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Research article

Some Grüss-type inequalities using generalized Katugampola fractional integral

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Abstract: The main objective of this paper is to obtain a generalization of some Grüss-type inequalities in case of functional bounds by using a generalized Katugampola fractional integral. We obtained new Grüss type inequalitys with functional bounds via the generalized fractional integral operators having same and different parameters. Results obtained are more generalized in nature.

Keywords: Grüss inequality; generalized fractional integral
Mathematics Subject Classification: 26A33, 26D10

1. Introduction

Fractional calculus is the study of integrations and derivatives in case of non-integer order, which is a generalized form of classical integrals and derivatives. The importance of fractional calculus is due to its multiple applications in several important scientific fields such as fluid dynamics, physics, computer networking, biology, control theory, signal processing, image processing and other fields. During the last few decades, fractional calculus have been studied extensively and one can observe a number of researchers have shown deep interest it, which led to the expansion and development of its concept by a number of authors.

Mathematical inequalities play an important role in a number of mathematical fields, especially those associated with finding the continuous dependence and uniqueness of solutions for fractional differential equations and others. This sensitive importance has stimulated a number of researchers recently to invent a number of useful inequalities.

In 1935, G. Grüss proved the renowned integral inequality [11] (see also [15]):

$$\left|\frac{1}{b-a}\int_a^b v(x)\,u(x)\,dx-\frac{1}{(b-a)^2}\int_a^b v(x)\,dx\int_a^b u(x)\,dx\right|\le\frac{1}{4}(M-m)(P-p), \tag{1.1}$$

where v, u are two integrable functions on $[a, b]$, satisfying the conditions

$$m \leq v(x) \leq M, \quad p \leq u(x) \leq P, \quad x \in [a, b], \quad m, M, p, P \in \mathbb{R}.$$

In recent years, the inequalities involving fractional calculus play a very important role in all mathematical fields which gave rise to important theories in mathematics, engineering, physics and other fields of science.

A remarkably large number inequalities of above type involving the special fractional integral (such as the Liouville, Riemann–Liouville, Erdélyi-Kober, Katugampola, Hadamard and Weyl types) have been investigated by many researchers and received considerable attention to it, see([2, 3, 5–10, 14, 17, 18, 21, 22]).

Grüss-type inequality has important applications in a number of mathematical fields, like an integral arithmetic mean, difference equations and h-integral arithmetic mean (see [1, 16]).

Dahmani et al. [4], in (2010), proved the following fractional version inequality by using Riemann–Liouville fractional integral

$$\left|\frac{x^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\alpha}(vu)(x) - \mathcal{J}^{\alpha}v(x)\mathcal{J}^{\alpha}u(x)\right| \leq \left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right)^{2}(M-m)(P-p), \tag{1.2}$$

for one parameter, and

$$\begin{aligned}
&\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}(vu)(x) + \frac{x^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}(vu)(x) - \mathcal{J}^{\alpha}v(x)\mathcal{J}^{\beta}u(x) - \mathcal{J}^{\beta}v(x)\mathcal{J}^{\alpha}u(x)\right)^{2} \\
&\leq \left[\left(M\frac{x^{\alpha}}{\Gamma(\alpha+1)} - \mathcal{J}^{\alpha}v(x)\right)\left(\mathcal{J}^{\beta}v(x) - m\frac{x^{\beta}}{\Gamma(\beta+1)}\right)\right. \\
&\quad \left. + \left(\mathcal{J}^{\alpha}v(x) - m\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{x^{\beta}}{\Gamma(\beta+1)} - \mathcal{J}^{\beta}v(x)\right)\right] \\
&\quad \times \left[\left(P\frac{x^{\alpha}}{\Gamma(\alpha+1)} - \mathcal{J}^{\alpha}u(x)\right)\left(\mathcal{J}^{\beta}u(x) - p\frac{x^{\beta}}{\Gamma(\beta+1)}\right)\right. \\
&\quad \left. + \left(\mathcal{J}^{\alpha}u(x) - p\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right)\left(P\frac{x^{\beta}}{\Gamma(\beta+1)} - \mathcal{J}^{\beta}u(x)\right)\right]
\end{aligned} \tag{1.3}$$

for two parameters, where v, u are two integrable functions on $[0, \infty)$, satisfying the conditions

$$m \leq v(x) \leq M, \quad p \leq u(x) \leq P, \quad x \in [0, \infty), \quad m, M, p, P \in \mathbb{R}. \tag{1.4}$$

In (2014), Tariboon et al. [20], replaced the constants which appeared as bounds of the functions v and u by four integrable functions on $[0, \infty)$, as

$$\varphi_1(x) \leq v(x) \leq \varphi_2(x) \quad and \quad \psi_1(x) \leq u(x) \leq \psi_2(x),$$

they obtained the inequality

$$\left|\frac{x^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\alpha}(vu)(x) - \mathcal{J}^{\alpha}v(x)\mathcal{J}^{\alpha}u(x)\right| \leq \sqrt{T(v, \varphi_1, \varphi_2)\,T(u, \psi_1, \psi_2)},$$

where $T(s,q,w)$ is defined by

$$\begin{aligned} T(s,q,w) &= \left(\mathcal{J}^{\alpha}\omega(x) - \mathcal{J}^{\alpha}s(x)\right)\left(\mathcal{J}^{\alpha}s(x) - \mathcal{J}^{\alpha}q(x)\right) \\ &+ \frac{x^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\alpha}(sq)(x) - \mathcal{J}^{\alpha}s(x)\mathcal{J}^{\alpha}q(x) \\ &+ \frac{x^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\alpha}(s\omega)(x) - \mathcal{J}^{\alpha}s(x)\mathcal{J}^{\alpha}\omega(x) \\ &- \frac{x^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\alpha}(q\omega)(x) + \mathcal{J}^{\alpha}q(x)\mathcal{J}^{\alpha}\omega(x). \end{aligned}$$

Motivated from above mentioned results, our purpose in this paper is to establish some new results on Grüss-type inequalities in case of functional bounds using the generalized Katugampola fractional integral.

2. Preliminaries

In this section, we give some definitions and properties will be used in our paper. For more details, please see Refs. [12, 13, 19].

Definition 2.1. *Consider the space $X_c^p(a,b)\,(c \in \mathbb{R}, 1 \le p \le \infty)$, of those complex valued Lebesgue measurable functions v on (a,b) for which the norm $\|v\|_{X_c^p} < \infty$, such that*

$$\|v\|_{X_c^p} = \left(\int_x^b |x^c v|^p \frac{dx}{x}\right)^{\frac{1}{p}}, \quad (1 \le p < \infty)$$

and

$$\|v\|_{X_c^\infty} = \sup ess_{x\in(a,b)} \left[x^c |v|\right].$$

In particular, when $c = 1/p$, the space $X_c^p(a,b)$ coincides with the space $L^p(a,b)$.

Definition 2.2. *The left- and right-sided fractional integrals of a function v where $v \in X_c^p(a,b)$, $\alpha > 0$, and $\beta, \rho, \eta, k \in \mathbb{R}$, are defined respectively by*

$${}^{\rho}\mathcal{J}_{a+;\eta,k}^{\alpha,\beta}v(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}}v(\tau)\,d\tau, \qquad 0 \le a < x < b \le \infty, \tag{2.1}$$

and

$${}^{\rho}\mathcal{J}_{b-;\eta,k}^{\alpha,\beta}v(x) = \frac{\rho^{1-\beta}x^{\rho\eta}}{\Gamma(\alpha)}\int_x^b \frac{\tau^{k+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}}v(\tau)\,d\tau, \qquad 0 \le a < x < b \le \infty, \tag{2.2}$$

if the integral exist.

To present and discuss our new results in this paper we use only the left-sided fractional integrals. The right sided fractional integrals can be proved similarly. Also we consider $a = 0$, in (2.1), to obtain

$${}^{\rho}\mathcal{I}_{\eta,k}^{\alpha,\beta}v(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}}v(\tau)\,d\tau.$$

The above fractional integral has the following Composition (index) formulae

$$ {}^{\rho}\mathcal{J}^{\alpha_1,\beta_1}_{a+;\eta_1,k_1}\; {}^{\rho}\mathcal{J}^{\alpha_2,\beta_2}_{a+;\eta_2,-\rho\eta_1} v = {}^{\rho}\mathcal{J}^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{a+;\eta_2,k_1} v, $$

$$ {}^{\rho}\mathcal{J}^{\alpha_1,\beta_1}_{b-;\eta_1,-\rho\eta_2}\; {}^{\rho}\mathcal{J}^{\alpha_2,\beta_2}_{b-;\eta_2,k_2} v = {}^{\rho}\mathcal{J}^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{a+;\eta_1,k_2} v. $$

For the convenience of establishing our results we define the following function as in [19]: let $x > 0$, $\alpha > 0, \rho, k, \beta, \eta \in \mathbb{R}$, then

$$ \Lambda^{\rho,\beta}_{x,k}(\alpha,\eta) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} \rho^{-\beta} x^{k+\rho(\eta+\alpha)}. $$

If $\eta = 0$, $a = 0$, $k = 0$ and taking the limit $\rho \to 1$, then the Definition (2.2) reduce to Liouville fractional integral and if $\eta = 0$, $k = 0$, with taking the limit $\rho \to 1$, then we can get Riemann-Liouville fractional integral. It is reduce to Weyl fractional integral, when $\eta = 0$, $a = -\infty$, $k = 0$ and taking the limit $\rho \to 1$. For Erdélyi-Kober fractional integral, we put $\beta = 0$, $k = -\rho(\alpha + \eta)$. We can also getting Katugampola fractional integral by taking $\beta = \alpha$, $k = 0$, $\eta = 0$. And finally Hadamard fractional integral when $\beta = \alpha$, $k = 0$, $\eta = 0^+$ and taking the limit $\rho \to 1$.

The definition (2.2) is more generalized and can be reduce to six cases by change its parameters with appropriate choice.

3. Main results

Now, we give our main results on Grüss-type inequalities in case of functional bounds.

Theorem 3.1. *Let v be an integrable function on $[0,\infty)$. Assume that there exist two integrable functions z_1, z_2 on $[0,\infty)$ such that*

$$ z_1(x) \le v(x) \le z_2(x) \qquad \forall x \in [0,\infty). \tag{3.1} $$

Then, for all $x > 0, \alpha > 0, \rho > 0, \delta > 0, \beta, \eta, k, \lambda \in \mathbb{R}$, we have

$$ \begin{aligned} &{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k} z_2(x)\; {}^{\rho}\mathcal{J}^{\delta,\lambda}_{\eta,k} v(x) + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k} v(x)\; {}^{\rho}\mathcal{J}^{\delta,\lambda}_{\eta,k} z_1(x) \\ &\ge {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k} v(x)\; {}^{\rho}\mathcal{J}^{\delta,\lambda}_{\eta,k} v(x) + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k} z_2(x)\; {}^{\rho}\mathcal{J}^{\delta,\lambda}_{\eta,k} z_1(x). \end{aligned} \tag{3.2} $$

Proof. From the condition (3.1), for all $\tau \ge 0$, $\sigma \ge 0$, we have

$$ (v(\sigma) - z_1(\sigma))(z_2(\tau) - v(\tau)) \ge 0. $$

Therefore

$$ v(\sigma) z_2(\tau) + z_1(\sigma) v(\tau) \ge v(\sigma) v(\tau) + z_1(\sigma) z_2(\tau). \tag{3.3} $$

Multiplying both sides of (3.3) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}$, where $\tau \in (0,x)$ and integrating with respect to τ over $(0,x)$, we get

$$ v(\sigma) \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} z_2(\tau)\, d\tau + z_1(\sigma) \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} v(\tau)\, d\tau $$

$$\geq v(\sigma)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}v(\tau)\,d\tau+z_{1}(\sigma)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}z_{2}(\tau)\,d\tau,$$

so we have

$$v(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)+z_{1}(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\geq v(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)+z_{1}(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x). \tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\delta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}$, where $\sigma\in(0,x)$, we obtain

$$\begin{aligned}&\frac{\rho^{1-\beta}x^{k}}{\Gamma(\delta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}v(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)+\frac{\rho^{1-\beta}x^{k}}{\Gamma(\delta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}z_{1}(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\&\geq\frac{\rho^{1-\beta}x^{k}}{\Gamma(\delta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}v(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}+\frac{\rho^{1-\beta}x^{k}}{\Gamma(\delta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}z_{1}(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x).\end{aligned} \tag{3.5}$$

Integrating both sides of (3.5) with respect to σ over $(0,x)$, we get

$$\begin{aligned}&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)\frac{\rho^{1-\lambda}x^{k}}{\Gamma(\delta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}v(\sigma)\,d\sigma+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\frac{\rho^{1-\lambda}x^{k}}{\Gamma(\delta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}z_{1}(\sigma)\,d\sigma\\&\geq{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\frac{\rho^{1-\lambda}x^{k}}{\Gamma(\delta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}v(\sigma)\,d\sigma+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)\frac{\rho^{1-\lambda}x^{k}}{\Gamma(\delta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\delta}}z_{1}(\sigma)\,d\sigma.\end{aligned}$$

Hence

$$\begin{aligned}&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}z_{1}(x)\\&\geq{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}z_{1}(x),\end{aligned}$$

which is inequality (3.2). □

Corollary 3.2. *Let z be an integrable function on $[0,\infty)$ satisfying $m\leq z(x)\leq M$, for all $x\in[0,\infty)$ and $m,M\in\mathbb{R}$. Then, for all $x>0$ and $\alpha>0,\rho>0,\delta>0,\beta,\eta,k,\lambda\in\mathbb{R}$, we have*

$$\begin{aligned}&M\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)+m\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\&\geq{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)+mM\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\,\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta).\end{aligned}$$

Remark 3.3. *If we put $\eta=0$, $k=0$, and taking the limit $\rho\to1$, then Theorem (3.1), reduces to Theorem 2 and Corollary (3.2), reduces to Corollary 3 in [20].*

Now we give the lemma required for proving our next theorem.

Lemma 3.1. *Let v,z_{1},z_{2} are integrable functions on $[0,\infty)$ satisfying the condition (3.1), then for all $x>0$ and $\alpha>0,\rho>0,\beta,\eta,k\in\mathbb{R}$, we have*

$$\begin{aligned}&\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v^{2}(x)-\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\right)^{2}\\&=\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{1}(x)\right)\\&\quad-\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\left[(z_{2}(x)-v(x))(v(x)-z_{1}(x))\right]\\&\quad+\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_{1}v)(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{1}(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\&\quad+\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_{2}v)(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\&\quad-\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_{1}z_{2})(x)+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{2}(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_{1}(x).\end{aligned} \tag{3.6}$$

Proof. For any $\tau, \sigma > 0$, we have

$$\begin{aligned}
&(z_2(\sigma) - v(\sigma))(v(\tau) - z_1(\tau)) + (z_2(\tau) - v(\tau))(v(\sigma) - z_1(\sigma)) \\
&- (z_2(\tau) - v(\tau))(v(\tau) - z_1(\tau)) - (z_2(\sigma) - v(\sigma))(v(\sigma) - z_1(\sigma)) \\
&= v^2(\tau) + v^2(\sigma) - 2v(\tau)v(\sigma) \\
&\quad + z_2(\sigma)v(\tau) + z_1(\tau)v(\sigma) - z_1(\tau)z_2(\sigma) + z_2(\tau)v(\sigma) + z_1(\sigma)v(\tau) - z_1(\sigma)z_2(\tau) \\
&\quad - z_2(\tau)v(\tau) + z_1(\tau)z_2(\tau) - z_1(\tau)v(\tau) - z_2(\sigma)v(\sigma) + z_1(\sigma)z_2(\sigma) - z_1(\sigma)v(\sigma).
\end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}}$, where $\tau \in (0, x)$ and integrating over $(0, x)$ with respect to the variable τ, we obtain

$$\begin{aligned}
&(z_2(\sigma) - v(\sigma))\left({}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x)\right) + (v(\sigma) - z_1(\sigma))\left({}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x)\right) \\
&- {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}\left[(z_2(\tau) - v(\tau))(v(\tau) - z_1(\tau))\right] - \left[(z_2(\sigma) - v(\sigma))(v(\sigma) - z_1(\sigma))\right]\Lambda^{\rho,\beta}_{x,k}(\alpha, \eta) \\
&= {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v^2(x) + v^2(\sigma)\Lambda^{\rho,\beta}_{x,k}(\alpha, \eta) - 2v(\sigma)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) \\
&\quad + z_2(\sigma)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) + v(\sigma)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x) - z_2(\sigma)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x) + v(\sigma)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x) \\
&\quad + z_1(\sigma)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) - z_1(\sigma)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_2 v)(x) + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_1 z_2)(x) \\
&\quad - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_1 v)(x) - \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)z_2(\sigma)v(\sigma) + \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)z_1(\sigma)z_2(\sigma) - \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)z_1(\sigma)v(\sigma).
\end{aligned} \tag{3.8}$$

Now multiplying both sides of (3.8) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho - \sigma^\rho)^{1-\alpha}}$, where $\sigma \in (0, x)$ and integrating over $(0, x)$ with respect to the variable σ, we obtain

$$\begin{aligned}
&\left({}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x)\right)\left({}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x)\right) \\
&+ \left({}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x)\right)\left({}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x)\right) \\
&- {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}\left[(z_2(x) - v(x))(v(x) - z_1(x))\right]\Lambda^{\rho,\beta}_{x,k}(\alpha, \eta) \\
&- {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}\left[(z_2(x) - v(x))(v(x) - z_1(x))\right]\Lambda^{\rho,\beta}_{x,k}(\alpha, \eta) \\
&= \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v^2(x) + \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v^2(x) - 2\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) \\
&\quad + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x) \\
&\quad + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x) + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}v(x) - {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_1(x)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}z_2(x) \\
&\quad - \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_2 v)(x) + \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_1 z_2)(x) - \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_1 v)(x) \\
&\quad - \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_2 v)(x) + \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_1 z_2)(x) - \Lambda^{\rho,\beta}_{x,k}(\alpha, \eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}(z_1 v)(x),
\end{aligned}$$

which yields the required identity (3.6). □

Our next result is on Grüss-type inequalities in case of functional bounds with same parameters.

Theorem 3.4. *Let v, u be two integrable functions on $[0, \infty)$. Suppose z_1, z_2, γ_1 and γ_2 be four integrable functions on $[0, \infty)$ satisfying the condition*

$$z_1(x) \leq v(x) \leq z_2(x) \quad \text{and} \quad \gamma_1(x) \leq u(x) \leq \gamma_2(x), \quad \forall x \in [0, \infty). \tag{3.9}$$

Then for all $x > 0$ and $\alpha > 0, \rho > 0, \beta, \eta, k \in \mathbb{R}$, we have

$$\left[\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(vu)(x) - \left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\right)\right]^2 \le T(v, z_1, z_2)\, T(u, \gamma_1, \gamma_2), \tag{3.10}$$

where $T(\varphi, \psi, \omega)$ as in [20], is defined by

$$\begin{aligned} T(\varphi,\psi,\omega) = {} & \left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\omega(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\varphi(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\varphi(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\psi(x)\right) \\ & + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\varphi\psi)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\varphi(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\psi(x) \\ & + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\varphi\omega)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\varphi(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\omega(x) \\ & - \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\psi\omega)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\psi(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\omega(x). \end{aligned}$$

Proof. Define

$$H(\tau,\sigma) := (v(\tau) - v(\sigma))(u(\tau) - u(\sigma)), \qquad \tau, \sigma \in (0,x), x > 0. \tag{3.11}$$

Multiplying both sides of (3.11) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}$, where $\tau \in (0,x)$ and integrating over $(0,x)$ with respect to the variable τ, we obtain

$$\begin{aligned} & \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} H(\tau,\sigma)\, d\tau \\ & := {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(uv)(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, v(\sigma)\, u(\sigma) - u(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) - v(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x). \end{aligned} \tag{3.12}$$

Now multiplying both sides of (3.12) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}$, where $\sigma \in (0,x)$ and integrating the resulting identity over $(0,x)$ with respect to the variable σ, we get

$$\begin{aligned} & \frac{\rho^{2(1-\beta)}x^{2k}}{2\Gamma^2(\alpha)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}} H(\tau,\sigma)\, d\tau d\sigma \\ & := \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(uv)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x). \end{aligned} \tag{3.13}$$

Applying the Cauchy-Schwarz inequality to (3.13), we can write

$$\begin{aligned} & \left(\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(uv)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\right)^2 \\ & \le \left(\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u^2(x) - \left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\right)^2\right) \\ & \quad \times \left(\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v^2(x) - \left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\right)^2\right). \end{aligned} \tag{3.14}$$

Since

$$\begin{aligned} (z_2(x) - v(x))(v(x) - z_1(x)) & \ge 0, \\ (\gamma_2(x) - u(x))(u(x) - \gamma_1(x)) & \ge 0, \end{aligned} \tag{3.15}$$

for all $x \in [0,\infty)$, we have

$$\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_2(x) - v(x))(v(x) - z_1(x)) \ge 0$$

and

$$\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\gamma_2(x)-u(x))(u(x)-\gamma_1(x))\geq 0.$$

Thus, from lemma (3.1), we have

$$\begin{aligned}
&\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v^2(x)-\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\right)^2\\
&\leq\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_1(x)\right)\\
&\quad+\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_1v)(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_1(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\
&\quad+\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_2v)(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\
&\quad-\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_1z_2)(x)+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_1(x)\\
&\quad=T(v,z_1,z_2)
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
&\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u^2(x)-\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\right)^2\\
&\leq\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\gamma_2(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\gamma_1(x)\right)\\
&\quad+\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\gamma_1u)(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\gamma_1(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\\
&\quad+\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\gamma_2u)(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\gamma_2(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\\
&\quad-\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\gamma_1\gamma_2)(x)+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\gamma_2(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\gamma_1(x)\\
&\quad=T(u,\gamma_1,\gamma_2)\,.
\end{aligned}\tag{3.17}$$

Combining the inequalities (3.16), (3.17) with inequality (3.14), we obtain inequality (3.10). □

Remark 3.5. *If we put* $T(v,z_1,z_2)=T(v,m,M)$ *and* $T(u,\gamma_1,\gamma_2)=T(v,p,P)$, *in Theorem (3.4), where* m,M,p,P *are constants, then inequality (3.10) reduces to*

$$\begin{aligned}
&\left|\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(vu)(x)-\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\right)\right|\\
&\leq\left(\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\right)^2(M-m)(P-p)\,,
\end{aligned}$$

which is a result given in [19].

Lemma 3.2. *Let* v,z_1,z_2 *are integrable functions on* $[0,\infty)$ *satisfying the condition (3.1), then for all* $x>0$ *and* $\alpha>0,\delta>0,\rho>0,\beta,\lambda,\eta,k\in\mathbb{R}$, *we have*

$$\begin{aligned}
&\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v^2(x)+\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v^2(x)-2\,{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\
&=\left({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}z_2(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_1(x)\right)\\
&\quad+\left({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}z_1(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\right)\\
&\quad-{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\left[(z_2(x)-v(x))(v(x)-z_1(x))\right]\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\\
&\quad-{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}\left[(z_2(x)-v(x))(v(x)-z_1(x))\right]\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\\
&\quad-{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}z_2(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)-{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x)
\end{aligned}\tag{3.18}$$

$$
\begin{aligned}
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) \\
&+ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x) \\
&+ \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_1 v)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_2 v)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_1 z_2)(x)\right] \\
&+ \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(z_1 v)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(z_2 v)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(z_1 z_2)(x)\right].
\end{aligned}
$$

Proof. In Lemma (3.1), multiplying both sides of (3.8) by $\frac{\rho^{1-\lambda}x^k}{\Gamma(\delta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\delta}}$, where $\sigma \in (0,x)$ and integrating the resulting identity over $(0,x)$ with respect to the variable σ, we obtain

$$
\begin{aligned}
&\left({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_2(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x)\right) \\
&+\left({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x)\right) \\
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\left[(z_2(x) - v(x))(v(x) - z_1(x))\right]\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta) \\
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}\left[(z_2(x) - v(x))(v(x) - z_1(x))\right]\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \\
&= \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v^2(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v^2(x) - 2\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) \\
&\quad + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x) \\
&\quad + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x) \\
&\quad - \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_2 v)(x) + \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_1 z_2)(x) - \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_1 v)(x) \\
&\quad - \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(z_2 v)(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1 z_2(x) - \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1 v(x),
\end{aligned}
$$

which gives (3.18) and proves the lemma. □

In our next theorem, we prove the result with different parameters. Here we use Lemma (3.2) to proving the result.

Theorem 3.6. *Let v, u be two integrable functions on $[0,\infty)$ and suppose z_1, z_2, γ_1 and γ_2 be four integrable functions on $[0,\infty)$ satisfying the condition (3.9), then for all $x > 0$ and $\alpha > 0$, $\delta > 0$, $\rho > 0$, $\beta, \lambda, \eta, k \in \mathbb{R}$, we have*

$$
\begin{aligned}
&\Big|\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(uv)(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(vu)(x) \\
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u(x)\Big| \\
&\le \sqrt{K(v, z_1, z_2)\, K(u, \gamma_1, \gamma_2)}\,,
\end{aligned}
\tag{3.19}
$$

where $K(\varphi, \psi, \omega)$ is defined by

$$
\begin{aligned}
&K(\varphi,\psi,\omega) \\
&= \left({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \omega(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \varphi(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \varphi(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \psi(x)\right) \\
&\quad + \left({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \varphi(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \psi(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \omega(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \varphi(x)\right) \\
&\quad - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \omega(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \varphi(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \varphi(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \omega(x) \\
&\quad - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \varphi(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \psi(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \psi(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \varphi(x)
\end{aligned}
$$

$$+ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}\omega(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\psi(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}\psi(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\omega(x)$$
$$+ \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\psi\varphi)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\omega\varphi)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\psi\omega)(x)\right]$$
$$+ \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(\psi\varphi)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(\omega\varphi)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(\psi\omega)(x)\right].$$

Proof. In Theorem (3.4), multiplying both sides of (3.12) by $\frac{\rho^{1-\lambda}x^k}{\Gamma(\delta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\delta}}$, where $\sigma \in (0,x)$ and integrating the resulting identity over $(0,x)$ with respect to the variable σ, we obtain

$$\begin{aligned}&\frac{\rho^{2-\beta-\lambda}x^{2k}}{\Gamma(\alpha)\Gamma(\delta)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\delta}}H(\tau,\sigma)\,d\tau d\sigma\\
&:= \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(uv)(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(vu)(x)\\
&\quad - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x).\end{aligned} \tag{3.20}$$

Applying Cauchy-Schwarz inequality for double integrals, we get

$$\begin{aligned}&\Big[\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(uv)(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}vu(x)\\
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\Big]^2\\
&\le \Big(\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v^2(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v^2(x) - 2\,{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\Big)\\
&\times\Big(\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u^2(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u^2(x) - 2\,{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)\Big).\end{aligned} \tag{3.21}$$

Since

$$(z_2(x) - v(x))(v(x) - z_1(x)) \ge 0$$

and

$$(\gamma_2(x) - u(x))(u(x) - \gamma_1(x)) \ge 0,$$

for all $x \in [0,\infty)$, we have

$$\begin{aligned}&\Big({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\left[(z_2(x)-v(x))(v(x)-z_1(x))\right]\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\\
&+ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}\left[(z_2(x)-v(x))(v(x)-z_1(x))\right]\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\Big)\\
&\ge 0\,,\end{aligned}$$

and

$$\begin{aligned}&\Big({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}\left[(\gamma_2(x)-u(x))(u(x)-\gamma_1(x))\right]\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\\
&+ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}\left[(\gamma_2(x)-u(x))(u(x)-\gamma_1(x))\right]\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\Big)\\
&\ge 0\,.\end{aligned}$$

Thus, from Lemma (3.2), we have

$$\begin{aligned}&\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v^2(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v^2(x) - 2\,{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\
&\le \Big({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}z_2(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\Big)\Big({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_1(x)\Big)\\
&\ + \Big({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}z_1(x)\Big)\Big({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\Big)\end{aligned}$$

$$
\begin{aligned}
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x) \\
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) \\
&+ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x) \\
&+ \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_1 v)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_2 v)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(z_1 z_2)(x)\right] \\
&+ \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(z_1 v)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(z_2 v)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(z_1 z_2)(x)\right] \\
&= K(v, z_1, z_2)
\end{aligned} \tag{3.22}
$$

and

$$
\begin{aligned}
&\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u^2(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u^2(x) - 2\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u(x) \\
&\le \left({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_2(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \gamma_1(x)\right) \\
&+ \left({}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_1(x)\right)\left({}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \gamma_2(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u(x)\right) \\
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \gamma_2(x) \\
&- {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \gamma_1(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u(x) \\
&+ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \gamma_1(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \gamma_2(x) \\
&+ \Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\gamma_1 u)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\gamma_2 u)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}(\gamma_1 \gamma_2)(x)\right] \\
&+ \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(\gamma_1 u)(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(\gamma_2 u)(x) - {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}(\gamma_1 \gamma_2)(x)\right] \\
&= K(u, \gamma_1, \gamma_2).
\end{aligned} \tag{3.23}
$$

From the inequalities (3.22), (3.23) and inequality (3.21), we obtain inequality (3.19). □

Now we give the following result.

Theorem 3.7. *Let v, u be two integrable functions on $[0,\infty)$ and suppose z_1, z_2, γ_1 and γ_2 be four integrable functions on $[0,\infty)$ satisfying the condition (3.9), then for all $x > 0$ and $\alpha > 0, \delta > 0, \rho > 0$, $\beta, \lambda, \eta, k \in \mathbb{R}$, the following inequalities holds:*

$$
\begin{aligned}
(a)\ & {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x) \\
&\ge {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x), \\
(b)\ & {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \gamma_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x) \\
&\ge {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} \gamma_2(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} u(x), \\
(c)\ & {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_2(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x) \\
&\ge {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_2(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x), \\
(d)\ & {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_1(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x) \\
&\ge {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} z_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} u(x) + {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda} \gamma_1(x)\, {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} v(x).
\end{aligned}
$$

Proof. To prove (a), from the condition (3.9), we have for $x \in [0,\infty)$ that

$$
(z_2(\tau) - v(\tau))(u(\sigma) - \gamma_1(\sigma)) \ge 0. \tag{3.24}
$$

Therefore

$$z_2(\tau)u(\sigma)+v(\tau)\gamma_1(\sigma)\geq z_2(\tau)\gamma_1(\sigma)+v(\tau)u(\sigma). \tag{3.25}$$

Multiplying both sides of (3.25) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}$, where $\tau\in(0,x)$ and integrating over $(0,x)$ with respect to the variable τ, we obtain

$$\begin{aligned}&u(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x)+\gamma_1(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\&\geq\gamma_1(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x)+u(\sigma)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x).\end{aligned} \tag{3.26}$$

Now multiplying both sides of (3.26) by $\frac{\rho^{1-\lambda}x^k}{\Gamma(\delta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\delta}}$, where $\sigma\in(0,x)$ and integrating the resulting inequality over $(0,x)$ with respect to the variable σ, we get the desired inequality (a).To prove (b), (c) and (d), we use the following inequalities:

$$(B)\qquad(\gamma_2(\tau)-u(\tau))(v(\sigma)-z_1(\sigma))\geq0,$$

$$(C)\qquad(z_2(\tau)-v(\tau))(u(\sigma)-\gamma_2(\sigma))\leq0,$$

$$(D)\qquad(z_1(\tau)-v(\tau))(u(\sigma)-\gamma_1(\sigma))\leq0.$$

□

The next corollary is a special case of Theorem (3.7).

Corollary 3.8. *Let v,u be two integrable functions on $[0,\infty)$ and suppose that there exist the constants n,N,m,M satisfying the condition*

$$m\leq v(x)\leq M\quad\text{and}\quad n\leq u(x)\leq N,\qquad\forall x\in[0,\infty),$$

, then for all $x>0$ and $\alpha>0,\delta>0,\rho>0,\beta,\lambda,\eta,k\in\mathbb{R}$, we have:

$$\begin{aligned}&(i)\ \ M\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)+n\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\\&\quad\geq nM\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x),\\&(ii)\ \ m\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x)+N\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\\&\quad\geq mN\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}u(x),\\&(iii)\ \ MN\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)\\&\quad\geq M\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)+N\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x),\\&(iv)\ \ mn\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)\\&\quad\geq m\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\delta,\lambda}u(x)+n\Lambda_{x,k}^{\rho,\lambda}(\delta,\eta)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}v(x).\end{aligned}$$

Remark 3.9. *If we put $\eta=0$, $k=0$, and taking the limit $\rho\to1$, then Theorem (3.7), reduces to Theorem 5 and Corollary (3.8), reduces to Corollary 6 in [20].*

Conflict of interest

All authors declare no conflict of interest in this paper.

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