehat{\eta}. \end{aligned}$$

Thus, $\widehat{\mathcal{W}}(h * k) + \widehat{\eta} > \widehat{1}$, implies $(h * k)_{\widehat{\eta}} q \widehat{\mathcal{W}}$, i.e., $h * k \in \langle\widehat{\mathcal{W}}\rangle_{\widehat{\eta}}$. Therefore, $\langle\widehat{\mathcal{W}}\rangle_{\widehat{\eta}}$ is a subalgebra of $\mathcal{P}$.

Conversely, suppose $\langle\widehat{\mathcal{W}}\rangle_{\widehat{\eta}}$ is a subalgebra of $\mathcal{P}$ for all $\widehat{\eta} \in (0.5, 1]^m$. Let $h, k \in \mathcal{P}$ such that

$$\widehat{\mathcal{W}}(h * k) < \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}.$$

Then,

$$\widehat{1} - \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\} < \widehat{1} - \widehat{\mathcal{W}}(h * k).$$

This implies

$$\sup\{\widehat{1} - \widehat{\mathcal{W}}(h), \widehat{1} - \widehat{\mathcal{W}}(k), \widehat{0.5}\} < \widehat{1} - \widehat{\mathcal{W}}(h * k).$$

Select some $\widehat{\eta} \in (0.5, 1]^m$ such that

$$\sup\{\widehat{1} - \widehat{\mathcal{W}}(h), \widehat{1} - \widehat{\mathcal{W}}(k), \widehat{0.5}\} < \widehat{\eta} \leq \widehat{1} - \widehat{\mathcal{W}}(h * k).$$

Then, $\widehat{\mathcal{W}}(h) + \widehat{\eta} > \widehat{1}$, $\widehat{\mathcal{W}}(k) + \widehat{\eta} > \widehat{1}$ and $\widehat{\mathcal{W}}(h * k) + \widehat{\eta} < \widehat{1}$. Thus, $h_{\widehat{\eta}} q \widehat{\mathcal{W}}$, $y_{\widehat{\eta}} q \widehat{\mathcal{W}}$, but $(h * k)_{\widehat{\eta}} \overline{q} \widehat{\mathcal{W}}$, i.e., $h, k \in \langle \widehat{\mathcal{W}} \rangle_{\widehat{\eta}}$, but $h * k \notin \langle \widehat{\mathcal{W}} \rangle_{\widehat{\eta}}$, a contradiction. Thus, $\widehat{\mathcal{W}}(h * k) \geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}$ for all $h, k \in \mathcal{P}$. This shows that $\widehat{\mathcal{W}}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$. □

Theorem 3.19. *An m-$\mathscr{PF}$ set $\widehat{\mathcal{W}}$ of $\mathcal{P}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$ if and only if $[\widehat{\mathcal{W}}]_{\widehat{\eta}} \neq \phi$ is a subalgebra of $\mathcal{P}$ for all $\widehat{\eta} \in (0, 1]^m$.*

Proof. Let $\widehat{\mathcal{W}}$ be an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$ and $\widehat{\eta} \in (0, 1]^m$. Let $h, k \in [\widehat{\mathcal{W}}]_{\widehat{\eta}}$, so we have

$$h_{\widehat{\eta}}, k_{\widehat{\eta}} \in \vee q \widehat{\mathcal{W}},$$

that is

$$\widehat{\mathcal{W}}(h) \geq \widehat{\eta} \text{ or } \widehat{\mathcal{W}}(h) + \widehat{\eta} \geq \widehat{1} \tag{3.4}$$

and

$$\widehat{\mathcal{W}}(k) \geq \widehat{\eta} \text{ or } \widehat{\mathcal{W}}(k) + \widehat{\eta} \geq \widehat{1}. \tag{3.5}$$

Case (1). If $\widehat{\eta} \in (0, 0.5]^m$, then $\widehat{1} - \widehat{\eta} \geq \widehat{0.5} \geq \widehat{\eta}$. It implies from (3.4) and (3.5) that

$$\widehat{\mathcal{W}}(h) \geq \widehat{\eta} \text{ and } \widehat{\mathcal{W}}(k) \geq \widehat{\eta}.$$

By hypothesis

$$\begin{aligned} \widehat{\mathcal{W}}(h * k) &\geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\} \\ &\geq \inf\{\widehat{\eta}, \widehat{\eta}, \widehat{0.5}\} \\ &= \inf\{\widehat{\eta}, \widehat{0.5}\} \\ &= \widehat{\eta}. \end{aligned}$$

Hence, $(h * k)_{\widehat{\eta}} \in \widehat{\mathcal{W}}$.
Case (2). If $\widehat{\eta} \in (0.5, 1]^m$, then $\widehat{1} - \widehat{\eta} < \widehat{0.5} < \widehat{\eta}$. It implies from (3.4) and (3.5) that

$$\widehat{\mathcal{W}}(h) > \widehat{1} - \widehat{\eta} \text{ and } \widehat{\mathcal{W}}(k) > \widehat{1} - \widehat{\eta}.$$

By hypothesis

$$\begin{aligned} \widehat{\mathcal{W}}(h * k) &\geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\} \\ &\geq \inf\{\widehat{1} - \widehat{\eta}, \widehat{1} - \widehat{\eta}, \widehat{0.5}\} \\ &= \inf\{\widehat{1} - \widehat{\eta}, \widehat{0.5}\} \\ &= \widehat{1} - \widehat{\eta}. \end{aligned}$$

Hence, $(h * k)_{\widehat{\eta}} q \widehat{\mathcal{W}}$. Thus, $(h * k)_{\widehat{\eta}} \vee q \widehat{\mathcal{W}}$, i.e., $(h * k) \in [\widehat{\mathcal{W}}]_{\widehat{\eta}}$. Therefore, $[\widehat{\mathcal{W}}]_{\widehat{\eta}}$ is a subalgebra of $\mathcal{P}$.

Conversely, Suppose that $[\widehat{\mathcal{W}}]_{\widehat{\eta}}$ is a subalgebra of $\mathcal{P}$ for all $\widehat{\eta} \in (0, 1]^m$. Assume $h, k \in \mathcal{P}$ such that

$$\widehat{\mathcal{W}}(h * k) < \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}.$$

Select $\widehat{\eta} \in (0, 1]^m$ such that

$$\widehat{\mathcal{W}}(h * k) < \widehat{\eta} \leq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}.$$

Then, $h_{\widehat{\eta}} \in \widehat{\mathcal{W}}, k_{\widehat{\eta}} \in \widehat{\mathcal{W}}$, but $(h * k)_{\widehat{\eta}} \overline{\in \vee q} \widehat{\mathcal{W}}$. Since $[\widehat{\mathcal{W}}]_{\widehat{\eta}}$ is a subalgebra of $\mathcal{P}$, we have $h * k \in [\widehat{\mathcal{W}}]_{\widehat{\eta}}$, a contradiction. Hence, $\widehat{\mathcal{W}}(h * k) \geq \inf\{\widehat{\mathcal{W}}(h), \widehat{\mathcal{W}}(k), \widehat{0.5}\}$ for all $h, k \in \mathcal{P}$. Thus, $\widehat{\mathcal{W}}$ is an m-polar $(\in, \in \vee q)$-fuzzy subalgebra of $\mathcal{P}$. □

4. Conclusions

The objective of this article is to establish a new concept of m-$\mathscr{PF}$ subalgebras in BCK/BCI-algebras $\mathcal{P}$, called m-polar (α,β)-fuzzy subalgebras, by using the notions of m-$\mathscr{PF}$ sets and m-$\mathscr{PF}$ points. As a special case of m-polar (α,β)-fuzzy subalgebras, we have presented the idea of m-polar $(\in, \in \vee q)$-fuzzy subalgebras, and investigated several related properties. Then, we have provided conditions for an m-$\mathscr{PF}$ set to be an m-polar $(q, \in \vee q)$-fuzzy subalgebra and an m-polar $(\in, \in \vee q)$-fuzzy subalgebra. Finally, we have discussed the relationship between crisp subalgebras and m-polar $(\in, \in \vee q)$-fuzzy subalgebras in $\mathcal{P}$ through level cut subsets. In our further research, we will focus on adopting this approach to some more algebraic structures, such as KU-algebras, UP-algebras, semigroups, KU-semigroups and Hemirings, and to some more complicated applications from the domains of information systems and computer sciences.

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Conflict of interest

The authors declare no conflict of interest.

References

1. Y. Imai, K. Iséki, On axiom systems of propositional calculi, Proc. Jpn. Acad. Ser. A Math. Sci., **42** (1966), 19–21.
2. K. Iséki, An algebra related with a propositional calculus, Proc. Jpn. Acad., **42** (1966), 26–29.
3. J. Meng, Y. B. Jun, BCK-Algebras, Kyung Moon Sa Co.: Seoul, Korea, 1994.
4. Y. Huang, BCI-Algebra, Science Press: Beijing, China, 2006.
5. L. A. Zadeh, Fuzzy sets, Inf. Control., **8** (1965), 338–353.
6. W. R. Zhang, Bipolar fuzzy sets and relations: A computational framework for cognitive and modeling and multiagent decision analysis, Proc. of IEEE conf., (1994), 305–309.
7. K. Hayat, T. Mahmood, B. Y. Cao, On bipolar anti fuzzy H-ideals in hemirings, Fuzzy Inf. Eng., **9** (2017), 1–19.
8. M. Zulfiqar, Some properties of n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra in BRK-algebras, An. St. Univ. Ovidius Constanta, **24** (2016), 301–320.
9. A. Al-Masarwah, A. G. Ahmad, Doubt bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras, J. Math. Anal., **9** (2018), 9–27.
10. A. Al-Masarwah, A. G. Ahmad, On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras, Eur. J. Pure Appl. Math., **11** (2018), 652–670.

11. A. Al-Masarwah, A.G. Ahmad, Novel concepts of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras, Int. J. Innov. Comput. Inf. Control, **14** (2018), 2025–2041.
12. A. Al-Masarwah, A.G. Ahmad, m-Polar (α,β)-fuzzy ideals in BCK/BCI-algebras, Symmetry **11** (2019), 44.
13. C. Jana, M. Pal, Generalized intuitionistic fuzzy ideals of BCK/BCI-algebras based on 3-valued logic and its computational study, Fuzzy Inf. Eng., **9** (2017), 455–478.
14. C. Jana, M. Pal, On (α,β)-Union-soft sets in BCK/BCI-algebras, Mathematics, **7** (2019), 252.
15. C. Jana, T. Senapati, K. P. Shum, et al., Bipolar fuzzy soft subalgebras and ideals of BCK/BCI-algebras based on bipolar fuzzy points, J. Intell. Fuzzy Syst., **37** (2019), 2785–2795.
16. C. Jana, M. Pal, Application of (α,β)-soft intersectional sets BCK/BCI-algebras, Int. J. Intell. Syst. Technol. Appl., **16** (2017), 269–288.
17. J. Chen, S. Li, S. Ma, et al., m-Polar fuzzy sets: An extension of bipolar fuzzy sets, Sci. World J., **2014** (2014), 8.
18. M. Akram, A. Farooq, K.P. Shum, On m-polar fuzzy lie subalgebras, Ital. J. Pure Appl. Math., **36** (2016), 445–454.
19. M. Akram, A. Farooq, m-polar fuzzy lie ideals of lie algebras, Quasigroups Relat. Syst., **24** (2016), 141– 150.
20. A. Farooq, G. Alia, M. Akram, On m-polar fuzzy groups, Int. J. Algebr. Stat., **5** (2016), 115–127.
21. A. Al-Masarwah, A.G. Ahmad, m-Polar fuzzy ideals of BCK/BCI-algebras, J. King Saud Univ. Sci., (2018), doi:10.1016/j.jksus.2018.10.002.
22. A. Al-Masarwah, A. G. Ahmad, On (complete) normality of m-pF subalgebras in BCK/BCI-algebras, AIMS Math., **4** (2019), 740–750.
23. M. kram, G. Ali, N. O. Alshehri, A new multi-attribute decision-making method based on m-polar fuzzy soft rough sets, Symmetry, **9** (2017), 271.
24. M. Abu Qamar, N. Hassan, Q-neutrosophic soft relation and its application in decision making, Entropy, **20** (2018), 172.
25. A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., **35** (1971), 512–517.
26. V. Murali, Fuzzy points of equivalent fuzzy subsets, Inform. Sci., **158** (2004), 277–288.
27. P. M. Pu, Y. M. Liu, Fuzzy topology I: Neighourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., **76** (1980), 571–599.
28. S. K. Bhakat, P. Das, $(\in,\in\vee q)$-fuzzy subgroups, Fuzzy Sets Syst., **80** (1996), 359–368.
29. W. A. Dudek, M. Shabir, M. Irfan Ali, (α,β)-fuzzy ideals of hemirings, Comput. Math. Appl., **58** (2009), 310–321.
30. A. Narayanan, T. Manikantan, $(\in,\in\vee q)$-fuzzy subnearrings and $(\in,\in\vee q)$-fuzzy ideals of nearrings, J. Appl. Math. Comput., **18** (2005), 419–430.
31. O. G. Xi, Fuzzy BCK-algebras, Math. Jpn., **36** (1991), 935–942.
32. Y. B. Jun, On (α,β)-fuzzy subalgebras of BCK/BCI-algebras, Bull. Korean Math. Soc., **42** (2005), 703–711.

33. G. Muhiuddin, A. M. Al-Roqi, Subalgebras of BCK/BCI-algebras based on (α, β)-type fuzzy sets, J. Comput. Anal. Appl., **18** (2015), 1057–1064.

34. C. Jana, T. Senapati, M. Pal, $(\in, \in \vee q)$-intuitionistic fuzzy BCI-subalgebras of a BCI-algebra, J. Intell. Fuzzy Syst., **31** (2016), 613–621.

35. C. Jana, M. Pal, A. B. Saeid, $(\in, \in \vee q)$-Bipolar fuzzy BCK/BCI-algebras, Missouri J. Math. Sci., **29** (2017), 139–160.

36. K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays. Math. Sci. Soc., **32** (2009), 361–373.

37. K. Iséki, S. Tanaka, An introduction to the theory of BCK-algebras, Math. Jpn., **23** (1978), 1–26.