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Research article

Boundedness of fractional integral operators containing Mittag-Leffler functions via (s, m)-convexity

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Abstract: The objective of this paper is to derive the bounds of fractional integral operators which contain Mittag-Leffler functions in the kernels. By using (s, m)-convex functions bounds of these operators are evaluated which lead to obtain their boundedness and continuity. Moreover the presented results can be used to get various results for known fractional integrals and functions deducible from (s, m)-convexity. Also a version of Hadamard type inequality is established for (s, m)-convex functions via generalized fractional integrals.

Keywords: convex function; (s, m)-convex function; Mittag-Leffler function; generalized fractional integral operators
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1. Introduction

Convex functions are useful in various aspects in diverse fields of mathematical sciences. They produce an elegant theory of convex analysis, see [22, 24, 27].

Definition 1. *[27] A function $f : I \to \mathbb{R}$ is said to be convex function, if the following inequality holds:*

$$f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b),$$

for all $a, b \in I$ and $t \in [0, 1]$.

Convex functions have been extended and generalized from their analytical interpretations. A generalization of convex function defined on right half of real line is called s-convex function given as follows:

Definition 2. *[16] Let* $s \in [0,1]$*. A function* $f : [0,\infty) \to \mathbb{R}$ *is said to be s-convex function in the second sense if*

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b),$$

holds for all $a, b \in [0,\infty)$ *and* $t \in [0,1]$.

Another generalization of convex function defined on right half of real line is called m-convex function given as follows:

Definition 3. *[2] A function* $f : [0,b] \to \mathbb{R}$ *is said to be m-convex function, where* $m \in [0,1]$ *and* $b > 0$*, if for every* $x, y \in [0,b]$ *and* $t \in [0,1]$ *we have*

$$f(ta + m(1-t)b) \leq tf(a) + m(1-t)f(b).$$

Aforementioned functions can be generalized by (s,m)-convex functions defined as follows:

Definition 4. *[2] A function* $f : [0,b] \to \mathbb{R}$ *is said to be* (s,m)*-convex function, where* $(s,m) \in [0,1]^2$ *and* $b > 0$*, if for every* $x, y \in [0,b]$ *and* $t \in [0,1]$ *we have*

$$f(ta + m(1-t)b) \leq t^s f(a) + m(1-t^s)f(b).$$

For some recent citations and utilizations of (s,m)-convex functions one can see [5,10,18,19,23,31] and references therein. Convex functions and related definitions have been widely used to develop the theory of inequalities and their applications. A huge amount of work by many authors had/has been dedicated to theory and applications of mathematical inequalities, see [22, 24, 27]. The aim of this paper is the study of boundedness, continuity of fractional integral operators containing Mittag-Leffler functions via (s,m)-convex functions.

The Mittag-Leffler function denoted by $E_\alpha(.)$ was introduced by Gosta Mittag-Leffler in 1903 [21]

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)},$$

where $t, \alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Gamma(.)$ is the gamma function.

In the solution of fractional integral equations and fractional differential equations the Mittag-Leffler function arises naturally. The Mittag-Leffler function is a direct generalization of some special functions. It was consequently explored by Wiman, Pollard, Humbert, Agarwal and Feller, see [15]. It is further generalized and extended by various authors, for details see [4, 15, 26, 28, 29]. Andrić et al. introduced the following extended Mittag-Leffler function:

Definition 5. *[3] Let* $\mu, \alpha, l, \gamma, c \in \mathbb{C}$*,* $\Re(\mu), \Re(\alpha), \Re(l) > 0$*,* $\Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0$*,* $\delta > 0$ *and* $0 < k \leq \delta + \Re(\mu)$*. Then the extended generalized Mittag-Leffler function* $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ *is defined by:*

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{1.1}$$

where β_p *is defined by*

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt$$

and $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

A derivative formula of the extended generalized Mittag-Leffler function is given in the following lemma.

Lemma 1. *[3] If* $m \in \mathbb{N}, \omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\alpha), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0, \delta > 0$ *and* $0 < k < \delta + \Re(\mu)$*, then*

$$\left(\frac{d}{dt}\right)^m [t^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu; p)] = t^{\alpha-m-1} E^{\gamma,\delta,k,c}_{\mu,\alpha-m,l}(\omega t^\mu; p) \quad \Re(\alpha) > m. \tag{1.2}$$

Remark 1. *The extended Mittag-Leffler function (1.1) produces the related functions defined in [25, 26, 28–30], see [32, Remark 1.3].*

Next we give the definition of the fractional integral operator containing the extended generalized Mittag-Leffler function (1.1).

Definition 6. *[3] Let* $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$*,* $\Re(\mu), \Re(\alpha), \Re(l) > 0$*,* $\Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0$*,* $\delta > 0$ *and* $0 < k \leq \delta + \Re(\mu)$*. Let* $f \in L_1[a, b]$ *and* $x \in [a, b]$*. Then the generalized fractional integral operators containing Mittag-Leffler function are defined by:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x; p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^\mu; p) f(t) dt, \tag{1.3}$$

and

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x; p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-x)^\mu; p) f(t) dt. \tag{1.4}$$

Remark 2. *The operators (1.3) and (1.4) produce in particular several kinds of known fractional integral operators, see [32, Remark 1.4]*

The classical Riemann-Liouville fractional integral operator is defined as follows:

Definition 7. *[30] Let* $f \in L_1[a, b]$*. Then Riemann-Liouville fractional integral operators of order* $\alpha \in \mathbb{C}$ $(\mathcal{R}(\alpha) > 0)$ *are defined as follows:*

$$I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t) dt, x > a, \tag{1.5}$$

$$I^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-x)^{\alpha-1} f(t) dt, x < b. \tag{1.6}$$

It can be noted that $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,0,a^+} f\right)(x; 0) = I^\alpha_{a^+} f(x)$ and $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,0,b^-} f\right)(x; 0) = I^\alpha_{b^-} f(x)$. From fractional integral operators (1.3) and (1.4), we have (see [13]):

$$J_{\alpha,a^+}(x; p) := \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} 1\right)(x; p) = (x-a)^\alpha E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(w(x-a)^\mu; p), \tag{1.7}$$

$$J_{\beta,b^-}(x; p) := \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} 1\right)(x; p) = (b-x)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta+1,l}(w(b-x)^\mu; p). \tag{1.8}$$

Now a days integral operators have been proved very useful in the advancement of mathematical inequalities. Recently, several authors have established fractional integral inequalities by utilizing different types of integral operators, see [1, 6–9, 11–14, 17, 20, 32] and references therein.

In the upcoming section upper bounds of generalized fractional integral operators are derived by using (s, m)-convexity, and some particular results are produced. By using these bounds continuity of these operators is established. Furthermore a modulus inequality is established for differentiable function f such that $|f'|$ is (s, m)-convex. By imposing an additional condition Hadamard type inequality is obtained for (s, m)-convex functions. Also the results of this paper are connected with already known results.

2. Main results

Theorem 1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a real valued function. If f is positive and (s, m)-convex, then for $\alpha, \beta \geq 1$, the following inequality holds for generalized fractional integral operators:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-}f\right)(x;p) \\ &\leq \left(\frac{f(a) + msf(x)}{s+1}\right)(x-a)J_{\alpha-1,a^+}(x;p) \\ &+ \left(\frac{f(b) + msf(x)}{s+1}\right)(b-x)J_{\beta-1,b^-}(x;p), x \in [a, b].\end{aligned} \tag{2.1}$$

Proof. Let $x \in [a, b]$. Then for $t \in [a, x)$ and $\alpha \geq 1$, one can has the following inequality:

$$(x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p) \leq (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p). \tag{2.2}$$

The function f is (s, m)-convex, therefore one can obtain

$$f(t) \leq \left(\frac{x-t}{x-a}\right)^s f(a) + m\left(1 - \left(\frac{t-a}{x-a}\right)^s\right)f(x). \tag{2.3}$$

By multiplying (2.2) and (2.3) and then integrating over $[a, x]$, we get

$$\begin{aligned}&\int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)f(t)dt \\ &\leq (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)\left(\frac{f(a)}{(x-a)^s}\int_a^x (x-t)^s dt\right. \\ &\left.+mf(x)\int_a^x \left(1 - \left(\frac{t-a}{x-a}\right)^s\right)dt\right),\end{aligned}$$

that is, the left integral operator satisfies the following inequality:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p) \leq (x-a)J_{\alpha-1,a^+}(x;p)\left(\frac{f(a) + msf(x)}{s+1}\right). \tag{2.4}$$

Now on the other hand for $t \in (x, b]$ and $\beta \geq 1$, one can has the following inequality:

$$(t-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p) \leq (b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p). \tag{2.5}$$

Again from (s, m)-convexity of f, we have

$$f(t) \leq \left(\frac{t-x}{b-x}\right)^s f(b) + m\left(1 - \left(\frac{b-t}{b-x}\right)^s\right)f(x). \tag{2.6}$$

By multiplying (2.5) and (2.6) and then integrating over $[x, b]$, we have

$$\begin{aligned}&\int_x^b (t-x)^{\beta-1} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p) f(t)dt\\ &\leq (b-x)^{\beta-1} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)\left(\frac{f(a)}{(b-x)^s}\int_x^b (t-x)^s dt\right.\\ &\left.+mf(x)\int_x^b\left(1-\left(\frac{b-t}{b-x}\right)^s\right)dt\right),\end{aligned}$$

that is, the right integral operator satisfies the following inequality:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} f\right)(x;p) \leq (b-x) J_{\beta-1,b^-}(x;p)\left(\frac{f(b)+msf(x)}{s+1}\right). \tag{2.7}$$

By Adding (2.4) and (2.7), the required inequality (2.1) is established. □

Some particular results are stated in the following corollaries.

Corollary 1. *If we set $\alpha = \beta$ in (2.1), then the following inequality is obtained:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x;p)\\ &\leq \left(\frac{f(a)+msf(x)}{s+1}\right)(x-a) J_{\alpha-1,a^+}(x;p)\\ &+\left(\frac{f(b)+msf(x)}{s+1}\right)(b-x) J_{\alpha-1,b^-}(x;p), x\in[a,b].\end{aligned} \tag{2.8}$$

Corollary 2. *Along with assumptions of Theorem 1, if $f \in L_\infty[a, b]$, then the following inequality is obtained:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} f\right)(x;p)\\ &\leq \frac{\|f\|_\infty(1+ms)}{s+1}\left[(x-a) J_{\alpha-1,a^+}(x;p) + (b-x) J_{\beta-1,b^-}(x;p)\right].\end{aligned} \tag{2.9}$$

Corollary 3. *For $\alpha = \beta$ in (2.9), we get the following result:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x;p)\\ &\leq \frac{\|f\|_\infty}{s+1}\left[(x-a) D_{\alpha-1,a^+}(x;p) + (b-x) D_{\alpha-1,b^-}(x;p)\right].\end{aligned} \tag{2.10}$$

Corollary 4. *For $s = 1$ in (2.9), we get the following result:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} f\right)(x;p)\\ &\leq \frac{\|f\|_\infty(1+m)}{2}\left[(x-a) J_{\alpha-1,a^+}(x;p) + (b-x) J_{\beta-1,b^-}(x;p)\right].\end{aligned} \tag{2.11}$$

Theorem 2. *With the assumptions of Theorem 1 if $f \in L_\infty[a,b]$, then operator defined in (1.3) and (1.4) are bounded and continuous.*

Proof. If $f \in L_\infty[a,b]$, then from (2.4) we have

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p)\right| \le \frac{2\|f\|_\infty(1+ms)|x-a|J_{\alpha-1,a^+}(x;p)}{s+1} \le \frac{2\|f\|_\infty(b-a)J_{\alpha-1,a^+}(b;p)(1+ms)}{s+1}, \tag{2.12}$$

that is

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p)\right| \le M\|f\|_\infty,$$

where $M = \frac{2(b-a)J_{\alpha-1,a^+}(b;p)(1+ms)}{s+1}$. Therefore $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p)$ is bounded also it is easy to see that it is linear, hence this is continuous operator. Also on the other hand from (2.7) we can obtain:

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-}f\right)(x;p)\right| \le K\|f\|_\infty,$$

where $K = \frac{2(b-a)J_{\beta-1,b^-}(a;p)(1+ms)}{s+1}$. Therefore $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-}f\right)(x;p)$ is bounded also it is linear, hence continuous. □

Theorem 3. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a real valued function. If f is differentiable and $|f'|$ is (s,m)-convex, then for $\alpha,\beta \ge 1$, the following fractional integral inequality for generalized integral operators (1.3) and (1.4) holds:*

$$\begin{aligned}
&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(x;p)\right. \\
&\left.-\left(J_{\alpha-1,a^+}(x;p)f(a) + J_{\beta-1,b^-}(x;p)f(b)\right)\right| \\
&\le \left(\frac{|f'(a)| + ms|f'(x)|}{s+1}\right)(x-a)J_{\alpha-1,a^+}(x;p) \\
&+ \left(\frac{|f'(b)| + ms|f'(x)|}{s+1}\right)(b-x)J_{\beta-1,b^-}(x;p),\ x \in [a,b].
\end{aligned} \tag{2.13}$$

Proof. As $x \in [a,b]$ and $t \in [a,x)$, by using (s,m)-convexity of $|f'|$, we have

$$|f'(t)| \le \left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(1 - \left(\frac{t-a}{x-a}\right)^s\right)|f'(x)|. \tag{2.14}$$

From (2.14), one can has

$$f'(t) \le \left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(1 - \left(\frac{t-a}{x-a}\right)^s\right)|f'(x)|. \tag{2.15}$$

The product of (2.2) and (2.15), gives the following inequality:

$$\begin{aligned}
&(x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^\mu;p)f'(t)dt \\
&\le (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^\mu;p)\left(\left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(1 - \left(\frac{t-a}{x-a}\right)^s\right)|f'(x)|\right).
\end{aligned} \tag{2.16}$$

After integrating above inequality over $[a, x]$, we get

$$\int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-t)^{\mu};p) f'(t)dt \tag{2.17}$$
$$\leq (x-a)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-a)^{\mu};p)\left(\frac{|f'(a)|}{(x-a)^s}\int_a^x (x-t)^s dt\right.$$
$$\left.+m|f'(x)|\int_a^x \left(1-\left(\frac{t-a}{x-a}\right)^s\right)dt\right)$$
$$= (x-a)^{\alpha} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-a)^{\mu};p)\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right).$$

The left hand side of (2.17) is calculated as follows:

$$\int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-t)^{\mu};p) f'(t)dt, \tag{2.18}$$

put $x - t = z$ that is $t = x - z$, also using the derivative property (1.2) of Mittag-Leffler function, we have

$$\int_0^{x-a} z^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega z^{\mu};p) f'(x-z)dz$$
$$= (x-a)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-a)^{\mu};p) f(a) - \int_0^{x-a} z^{\alpha-2} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega z^{\mu};p) f(x-z)dz,$$

now put $x - z = t$ in second term of the right hand side of the above equation and then using (1.3), we get

$$\int_0^{x-a} z^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega z^{\mu};p) f'(x-z)dz$$
$$= (x-a)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-a)^{\mu};p) f(a) - \left(\epsilon_{\mu,\alpha+1,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x;p).$$

Therefore (2.17) takes the following form:

$$\left(J_{\alpha-1,a^+}(x;p)\right) f(a) - \left(\epsilon_{\mu,\alpha+1,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x;p) \tag{2.19}$$
$$\leq (x-a) J_{\alpha-1,a^+}(x;p)\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right).$$

Also from (2.14), one can has

$$f'(t) \geq -\left(\left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(1-\left(\frac{t-a}{x-a}\right)^s\right)|f'(x)|\right). \tag{2.20}$$

Following the same procedure as we did for (2.15), one can obtain:

$$\left(\epsilon_{\mu,\alpha+1,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x;p) - J_{\alpha-1,a^+}(x;p) f(a) \tag{2.21}$$
$$\leq (x-a) J_{\alpha-1,a^+}(x;p)\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right).$$

From (2.19) and (2.21), we get

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p)-J_{\alpha-1,a^+}(x;p)f(a)\right| \\ &\leq (x-a)J_{\alpha-1,a^+}(x;p)\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right).\end{aligned} \tag{2.22}$$

Now we let $x\in[a,b]$ and $t\in(x,b]$. Then by using (s,m)-convexity of $|f'|$ we have

$$|f'(t)|\leq\left(\frac{t-x}{b-x}\right)^s|f'(b)|+m\left(1-\left(\frac{b-t}{b-x}\right)^s\right)|f'(x)|. \tag{2.23}$$

on the same lines as we have done for (2.2), (2.15) and (2.20) one can get from (2.5) and (1.7), the following inequality:

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(x;p)-J_{\beta-1,b^-}(x;p)f(b)\right| \\ &\leq (b-x)J_{\beta-1,b^-}(x;p)\left(\frac{|f'(b)|+ms|f'(x)|}{s+1}\right).\end{aligned} \tag{2.24}$$

From inequalities (2.22) and (2.24) via triangular inequality (2.13) is obtained. □

Corollary 5. *If we put $\alpha=\beta$ in (2.13), then the following inequality is obtained:*

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,b^-}f\right)(x;p)\right. \\ &\left.-\left(J_{\alpha-1,a^+}(x;p)f(a)+J_{\alpha-1,b^-}(x;p)f(b)\right)\right| \\ &\leq\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right)(x-a)J_{\alpha-1,a^+}(x;p) \\ &+\left(\frac{|f'(b)|+ms|f'(x)|}{s+1}\right)(b-x)J_{\alpha-1,b^-}(x;p),\ x\in[a,b].\end{aligned} \tag{2.25}$$

It is easy to prove the next lemma which will be helpful to produce Hadamard type estimations for the generalized fractional integral operators.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be (s,m)-convex function. If f is $f(\frac{a+mb-x}{m})=f(x)$ and $(s,m)\in[0,1]^2$, then the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right)\leq\frac{(1+m)f(x)}{2^s}. \tag{2.26}$$

Proof. For $t\in[0,1]$ we have

$$\frac{a+mb}{2}=\frac{(1-t)a+mtb}{2}+\frac{ta+m(1-t)b}{2}. \tag{2.27}$$

As f is (s,m)-convex function, we have

$$f\left(\frac{a+mb}{2}\right)\leq\frac{f((1-t)a+mtb)}{2^s}+\frac{mf(\frac{a+mb-x}{m})}{2^s}. \tag{2.28}$$

Let $x = a(1-t) + mtb$. Then we have $a + mb - x = ta + m(1-t)b$.

$$f\left(\frac{a+mb}{2}\right) \leq \frac{f(x)}{2^s} + m\frac{f(\frac{a+mb-x}{m})}{2^s}. \tag{2.29}$$

Hence by using $f(\frac{a+mb-x}{m}) = f(x)$, the inequality (2.26) can be obtained. □

Theorem 4. *Let $f : [a,b] \longrightarrow \mathbb{R}$, $a > b$, be a real valued function. If f is positive, (s,m)-convex and $f(a+mb-x) = f(x)$, then for $\alpha, \beta > 0$, the following inequality holds for generalized fractional integral operators:*

$$\begin{aligned} &\frac{2^s}{1+m} f\left(\frac{a+mb}{2}\right)\left[J_{\beta+1,b^-}(a;p) + J_{\alpha+1,a^+}(b;p)\right] \\ &\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-} f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(b;p) \\ &\leq \left[J_{\beta-1,b^-}(a;p) + J_{\alpha-1,a^+}(b;p)\right](b-a)^2\left(\frac{f(b)+msf(a)}{s+1}\right). \end{aligned} \tag{2.30}$$

Proof. For $x \in [a,b]$, we have

$$(x-a)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^\mu;p) \leq (b-a)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-a)^\mu;p), \beta > 0. \tag{2.31}$$

As f is (s,m)-convex so for $x \in [a,b]$, we have:

$$f(x) \leq \left(\frac{x-a}{b-a}\right)^s f(b) + m\left(1 - \left(\frac{b-x}{b-a}\right)^s\right)f(a). \tag{2.32}$$

By multiplying (2.31) and (2.32) and then integrating over $[a,b]$, we get

$$\begin{aligned} &\int_a^b (x-a)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^\mu;p)f(x)dx \\ &\leq (b-a)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-a)^\mu;p)\left(\frac{f(b)}{(b-a)^s}\int_a^b (x-a)^s dx + mf(a)\int_a^b \left(1 - \left(\frac{b-x}{b-a}\right)^s\right)dx\right). \end{aligned}$$

From which we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-} f\right)(a;p) \leq (b-a)^{\beta+1} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-a)^\mu;p)\left(\frac{f(b)+msf(a)}{s+1}\right), \tag{2.33}$$

that is

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-} f\right)(a;p) \leq (b-a)^2 J_{\beta-1,b^-}(a;p)\left(\frac{f(b)+msf(a)}{s+1}\right). \tag{2.34}$$

Now on the other hand for $x \in [a,b]$, we have

$$(b-x)^\alpha E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^\mu;p) \leq (b-a)^\alpha E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-a)^\mu;p),\ \alpha > 0. \tag{2.35}$$

By multiplying (2.32) and (2.35) and then integrating over $[a,b]$, we get

$$\int_a^b (b-x)^\alpha E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^\mu;p)f(x)dx$$

$$\leq (b-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-a)^{\mu};p)\left(\frac{f(b)}{(b-a)^s}\int_a^b (x-a)^s dx + mf(a)\int_a^b \left(1-\left(\frac{b-x}{b-a}\right)^s\right)dx\right).$$

From which we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(b;p) \leq (b-a)^{\alpha+1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-a)^{\mu};p)\left(\frac{f(b)+msf(a)}{s+1}\right), \tag{2.36}$$

that is

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(b;p) \leq (b-a)^2 J_{\alpha-1,a^+}(b;p)\left(\frac{f(b)+msf(a)}{s+1}\right). \tag{2.37}$$

Adding (2.34) and (2.37), we get;

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(b;p) \\ &\leq \left[J_{\beta-1,b^-}(a;p) + J_{\alpha-1,a^+}(b;p)\right](b-a)^2\left(\frac{f(b)+msf(a)}{s+1}\right).\end{aligned} \tag{2.38}$$

Multiplying (2.26) with $(x-a)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^{\mu};p)$ and integrating over $[a,b]$, we get

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)\int_a^b (x-a)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^{\mu};p)dx \\ &\leq \frac{1+m}{2^s}\int_a^b (x-a)^{\beta}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^{\mu};p)f(x)dx.\end{aligned} \tag{2.39}$$

By using (1.4) and (1.7), we get

$$f\left(\frac{a+mb}{2}\right)J_{\beta+1,b^-}(a;p) \leq \frac{1+m}{2^s}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(a;p). \tag{2.40}$$

By multiplying (2.26) with $(b-x)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^{\mu};p)$ and integrating over $[a,b]$, also using (1.3) and (1.7), we get

$$f\left(\frac{a+mb}{2}\right)J_{\alpha+1,a^+}(b;p) \leq \frac{1+m}{2^s}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(b;p). \tag{2.41}$$

By adding (2.40) and (2.41), we get;

$$\begin{aligned}&\frac{2^s}{1+m}f\left(\frac{a+mb}{2}\right)\left[J_{\beta+1,b^-}(a;p) + J_{\alpha+1,a^+}(b;p)\right] \\ &\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(b;p).\end{aligned} \tag{2.42}$$

By combining (2.38) and (2.42), inequality (2.30) can be obtained. □

Corollary 6. *If we put $\alpha = \beta$ in (2.30), then the following inequality is obtained:*

$$\begin{aligned}&\frac{2^s}{1+m}f\left(\frac{a+mb}{2}\right)[J_{\alpha+1,b^-}(a;p) + J_{\alpha+1,a^+}(b;p)] \\ &\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,b^-}f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(b;p) \\ &\leq [J_{\alpha-1,b^-}(a;p) + J_{\alpha-1,a^+}(b;p)](b-a)^2\left(\frac{f(b)+msf(a)}{s+1}\right).\end{aligned} \tag{2.43}$$

3. Conclusion

This work deals with the boundedness of generalized fractional integral operators given in (1.3) and (1.4), by using (s, m)-convex functions. The results of this paper provide the boundedness and continuity of several known integral operators defined in [25, 26, 28–30]. By applying (s, m)-convexity of functions f and $|f'|$, variable bounds of sum of left and right definitions of these operators are obtained, while by imposing an additional condition a Hadamard inequality is proved. All the results hold for convex, m-convex and s-convex functions and for integral operators given in [25, 26, 28–30]. The reader can obtain results for s-convex functions and for convex functions proved in [11]. The method adopted in this paper can be applied to derive bounds of other kinds of well known integral operators already exist in literature.

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Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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