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Research article

Radial stationary solutions to a class of wave system as well as their asymptotical behavior

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Abstract: For a stationary version to a class of wave system

$$\begin{cases} -\left(a_1 + b_1 \int_{\mathbb{R}^3} |\nabla u|^2 dx + c \int_{\mathbb{R}^3} |\nabla v|^2 dx\right) \Delta u + u = \frac{p}{Q}|u|^{p-2}u|v|^q, \\ -\left(a_2 + b_2 \int_{\mathbb{R}^3} |\nabla v|^2 dx + c \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta v + v = \frac{q}{Q}|u|^p|v|^{q-2}v, \end{cases}$$

$u, v \in H_r^1(\mathbb{R}^3)$, by establishing a variant variational identity and constraint set, we prove that for $a_s > 0$, $b_s > 0$, $(s = 1,\ 2)$, $c \geq 0$ and $p > 1$, $q > 1$ with $Q := p+q \in (2, 6)$, the system admits a positive radially symmetric ground state solution in $H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$. Moreover, for any fixed $a_1 > 0$ and $a_2 > 0$, as $b_1^2 + b_2^2 + c^2 \to 0$, this solution converges to a positive radially symmetric solution to

$$-a_1\Delta u + u = \frac{p}{Q}|u|^{p-2}u|v|^q, \quad -a_2\Delta v + v = \frac{q}{Q}|u|^p|v|^{q-2}v, \quad u,\ v \in H_r^1(\mathbb{R}^3).$$

Keywords: system with Kirchhoff term; radially symmetric solutions; variant variational identity
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1. Introduction and main result

Let $H^1(\mathbb{R}^3) := W^{1,2}(\mathbb{R}^3)$ be the usual Hilbert space. Denoted by $H_r^1(\mathbb{R}^3)$ the subspace of $H^1(\mathbb{R}^3)$ which contains all radially symmetric functions in $H^1(\mathbb{R}^3)$. In this paper we are concerned with the existence of positive radial solutions to the following coupled Kirchhoff type system:

$$\begin{cases} -\left(a_1 + b_1 \int_{\mathbb{R}^3} |\nabla u|^2 dx + c \int_{\mathbb{R}^3} |\nabla v|^2 dx\right) \Delta u + u = \dfrac{p}{Q} |u|^{p-2} u |v|^q, \\ -\left(a_2 + b_2 \int_{\mathbb{R}^3} |\nabla v|^2 dx + c \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta v + v = \dfrac{q}{Q} |u|^p |v|^{q-2} v, \\ u, v \in H_r^1(\mathbb{R}^3), \quad u := u(x),\ v := v(x),\ x \in \mathbb{R}^3 \end{cases} \tag{1.1}$$

where $a_1 > 0, a_2 > 0; b_1 > 0, b_2 > 0, c \geq 0$ and $p > 1, q > 1$ with $2 < Q := p + q < 6$. The number 6 is the critical exponent of the embedding $H^1(\mathbb{R}^3) \to L^s(\mathbb{R}^3)$ $(2 \leq s \leq 6)$.

Such kind of Kirchhoff system is originated from the study of Kirchhoff string [8] and recent work by Matsuyama and Ruzhansky [20]. In [20], the authors propose the following Kirchhoff systems

$$\begin{cases} \partial_t^2 \varphi_1 - \rho_1 \left(1 + \int_{\mathbb{R}^3} |\nabla \varphi_1|^2 dx + c \int_{\mathbb{R}^3} |\nabla \varphi_2|^2 dx\right) \Delta \varphi_1 + P_1(x, D_x) \varphi_1 = 0, \\ \partial_t^2 \varphi_2 - \rho_1 \left(1 + \int_{\mathbb{R}^3} |\nabla \varphi_2|^2 dx + c \int_{\mathbb{R}^3} |\nabla \varphi_1|^2 dx\right) \Delta \varphi_2 + P_2(x, D_x) \varphi_2 = 0 \\ \varphi_1 := \varphi_1(t, x), \quad \varphi_2 := \varphi_2(t, x), \quad t > 0,\ x \in \mathbb{R}^3. \end{cases} \tag{$Ksys$}$$

In [20], analytic methods are used to study the Cauchy problem of $(Ksys)$. Stationary version related to $(Ksys)$ with nonlinear perturbation has attracted more and more attentions. In [26], the authors study the existence of solutions to the following system

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x) u = \dfrac{\partial F(x, u, v)}{\partial u}, \\ -\left(c + d \int_{\mathbb{R}^N} |\nabla v|^2 dx\right) \Delta v + V(x) v = \dfrac{\partial F(x, u, v)}{\partial v}, \\ u(x) \to 0 \quad \text{and} \quad v(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

Besides some other conditions, the authors [26] assume that the nonlinear perturbation $F(x, u, v)$ satisfies "4-AR condition" for system in the sense that

$$u \frac{\partial F(x, u, v)}{\partial u} + v \frac{\partial F(x, u, v)}{\partial v} \geq 4 F(x, u, v), \quad \forall\ x \in \mathbb{R}^N,\ (u, v) \in \mathbb{R}^2.$$

We point out that a natural and important case of $F(x, u, v) = |u|^p |v|^q$ with $p > 1, q > 1$ and $p + q < 4$ was *not* covered by the results of [26]. In [19], the author considers the following system

$$\begin{cases} -\left(\epsilon^2 a + \epsilon b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x) u = \dfrac{1}{\mu} \dfrac{\partial F(u, v)}{\partial u}, \\ -\left(\epsilon^2 a + \epsilon b \int_{\mathbb{R}^N} |\nabla v|^2 dx\right) \Delta v + V(x) v = \dfrac{1}{\mu} \dfrac{\partial F(u, v)}{\partial v}, \\ u, \quad v \in H^1(\mathbb{R}^N). \end{cases}$$

The author assumes that $\mu > 4$ and for $s > 0$, $F(su, sv) = s^\mu F(u, v)$ and prove the existence of solutions by variational methods. Again, the case of $F(u, v) = |u|^p |v|^q$ with $p > 1, q > 1$ and $p + q \leq 4$ is not studied, either.

The main purpose of the present paper is to study the existence of positive radial solutions to (1.1) for all $p > 1$, $q > 1$ with $2 < p+q < 6$. By a solution to (1.1), we mean a critical point of the following functional

$$\mathcal{I}(u,v) = \frac{1}{2}\int\left(a_1|\nabla u|^2 + a_2|\nabla v|^2\right)dx + \frac{1}{2}\int\left(|u|^2 + |v|^2\right)dx - \frac{1}{Q}\int |u|^p|v|^q dx$$
$$+ \frac{1}{4}\left(b_1\left(\int |\nabla u|^2 dx\right)^2 + b_2\left(\int |\nabla v|^2 dx\right)^2 + 2c\int |\nabla u|^2 dx \int |\nabla v|^2 dx\right)$$

defined on $H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$. According to the Sobolev embedding theorem, the functional $\mathcal{I}$ is well defined and C^1. It is easily to see that $(0,0)$ is a solution to (1.1), which is usually called trivial solution. We call (u,v) a semitrivial solution if $\mathcal{I}'(u,v) = (0,0)$ and $u \neq 0$, $v = 0$ or $u = 0$, $v \neq 0$. If $\mathcal{I}'(u,v) = (0,0)$ and $u \neq 0$, $v \neq 0$, then (u,v) is called a nontrivial solution. If (u,v) is a solution to (1.1) and $u > 0$, $v > 0$, then (u,v) is called a positive solution.

Definition 1.1. *Denote $H := H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$. A nontrivial solution $(u,v) \in H$ is called a positive radially symmetric ground state solution to (1.1) if $\mathcal{I}(u,v) \leq \mathcal{I}(\bar{u},\bar{v})$ for any $(\bar{u},\bar{v}) \in H\backslash\{(0,0)\}$ and $\mathcal{I}'(\bar{u},\bar{v}) = (0,0)$.*

The first result of the present paper is the following theorem.

Theorem 1.2. *Assume that $a_1 > 0$, $a_2 > 0$, $b_1 > 0$, $b_2 > 0$, $c \geq 0$; $p > 1$, $q > 1$ and $2 < p+q < 6$. Then the system (1.1) has a positive radially symmetric ground state solution $(u,v) \in H$.*

We emphasize that, as a corollary of Theorem 1.2, when $c = 0$, Theorem 1.2 generalizes results of [19,26] in the sense that we can get the existence of solutions in the case of $F(u,v) = |u|^p|v|^q$, $p > 1$, $q > 1$ and $2 < p+q < 4$.

We also point out that, when $a_1 = a_2$, $b_1 = b_2$, $c = 0$ and $u = v$, (1.1) becomes a semilinear elliptic equation with Kirchhoff term. In the past ten years, a lot of mathematicians have made contributions to the existence and multiplicity of solutions to

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^3,\ u \in H^1(\mathbb{R}^3), \tag{SK}$$

see e. g. [1,7,9,10,12,14–16,25] and some related background [6,13,17]. In the process of studying (SK), to overcome the difficulties created by the Kirchhoff trem, authors usually need to assume that $f(x,u)$ satisfies either 4−superlinear at infinity in the sense that

$$\lim_{|u|\to+\infty} \frac{\int_0^u f(x,s)ds}{|u|^4} = +\infty \quad \text{uniformly in} \quad x \in \mathbb{R}^3$$

or satisfies '4-Ambrosetti-Rabinowitz condition' with the version
(AR) there is $\mu > 4$ such that $0 < \mu\int_0^u f(x,s)ds \leq f(x,u)u$ for all $u \neq 0$.
Recently, for single equation with a Kirchhoff term, such kind of results has been extended to fractional Kirchhoff type equation or p−Kirchhoff equations, see e. g. [2,4,5,21,22] as well as the references therein. Therefore a special case (i. e., $a_1 = a_2$, $b_1 = b_2$ and $u = v$) of Theorem 1.2 can be a complement to the results of [11,12,18,24] since we can get solutions for nonlinear growth in the full range of $(2,6)$.

Noticing that when $b_1 = 0$, $b_2 = 0$ and $c = 0$, (1.1) is the usual semilinear elliptic system. While for $b_1, b_2 > 0$ and $c > 0$, any solution to (1.1) usually depends on b_1, b_2 and c. A natural and interesting question is: what is the asymptotical behavior of the solution (u, v) as $|b_1|^2 + |b_2|^2 + |c|^2 \to 0$. To answer this question precisely, we denote this (u, v) by $(u_{b_1,b_2,c}, v_{b_1,b_2,c})$ and the (1.1) by $(1.1)_{b_1,b_2,c}$. Then we can prove the following theorem.

Theorem 1.3. *Let $p > 1$, $q > 1$ and $2 < p + q < 6$. For any fixed $a_1 > 0$ and $a_2 > 0$, if sequences $b_1^{(n)} > 0$, $b_2^{(n)} > 0$, $c^{(n)} \geq 0$, and $|b_1^{(n)}|^2 + |b_2^{(n)}|^2 + |c^{(n)}|^2 \to 0$ as $n \to \infty$, then the sequence of solutions $(u_{b_1^{(n)},b_2^{(n)},c^{(n)}}, v_{b_1^{(n)},b_2^{(n)},c^{(n)}})$ to $(1.1)_{b_1^{(n)},b_2^{(n)},c^{(n)}}$ converges to a positive radially symmetric solution to $(1.1)_{0,0,0}$.*

In order to prove Theorem 1.2 and Theorem 1.3, we use variational methods. But the methods used in [11, 12, 19, 26] to deal with the case of 4−superlinear or 4−(AR) condition can not be applied here (since our $p+q$ may be less than 4). The main strategy of the present paper is to establish a constrained set $\mathcal{M}$ and then minimize the functional $\mathcal{I}$ over $\mathcal{M}$, see (2.1) in Section 2. And we manage to prove that the minimum d of $\mathcal{I}|_{\mathcal{M}}$ can be solved and the minimizer is a solution to (1.1) as required. This idea is inspired from Ruiz [23] where the author studied a class of Schrödinger-Poisson system. But in the present paper, we have to overcome the Kirchhoff term and the coupling between u and v.

This paper is organized as follows. In Section 2, we study the variational structure of (1.1) and establish a set $\mathcal{M}$, proving that this set is indeed a manifold and can be a natural constraint of the functional $\mathcal{I}$, see Lemma 2.6 and Lemma 2.7. In Section 3, we prove Theorem 1.2. In Section 4, we study the asymptotical behavior of solutions obtained in Theorem 1.2 with respect to b_1, b_2 and c, where we finish the proof of Theorem 1.3.

Notation. Throughout the paper, all integrals are taken over $\mathbb{R}^3$ unless specified. $L^s(\mathbb{R}^3)$ $(1 \leq s < +\infty)$ is the usual Lebesgue space with the standard norm $|u|_s$. For $a_1 > 0$, $a_2 > 0$, we use a norm on H: $\|u\|^2 := \int \left(a_1|\nabla u|^2 + u^2 + a_2|\nabla v|^2 + v^2\right) dx$ whose inner product is $\langle (u_1, v_1), (u_2, v_2)\rangle = \int (a_1\nabla u_1\nabla u_2 + u_1u_2 + a_2\nabla v_1\nabla v_2 + v_1v_2)\, dx$.

2. Variational structure

In this section, we establish variational framework of (1.1). Since we only assume that $p > 1$ and $q > 1$ and $2 < Q := p + q < 6$, the standard Nehari type constraint can not be applied here. As we pointed out in the introduction, our strategy is to construct a set $\mathcal{M}$ which is a manifold and prove that this set can be a natural constraint. Then we define a suitable minimiztion problem and prove that the minimizer can be achieved. We start with the following Pohozaev identity.

Lemma 2.1. *Let $(u, v) \in H$ be a solution to (1.1). Then $\mathcal{P}(u, v) = 0$, where*

$$\mathcal{P}(u, v) := \int \left(a_1|\nabla u|^2 + a_2|\nabla v|^2\right) dx + 3\int \left(|u|^2 + |v|^2\right) dx - \frac{6}{Q}\int |u|^p|v|^q dx + \left(b_1\left(\int |\nabla u|^2 dx\right)^2 + 2c\int |\nabla u|^2 dx \int |\nabla v|^2 dx + b_2\left(\int |\nabla v|^2 dx\right)^2\right).$$

Proof. We only prove it formally. Let $(u, v) \in H$ be a solution to (1.1). Multiplying the first equation of the system (1.1) by $x \cdot \nabla u$ and the second equation by $x \cdot \nabla v$ respectively, and integrating by parts, we get that

$$\left(a_1 + b_1 \int_{\mathbb{R}^3} |\nabla u|^2 dx + c \int_{\mathbb{R}^3} |\nabla v|^2 dx\right) \int \nabla u \nabla (x \cdot \nabla u) dx + \int u(x \cdot \nabla u) dx$$
$$= \frac{p}{Q} \int |u|^{p-2} u |v|^q (x \cdot \nabla u) dx;$$
$$\left(a_2 + b_2 \int_{\mathbb{R}^3} |\nabla v|^2 dx + c \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \int \nabla v \nabla (x \cdot \nabla v) dx + \int v(x \cdot \nabla v) dx$$
$$= \frac{q}{Q} \int |u|^p |v|^{q-2} v (x \cdot \nabla v) dx.$$

By simple computation, we also have that

$$\int \nabla u \nabla (x \cdot \nabla u) dx = -\frac{1}{2} \int |\nabla u|^2 dx, \qquad \int \nabla v \nabla (x \cdot \nabla v) dx = -\frac{1}{2} \int |\nabla v|^2 dx,$$
$$\int u(x \cdot \nabla u) dx = -\frac{3}{2} \int |u|^2 dx, \qquad \int v(x \cdot \nabla v) dx = -\frac{3}{2} \int |v|^2 dx,$$

and

$$p \int |u|^{p-2} u |v|^q (x \cdot \nabla u) dx = -3 \int |u|^p |v|^q dx - q \int |u|^p |v|^{q-2} v (x \cdot \nabla v) dx.$$

Combining the above equalities, we deduce that

$$\int \left(a_1 |\nabla u|^2 + a_2 |\nabla v|^2\right) dx + 3 \int \left(|u|^2 + |v|^2\right) dx - \frac{6}{Q} \int |u|^p |v|^q dx$$
$$+ \left(b_1 \left(\int |\nabla u|^2 dx\right)^2 + 2c \int |\nabla u|^2 dx \int |\nabla v|^2 dx + b_2 \left(\int |\nabla v|^2 dx\right)^2\right) = 0.$$

The proof is complete. □

Next, we define $\mathcal{G}(u, v) := \frac{1}{4}\langle \mathcal{I}'(u, v), (u, v)\rangle + \frac{1}{4}\mathcal{P}(u, v)$. Then

$$\mathcal{G}(u, v) := \frac{1}{2}\mathcal{A}(u, v) + C(u, v) + \frac{1}{2}\mathcal{B}(u, v) - \frac{Q+6}{4Q} \int |u|^p |v|^q dx,$$

where $\mathcal{A}(u, v) := \int \left(a_1 |\nabla u|^2 + a_2 |\nabla v|^2\right) dx$, $C(u, v) := \int \left(|u|^2 + |v|^2\right) dx$ and $\mathcal{B}(u, v) := b_1 \left(\int |\nabla u|^2 dx\right)^2 + 2c \int |\nabla u|^2 dx \int |\nabla v|^2 dx + b_2 \left(\int |\nabla v|^2 dx\right)^2$. Define

$$\mathcal{M} := \{(u, v) \in H \backslash \{(0, 0)\} : \mathcal{G}(u, v) = 0\}.$$

Clearly if $(u, 0) \in \mathcal{M}$, then $u = 0$ and if $(0, v) \in \mathcal{M}$, then $v = 0$. Hence

$$\mathcal{M} = \{(u, v) \in H \ : \ \mathcal{G}(u, v) = 0 \text{ and } u \neq 0, \ v \neq 0\}.$$

We define the following minimization problem

$$d := \inf\{\mathcal{I}(u, v) \ : \ (u, v) \in \mathcal{M}\}. \tag{2.1}$$

To study this minimization problem, we firstly characterize the properties of the constrained set $\mathcal{M}$.

Proposition 2.2. *Suppose the conditions of Theorem 1.2 hold. For any* $u,\ v \in H_r^1(\mathbb{R}^3)\backslash\{0\}$*, there is a unique* $t := t(u,v) > 0$ *such that* $(u^t, v^t) \in \mathcal{M}$*, where* $u^t(x) := t^{\frac{1}{4}}u(t^{-\frac{1}{2}}x)$*,* $v^t(x) := t^{\frac{1}{4}}v(t^{-\frac{1}{2}}x)$*. Particularly, the* $\mathcal{M}$ *is not empty.*

Before proving Proposition 2.2, we give a lemma.

Lemma 2.3. *Let* $\alpha, \beta, \gamma, \delta$ *be positive constants and* $Q \in (2,6)$*. For* $t \geq 0$*, we define* $f(t) := \alpha t + \beta t^2 + \gamma t^2 - \delta t^{\frac{Q+6}{4}}$*. Then* f *has a unique critical point which corresponds to its maximum.*

Proof. For $t \geq 0$, we compute directly that

$$f'(t) = \alpha + 2\beta t + 2\gamma t - \frac{Q+6}{4}\delta t^{\frac{Q+2}{4}},$$
$$f''(t) = 2\beta + 2\gamma - \frac{Q+6}{4}\frac{Q+2}{4}\delta t^{\frac{Q-2}{4}}.$$

Since f'' is strictly decreasing and $f''(0) = 2\beta + 2\gamma > 0$, there exists $t_2 > 0$ such that $f''(t_2) = 0$ and $f''(t)(t_2 - t) > 0$ for $t \neq t_2$.

Since $f'(0) = \alpha > 0$ and f' is increasing for $t < t_2$, f' takes positive values at least for $t \in [0, t_2]$. For $t > t_2$, f' decreases, tending to $-\infty$. Then there exists $t_0 > t_2$ such that $f'(t_0) = 0$ and $f'(t)(t_0 - t) > 0$ for $t \neq t_0$.

In conclusion, t_0 is the unique critical point of f and corresponds to its maximum as $\frac{Q+6}{4} > 2$. □

We are now in a position to prove Proposition 2.2.

Proof of Proposition 2.2. For any $u, v \in H_r^1(\mathbb{R}^3)\backslash\{0\}$ and any $t > 0$, we define $u^t(x) := t^{\frac{1}{4}}u(t^{-\frac{1}{2}}x)$ and $v^t(x) := t^{\frac{1}{4}}v(t^{-\frac{1}{2}}x)$. Then by direct computation, there hold $\mathcal{A}(u^t, v^t) = t\mathcal{A}(u,v)$, $C(u^t, v^t) = t^2 C(u,v)$, $\mathcal{B}(u^t, v^t) = t^2\mathcal{B}(u,v)$ and $\int |u^t|^p |v^t|^q dx = t^{\frac{Q+6}{4}} \int |u|^p |v|^q dx$. Therefore

$$\mathcal{I}(u^t, v^t) = \frac{t}{2}\mathcal{A}(u,v) + \frac{t^2}{2}C(u,v) + \frac{t^2}{4}\mathcal{B}(u,v) - \frac{1}{Q}t^{\frac{Q+6}{4}} \int |u|^p|v|^q dx.$$

Denote $g(t) := \mathcal{I}(u^t, v^t)$. Then g is positive for small t and tends to $-\infty$ as $t \to +\infty$ because $\frac{Q+6}{4} > 2$. From Lemma 2.3, $g(t)$ has a unique critical point $t(u,v)$ (here and after, $t(u,v)$ means t depends on u and v), corresponding to its maximum. Denoting this $t(u,v)$ by t_0, then we have that

$$g'(t_0) = \frac{1}{2}\mathcal{A}(u,v) + t_0 C(u,v) + \frac{t_0}{2}t_0\mathcal{B}(u,v) - \frac{Q+6}{4Q}t_0^{\frac{Q+2}{4}} \int |u|^p|v|^q dx = 0.$$

Moreover from

$$\begin{aligned}\mathcal{G}(u^{t_0}, v^{t_0}) &= \frac{1}{2}\mathcal{A}(u^{t_0}, v^{t_0}) + C(u^{t_0}, v^{t_0}) + \frac{1}{2}\mathcal{B}(u^{t_0}, v^{t_0}) - \frac{Q+6}{4Q}\int |u^{t_0}|^p|v^{t_0}|^q dx \\ &= t_0 g'(t_0) = 0,\end{aligned}$$

we deduce that $(u^{t_0}, v^{t_0}) \in \mathcal{M}$. This proves the proposition and particularly $\mathcal{M}$ is not empty. □

Remark 2.4. *From the definition of* $\mathcal{G}(u,v)$ *and Lemma 2.1, we know that if* (u,v) *is a solution to (1.1), then the unique* $t(u,v)$ *defined as above satisfies* $t(u,v) = 1$.

Lemma 2.5. *For any $u, v \in H_r^1(\mathbb{R}^3)\backslash\{0\}$, if $\mathcal{G}(u, v) < 0$, then $t(u, v) \in (0, 1)$, where the $t(u, v)$ is defined as in the proof of Proposition 2.2.*

Proof. For $u, v \in H_r^1(\mathbb{R}^3)\backslash\{0\}$, and let $t_0 := t(u, v)$ be defined by Proposition 2.2. From

$$\mathcal{G}(u, v) = \frac{1}{2}\mathcal{A}(u, v) + C(u, v) + \frac{1}{2}\mathcal{B}(u, v) - \frac{Q+6}{4Q}\int |u|^p|v|^q dx < 0$$

and

$$\mathcal{G}(u^{t_0}, v^{t_0}) = \frac{t_0}{2}\mathcal{A}(u, v) + t_0^2 C(u, v) + \frac{t_0^2}{2}\mathcal{B}(u, v) - \frac{Q+6}{4Q}t_0^{\frac{Q+6}{4}}\int |u|^p|v|^q dx = 0,$$

we obtain that

$$\frac{1}{2}\left(t_0^{\frac{p+6}{4}} - t_0\right)\mathcal{A}(u, v) + \left(t_0^{\frac{p+6}{4}} - t_0^2\right)C(u, v) + \frac{1}{2}\left(t_0^{\frac{p+6}{4}} - t_0^2\right)\mathcal{B}(u, v) < 0,$$

which implies $t_0 < 1$. Therefore $t_0 := t(u, v) \in (0, 1)$. □

Lemma 2.6. *Suppose the conditions of Theorem 1.2 hold. Then the $\mathcal{M}$ is bounded away from zero and $\mathcal{M}$ is a C^1 manifold.*

Proof. Firstly, for any $(u, v) \in \mathcal{M}$, we deduce from $\mathcal{G}(u, v) = 0$ and Sobolev inequality that there is $M_1 > 0$,

$$\begin{aligned}\frac{1}{2}\mathcal{A}(u, v) + C(u, v) &\le \frac{1}{2}\mathcal{A}(u, v) + C(u, v) + \frac{1}{2}\mathcal{B}(u, v)\\ &= \frac{Q+6}{4Q}\int |u|^p|v|^q dx \le M_1\left(\mathcal{A}(u, v) + C(u, v)\right)^{\frac{Q}{2}}.\end{aligned}$$

Hence there is $M_2 > 0$ such that $\mathcal{A}(u, v) + C(u, v) \ge M_2$. This proves that $\mathcal{M}$ is bounded away from zero.

Secondly, we will prove that for any $(u, v) \in \mathcal{M}$, $\mathcal{G}'(u, v) \neq (0, 0)$. Arguing by a contradiction, we assume that there is $(u_0, v_0) \in \mathcal{M}$ such that $\mathcal{G}'(u_0, v_0) = (0, 0)$. Then in a weak sense, (u_0, v_0) satisfies

$$\begin{cases} -\left(a_1 + 2b_1\int |\nabla u_0|^2 dx + 2c\int |\nabla v_0|^2 dx\right)\Delta u_0 + 2u_0 = \dfrac{p(Q+6)}{4Q}|u_0|^{p-2}u_0|v_0|^q, \\ -\left(a_2 + 2b_2\int |\nabla v_0|^2 dx + 2c\int |\nabla u_0|^2 dx\right)\Delta v_0 + 2v_0 = \dfrac{q(Q+6)}{4Q}|u_0|^p|v_0|^{q-2}v_0, \end{cases} \tag{2.2}$$

Setting $h_0 := \mathcal{I}(u_0, v_0)$ and $i := \int\left(a_1|\nabla u_0|^2 + a_2|\nabla v_0|^2\right)dx$, $j := \int\left(|u_0|^2 + |v_0|^2\right)dx$, $k := b_1\left(\int |\nabla u_0|^2 dx\right)^2 + 2c\int |\nabla u_0|^2 dx\int |\nabla v_0|^2 dx + b_2\left(\int |\nabla v_0|^2 dx\right)^2$ and $e := \int |u_0|^p|v_0|^q dx$. Then we have from $h_0 = \mathcal{I}(u_0, v_0)$ that

$$\frac{1}{2}i + \frac{1}{2}j + \frac{1}{4}k - \frac{1}{Q}e = h_0. \tag{2.3}$$

The $(u_0, v_0) \in \mathcal{M}$ implies that

$$\frac{1}{2}i + j + \frac{1}{2}k - \frac{Q+6}{4Q}e = 0. \tag{2.4}$$

Multiplyling the first equation in the system (2.2) by u and the second equation in (2.2) by v, respectively, and integrating by parts, we obtain that

$$\begin{aligned}
&\left(a_1 + 2b_1\int|\nabla u_0|^2dx + 2c\int|\nabla v_0|^2dx\right)\int|\nabla u_0|^2dx + 2\int|u_0|^2dx \\
&\quad = \frac{p(Q+6)}{4Q}\int|u_0|^p|v_0|^qdx, \\
&\left(a_2 + 2b_2\int|\nabla v_0|^2dx + 2c\int|\nabla u_0|^2dx\right)\int|\nabla v_0|^2dx + 2\int|v_0|^2dx \\
&\quad = \frac{q(Q+6)}{4Q}\int|u_0|^p|v_0|^q,
\end{aligned}$$

Hence one gets that

$$i + 2j + 2k - \frac{Q+6}{4}e = 0. \tag{2.5}$$

Since (u_0, v_0) is a weak solution of (2.2), we deduce by a Pohozaev type argument and use the definition of i, j, k and e that

$$\frac{1}{2}i + 3j + k - \frac{3(Q+6)}{4Q}e = 0. \tag{2.6}$$

Now solving the equations (2.3), (2.4), (2.5) and (2.6) as the following: multiplying (2.6) by 2 and minus (2.5), one deduces that

$$4j + \left(\frac{Q+6}{4} - \frac{6(Q+6)}{4Q}\right)e = 0. \tag{2.7}$$

Multiplying (2.4) by 4 and minus (2.5), one has that

$$i + 2j = \frac{Q+6}{Q} - \frac{Q+6}{4}e. \tag{2.8}$$

It is now deduced from (2.7) and (2.8) that

$$i = \frac{2-Q}{8Q}(Q+6)e < 0, \tag{2.9}$$

which is a contradiction because $i > 0$, $e > 0$ and $Q > 0$. This proves the lemma. □

Lemma 2.7. *Under the conditions of Theorem 1.2, the $\mathcal{M}$ is a natural constraint in the following sense: if $(u_0, v_0) \in \mathcal{M}$ is a critical point of $\mathcal{I}|_{\mathcal{M}}$, then the $(u_0, v_0) \in \mathcal{M}$ is also a critical point of $\mathcal{I}$ on H.*

Proof. Suppose that $(u_0, v_0) \in \mathcal{M}$ is a critical point of $\mathcal{I}|_{\mathcal{M}}$, then in a weak sense, there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $\mathcal{I}'(u_0, v_0) = \lambda\mathcal{G}'(u_0, v_0)$. Therefore in a weak sense, the (u_0, v_0) satisfies

$$\begin{cases}
-\left(a_1 + b_1\int|\nabla u_0|^2dx + c\int|\nabla v_0|^2dx\right)\Delta u_0 + u_0 - \frac{p}{Q}|u_0|^{p-2}u_0|v_0|^q \\
= \lambda\left(-\left(a_1 + 2b_1\int|\nabla u_0|^2dx + 2c\int|\nabla v_0|^2dx\right)\Delta u_0 + 2u_0 - \frac{p(Q+6)}{4Q}|u_0|^{p-2}u_0|v_0|^q\right), \\
-\left(a_2 + b_2\int|\nabla v_0|^2dx + c\int|\nabla u_0|^2dx\right)\Delta v_0 + v_0 - \frac{q}{Q}|u_0|^p|v_0|^{q-2}v_0 \\
= \lambda\left(-\left(a_2 + 2b_2\int|\nabla v_0|^2dx + 2c\int|\nabla u_0|^2dx\right)\Delta v_0 + 2v_0 - \frac{q(Q+6)}{4Q}|u_0|^p|v_0|^{q-2}v_0\right).
\end{cases} \tag{2.10}$$

Which is equivalent to the following system

$$\begin{cases} -\left((\lambda-1)a_1+(2\lambda-1)b_1\displaystyle\int|\nabla u_0|^2dx+(2\lambda-1)c\int|\nabla v_0|^2dx\right)\Delta u_0 \\ \qquad +(2\lambda-1)u_0=\left(\dfrac{p(Q+6)}{4Q}\lambda-\dfrac{p}{Q}\right)|u_0|^{p-2}u_0|v_0|^q, \\ -\left((\lambda-1)a_2+(2\lambda-1)b_2\displaystyle\int|\nabla v_0|^2dx+(2\lambda-1)c\int|\nabla u_0|^2dx\right)\Delta v_0 \\ \qquad +(2\lambda-1)v_0=\left(\dfrac{q(Q+6)}{4Q}\lambda-\dfrac{q}{Q}\right)|u_0|^p|v_0|^{q-2}v_0. \end{cases} \tag{2.11}$$

Claim: $\lambda=0$.

In order to prove this claim, we denote $d_0=\mathcal{I}(u_0,v_0)$. Then Lemma 2.6 implies that $d_0>0$. Set

$$i:=\int\left(a_1|\nabla u_0|^2+a_2|\nabla v_0|^2\right)dx,\qquad j:=\int\left(|u_0|^2+|v_0|^2\right)dx,$$

$$k:=b_1\left(\int|\nabla u_0|^2dx\right)^2+2c\int|\nabla u_0|^2dx\int|\nabla v_0|^2dx+b_2\left(\int|\nabla v_0|^2dx\right)^2$$

and $e:=\int|u_0|^p|v_0|^qdx$. Firstly, from $d_0=\mathcal{I}(u_0,v_0)$ and $\mathcal{G}(u_0,v_0)=0$, we have that

$$\frac{1}{2}i+\frac{1}{2}j+\frac{1}{4}k-\frac{1}{Q}e=d_0 \tag{2.12}$$

and

$$\frac{1}{2}i+j+\frac{1}{2}k-\frac{Q+6}{4Q}e=0. \tag{2.13}$$

Secondly, multiplying the first equation in the system (2.11) by u_0 and integrating by parts, we have that

$$\left((\lambda-1)a_1+(2\lambda-1)b_1\int_{\mathbb{R}^3}|\nabla u_0|^2dx+(2\lambda-1)c\int_{\mathbb{R}^3}|\nabla v_0|^2dx\right)\int|\nabla u_0|^2dx$$
$$+(2\lambda-1)\int|u_0|^2dx=\left(\frac{p(Q+6)}{4Q}\lambda-\frac{p}{Q}\right)\int|u_0|^p|v_0|^qdx;$$

multiplying the second equation in (2.11)by v_0 and integrating by parts, we get that

$$\left((\lambda-1)a_2+(2\lambda-1)b_2\int_{\mathbb{R}^3}|\nabla v_0|^2dx+(2\lambda-1)c\int_{\mathbb{R}^3}|\nabla u_0|^2dx\right)\int|\nabla v_0|^2dx$$
$$+(2\lambda-1)\int|v_0|^2dx=\left(\frac{q(Q+6)}{4Q}\lambda-\frac{q}{Q}\right)\int|u_0|^p|v_0|^qdx.$$

Combining the above two equalities, we deduce that

$$(\lambda-1)i+(2\lambda-1)j+(2\lambda-1)k-\left(\frac{Q+6}{4}\lambda-1\right)e=0. \tag{2.14}$$

Since (u_0, v_0) is a weak solution of (2.11), we deduce by a Phozaev type argument that

$$\frac{1}{2}(\lambda-1)i+\frac{3}{2}(2\lambda-1)j+\frac{1}{2}(2\lambda-1)k-\frac{3}{Q}\left(\frac{Q+6}{4}\lambda-1\right)e=0. \tag{2.15}$$

Now solving the linear system (2.12), (2.13), (2.14) and (2.15). Denoted the coefficient matrix by M

$$M=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{Q} \\ \frac{1}{2} & 1 & \frac{1}{2} & -\frac{Q+6}{4Q} \\ \lambda-1 & 2\lambda-1 & 2\lambda-1 & 1-\frac{Q+6}{4}\lambda \\ \frac{1}{2}(\lambda-1) & \frac{3}{2}(2\lambda-1) & \frac{1}{2}(2\lambda-1) & \frac{3}{Q}(1-\frac{Q+6}{4}\lambda) \end{pmatrix}.$$

By taking elementary transformation to the matrix, we can deduce the determinent

$$\det M=\frac{(Q+2)(2-Q)}{64Q}\lambda(2\lambda-1).$$

If $\det M \neq 0$, then by Cramer rule, the linear system has a unique solution and

$$e=\frac{d_0}{\det M}\lambda(2\lambda-1)=\frac{64Qd_0}{(Q+2)(2-Q)},$$

which is a contradiction since $Q := p+q > 2$, $d_0 > 0$ and $e > 0$.

Therefore $\det M = 0$, which implies that

$$\lambda = 0 \text{ or } \lambda=\frac{1}{2}.$$

If $\lambda = \frac{1}{2}$, then (2.13) becomes

$$-\frac{1}{2}i-\frac{Q-2}{8}e=0,$$

which is also a contradiction since $Q > 2$, $i > 0$ and $e > 0$. Therefore $\lambda = 0$ and we prove the claim. Hence $\mathcal{I}'(u_0, v_0) = (0, 0)$. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Our strategy is to prove that the minimization problem

$$d=\inf\{\mathcal{I}(u,v)\ :\ (u,v)\in\mathcal{M}\}$$

defined in (2.1) can be solved. And then using Lemma 2.7 to show that the minimizer is a positive solution to (1.1), which is radially symmetric. Keep the definition of the functional $\mathcal{G}(u, v)$ in mind:

$$\mathcal{G}(u,v):=\frac{1}{2}\mathcal{A}(u,v)+C(u,v)+\frac{1}{2}\mathcal{B}(u,v)-\frac{Q+6}{4Q}\int|u|^p|v|^q dx,$$

where $\mathcal{A}(u,v)=\int\left(a_1|\nabla u|^2+a_2|\nabla v|^2\right)dx$, $C(u,v)=\int\left(|u|^2+|v|^2\right)dx$ and $\mathcal{B}(u,v) = b_1\left(\int|\nabla u|^2dx\right)^2 + b_2\left(\int|\nabla v|^2dx\right)^2 + 2c\int|\nabla u|^2dx\int|\nabla v|^2dx$. We will prove the following proposition.

Proposition 3.1. *Under the conditions of Theorem 1.2, the minimum d defined by (2.1) is achieved.*

Proof. Let $(u_n, v_n) \in \mathcal{M}$ be such that $\mathcal{I}(u_n, v_n) \to d$ as $n \to \infty$. Then for n large enough, we have that

$$d + o(1) = \frac{1}{2}\mathcal{A}(u_n, v_n) + \frac{1}{2}C(u_n, v_n) + \frac{1}{4}\mathcal{B}(u_n, v_n) - \frac{1}{Q}\int |u_n|^p|v_n|^q dx; \tag{3.1}$$

$$\frac{1}{2}\mathcal{A}(u_n, v_n) + C(u_n, v_n) + \frac{1}{2}\mathcal{B}(u_n, v_n) - \frac{Q+6}{4Q}\int |u_n|^p|v_n|^q dx = 0. \tag{3.2}$$

Combining (3.1) and (3.2), we deduce that

$$d + o(1) = \left(\frac{1}{2} - \frac{2}{Q+6}\right)\mathcal{A}(u_n, v_n) + \left(\frac{1}{2} - \frac{4}{Q+6}\right)C(u_n, v_n) + \left(\frac{1}{4} - \frac{2}{Q}\right)\mathcal{B}(u_n, v_n). \tag{3.3}$$

Since $Q := p + q > 2$, we know that $\frac{1}{2} > \frac{2}{Q+6}$, $\frac{1}{2} > \frac{4}{Q+6}$ and $\frac{1}{4} > \frac{2}{Q}$. Therefore $\mathcal{A}(u_n, v_n) + C(u_n, v_n)$ is bounded from above.

Going if necessary to a subsequence, still denoted by $\{(u_n, v_n)\}$, we may assume that $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$ weakly in H. Since $H_r^1(\mathbb{R}^3) \to L_r^s(\mathbb{R}^3)$ is compact for any $s \in (2, 6)$, we have that

$$\int |u_n|^p|v_n|^q dx \to \int |\tilde{u}|^p|\tilde{v}|^q dx \quad \text{as} \quad n \to \infty.$$

Using $\mathcal{G}(u_n, v_n) = 0$ and $\mathcal{M}$ is bounded away from zero, we obtain that $\tilde{u} \neq 0$ and $\tilde{v} \neq 0$. In the following, we will prove that $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ strongly in H and then $(\tilde{u}, \tilde{v}) \in \mathcal{M}$.

Denote $w_n := u_n - \tilde{u}$ and $z_n := v_n - \tilde{v}$. Supposing that as $n \to \infty$, $\mathcal{A}(w_n, z_n) + C(w_n, z_n) \not\to 0$, then from Brezis-Lieb lemma [3]

$$\begin{aligned}
0 = \mathcal{G}(u_n, v_n) &= \frac{1}{2}\mathcal{A}(w_n, z_n) + \frac{1}{2}\mathcal{A}(\tilde{u}, \tilde{v}) + C(w_n, z_n) + C(\tilde{u}, \tilde{v}) - \frac{Q+6}{4Q}\int |u_n|^p|v_n|^q dx \\
&\quad + \frac{1}{2}\left(b_1\left(\int |\nabla u_n|^2 dx\right)^2 + b_2\left(\int |\nabla v_n|^2 dx\right)^2 + 2c\int |\nabla u_n|^2 dx \int |\nabla v_n|^2 dx\right) \\
&\geq \frac{1}{2}\mathcal{A}(\tilde{u}, \tilde{v}) + C(\tilde{u}, \tilde{v}) + \frac{1}{2}\mathcal{B}(\tilde{u}, \tilde{v}) - \frac{Q+6}{4Q}\int |\tilde{u}|^p|\tilde{v}|^q dx \\
&\quad + \frac{1}{2}\mathcal{A}(w_n, z_n) + \frac{1}{2}\mathcal{B}(w_n, z_n) + C(w_n, z_n) \\
&= \mathcal{G}(\tilde{u}, \tilde{v}) + \frac{1}{2}\mathcal{A}(w_n, z_n) + \frac{1}{2}\mathcal{B}(w_n, z_n) + C(w_n, z_n),
\end{aligned}$$

which implies that $\mathcal{G}(\tilde{u}, \tilde{v}) < 0$, then according to Proposition 2.2 and Lemma 2.5, we have a unique $\tilde{t} := t(\tilde{u}, \tilde{v}) \in (0, 1)$ such that $(\tilde{u}^{\tilde{t}}, \tilde{v}^{\tilde{t}}) \in \mathcal{M}$, where $\tilde{u}^{\tilde{t}}(x) := \tilde{t}^{\frac{1}{4}}\tilde{u}(\tilde{t}^{-\frac{1}{2}}x)$ and $\tilde{v}^{\tilde{t}}(x) := \tilde{t}^{\frac{1}{4}}\tilde{v}(\tilde{t}^{-\frac{1}{2}}x)$.

As $\{(u_n, v_n)\} \subset \mathcal{M}$ is a minimizing sequence, we deduce from $\mathcal{G}(u_n, v_n) = 0$ that

$$\begin{aligned} d + o(1) =& \mathcal{I}(u_n, v_n) = \frac{1}{4}\mathcal{A}(u_n, v_n) + \frac{Q-2}{8Q}\int |u_n|^p|v_n|^q dx \\ \geq& \frac{1}{4}\mathcal{A}(\tilde{u}, \tilde{v}) + \frac{Q-2}{8Q}\int |\tilde{u}|^p|\tilde{v}|^q dx \\ >& \frac{1}{4}\tilde{t}\mathcal{A}(\tilde{u}, \tilde{v}) + \frac{Q-2}{8Q}\tilde{t}^{\frac{p+6}{4}}\int |\tilde{u}|^p|\tilde{v}|^q dx \\ =& \frac{1}{4}\mathcal{A}(\tilde{u}^{\tilde{t}}, \tilde{v}^{\tilde{t}}) + \frac{Q-2}{8Q}\int |\tilde{u}^{\tilde{t}}|^p|\tilde{v}^{\tilde{t}}|^q dx \\ =& \mathcal{I}(\tilde{u}^{\tilde{t}}, \tilde{v}^{\tilde{t}}), \end{aligned}$$

which is a contradiction as $(\tilde{u}^{\tilde{t}}, \tilde{v}^{\tilde{t}}) \in \mathcal{M}$.

Hence $\mathcal{A}(w_n, z_n) + C(w_n, z_n) \to 0$ as $n \to \infty$. Therefore $(\tilde{u}, \tilde{v}) \in \mathcal{M}$ and $(\tilde{u}, \tilde{v})$ is a minimizer of $\mathcal{I}|_{\mathcal{M}}$. □

Proof of Theorem 1.2. From Proposition 3.1, we have a $(\tilde{u}, \tilde{v}) \in \mathcal{M}$ such that $d = \mathcal{I}(\tilde{u}, \tilde{v})$. By Lemma 2.7, the $(\tilde{u}, \tilde{v})$ is a critical point of $\mathcal{I}$ and hence a solution to (1.1). Using a standard argument, we know that $(\tilde{u}, \tilde{v})$ is a positive radially symmetric ground state solution to (1.1). The proof is complete.

4. Asymptotical behavior of solutions as $(b_1)^2 + (b_2)^2 + c^2 \to 0$

From previous section, we know that for any $a_1 > 0$, $a_2 > 0$, $b_1 > 0$, $b_2 > 0$, $c \geq 0$ and $p > 1$, $q > 1$ with $Q := p + q \in (2, 6)$, (1.1) has a positive radially symmetric ground state solution $(\tilde{u}, \tilde{v})$. In this section, for any fixed $a_1 > 0$ and $a_2 > 0$, we will study how this ground state solution depends on b_1, b_2 and c. To emphasize the role of b_1, b_2 and c, we write the system (1.1) as

$$\begin{cases} -\left(a_1 + b_1\int_{\mathbb{R}^3} |\nabla u|^2 dx + c\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)\Delta u + u = \dfrac{p}{Q}|u|^{p-2}u|v|^q, \\ -\left(a_2 + b_2\int_{\mathbb{R}^3} |\nabla v|^2 dx + c\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta v + v = \dfrac{q}{Q}|u|^p|v|^{q-2}v, \\ u := u(x),\ v := v(x)\ \in H_r^1(\mathbb{R}^3), \quad x \in \mathbb{R}^3. \end{cases} \tag{1.1$_{b_1,b_2,c}$}$$

The solution obtained in Theorem 1.2 is denoted by $(u_{b_1,b_2,c}, v_{b_1,b_2,c})$. We write $\mathcal{M}$ as $\mathcal{M}_{b_1,b_2,c}$, the functional $\mathcal{I}$ as $\mathcal{I}_{b_1,b_2,c}$, and the functional $\mathcal{G}$ as $\mathcal{G}_{b_1,b_2,c}$.

Lemma 4.1. *For any fixed $a_1 > 0$ and $a_2 > 0$, let $b_1,\ b_2,\ c \in (0, 1]$ and $p > 1$, $q > 1$ with $Q := p + q \in (2, 6)$. Denoted by $(u_{b_1,b_2,c}, v_{b_1,b_2,c})$ the solution obtained in Theorem 1.2. Then $\{(u_{b_1,b_2,c}, v_{b_1,b_2,c})\}$ is uniformly bounded in H with respect to $b_1,\ b_2,\ c \in (0, 1]$.*

Proof. For any $b_1,\ b_2,\ c \in (0, 1]$, choosing nonzero radial functions $\phi, \psi \in H_r^1(\mathbb{R}^3) \cap C_0^\infty(\mathbb{R}^3)$ and defining $\phi^t(x) := t^{\frac{1}{4}}\phi(t^{-\frac{1}{2}}x)$, $\psi^t(x) := t^{\frac{1}{4}}\psi(t^{-\frac{1}{2}}x)$, $t > 0$, then by direct computation, we obtain that

$$\begin{aligned}\mathcal{G}_{b_1,b_2,c}(\phi^t,\psi^t) &= \frac{t}{2}\mathcal{A}(\phi,\psi) + t^2 C(\phi,\psi) + \frac{t^2}{2}\mathcal{B}(\phi,\psi) - \frac{Q+6}{4Q}t^{\frac{Q+6}{4}}\int|\phi|^p|\psi|^q dx \\ &\le \frac{t}{2}\mathcal{A}(\phi,\psi) + t^2 C(\phi,\psi) - \frac{Q+6}{4Q}t^{\frac{Q+6}{4}}\int|\phi|^p|\psi|^q dx \\ &\quad + \frac{t^2}{2}\left(\left(\int|\nabla\phi|^2dx\right)^2 + \left(\int|\nabla\psi|^2dx\right)^2 + 2\int|\nabla\phi|^2dx\int|\nabla\psi|^2dx\right).\end{aligned}$$

Hence by the last inequality and $\frac{Q+6}{4} > 2$, we have a $t_3 > 0$ such that $\mathcal{G}_{b_1,b_2,c}(\phi^{t_3},\psi^{t_3}) < 0$, where t_3 is independent of b_1, b_2 and c. And then (ϕ^{t_3},ψ^{t_3}) is independent of b_1, b_2 and c, either. Denote $w(x) := \phi^{t_3}(x)$ and $z(x) := \psi^{t_3}(x)$. By Proposition 2.2 and Lemma 2.5, we get a $t_0 := t(w,z) \in (0,1)$ such that $\mathcal{G}_{b_1,b_2,c}(w^{t_0},z^{t_0}) = 0$, where $w^{t_0}(x) := t_0^{\frac{1}{4}}w(t_0^{-\frac{1}{2}}x)$ and $z^{t_0}(x) := t_0^{\frac{1}{4}}z(t_0^{-\frac{1}{2}}x)$. From this, we deduce that

$$\begin{aligned}\mathcal{I}_{b_1,b_2,c}(w^{t_0},z^{t_0}) &= \frac{1}{4}\mathcal{A}(w^{t_0},z^{t_0}) + \frac{Q-2}{8Q}\int|w^{t_0}|^p|z^{t_0}|^q dx \\ &= \frac{t_0}{4}\mathcal{A}(w,z) + \frac{Q-2}{8Q}t_0^{\frac{Q+6}{4}}\int|w|^p|z|^q dx \\ &< \frac{1}{4}\mathcal{A}(w,z) + \frac{Q-2}{8Q}\int|w|^p|z|^q dx := M_4.\end{aligned}$$

In here M_4 is a positive constant. Since neither w nor z depends on b_1, b_2 and c, M_4 does not depend on any one of b_1, b_2 and c, either.

Next let $\{(u_{b_1,b_2,c}, v_{b_1,b_2,c})\}$ be a minimizer of $\mathcal{I}_{b_1,b_2,c}$ under the constraint of $\mathcal{M}_{b_1,b_2,c}$. Then $\mathcal{I}_{b_1,b_2,c}(u_{b_1,b_2,c}, v_{b_1,b_2,c}) \le \mathcal{I}_{b_1,b_2,c}(w^{t_0},z^{t_0}) < M_4$. Using $\mathcal{G}_{b_1,b_2,c}(w^{t_0},z^{t_0}) = 0$, we get that

$$\begin{aligned}M_4 > \mathcal{I}_{b_1,b_2,c}(u_{b_1,b_2,c}, v_{b_1,b_2,c}) &= \frac{Q+2}{2(Q+6)}\mathcal{A}(u_{b_1,b_2,c}, v_{b_1,b_2,c}) \\ &+ \frac{Q-2}{2(Q+6)}C(u_{b_1,b_2,c}, v_{b_1,b_2,c}) + \frac{Q-2}{4(Q+6)}\mathcal{B}(u_{b_1,b_2,c}, v_{b_1,b_2,c}).\end{aligned}$$

As $Q > 2$, we deduce that $\mathcal{A}(u_{b_1,b_2,c}, v_{b_1,b_2,c}) + C(u_{b_1,b_2,c}, v_{b_1,b_2,c})$ is uniformly bounded with respect to $b_1,\ b_2,\ c \in (0,1]$. This proves the lemma. □

Proof of Theorem 1.3. For the sequences $b_1^{(n)}$, $b_2^{(n)}$ and $c^{(n)}$, we may assume that for all $n = 1,\ 2,\cdots$, $b_1^{(n)} < 1$, $b_2^{(n)} < 1$ and $c^{(n)} < 1$. To simplify notations, we denote

$$u^{(n)}(x) := u_{b_1^{(n)},b_2^{(n)},c^{(n)}}(x) \ \text{ and } v^{(n)}(x) := v_{b_1^{(n)},b_2^{(n)},c^{(n)}}(x).$$

From Theorem 1.2 and Lemma 4.1, we know that $\{(u^{(n)}, v^{(n)})\}$ is bounded in H. Going if necessary to a subsequence, we may assume that

$$(u^{(n)}(x), v^{(n)}) \rightharpoonup (U_0, V_0) \quad \text{weakly in} \quad H.$$

Hence (U_0, V_0) is a weak solution to

$$\begin{cases} -a_1\Delta u + u = \dfrac{p}{Q}|u|^{p-2}u|v|^q, \\ -a_2\Delta v + v = \dfrac{q}{Q}|u|^p|v|^{q-2}v, \\ u,\ v \in H_r^1(\mathbb{R}^3), \quad x \in \mathbb{R}^3. \end{cases} \qquad (1.1)_{0,0,0}$$

Since $H_r^1(\mathbb{R}^3) \to L_r^s(\mathbb{R}^3)$ is compact for any $2 < s < 6$, we have that

$$\int |u^{(n)}|^p |v^{(n)}|^q dx \to \int |U_0|^p |V_0|^q dx \quad \text{as} \quad n \to \infty.$$

Using $\mathcal{I}'_{0,0,0}(U_0, V_0) = 0$ and $\mathcal{I}'_{b_1^{(n)}, b_2^{(n)}, c^{(n)}}(u^{(n)}, v^{(n)}) = 0$, we get that

$$\begin{aligned}
0 &= \langle (\mathcal{I}'_{b_1^{(n)}, b_2^{(n)}, c^{(n)}}(u^{(n)}, v^{(n)}) - \mathcal{I}'_{0,0,0}(U_0, V_0)), (u^{(n)} - U_0, v^{(n)} - V_0) \rangle \\
&= \int \left(a_1 |\nabla u^{(n)} - \nabla U_0|^2 + |u^{(n)} - U_0|^2 \right) dx + b_1^{(n)} \int |\nabla u^{(n)}|^2 dx \int \nabla u^{(n)} \nabla (u^{(n)} - U_0) dx \\
&\quad - \frac{p}{Q} \int |u^{(n)}|^{p-2} u^{(n)} (u^{(n)} - U_0) |v^{(n)}|^q dx + \frac{p}{Q} \int |U_0|^{p-2} U_0 (u^{(n)} - U_0) |V_0|^q dx. \\
&\quad + \int \left(a_2 |\nabla v^{(n)} - \nabla V_0|^2 + |v^{(n)} - V_0|^2 \right) dx + b_2^{(n)} \int |\nabla v^{(n)}|^2 dx \int \nabla v^{(n)} \nabla (v^{(n)} - V_0) dx \\
&\quad - \frac{q}{Q} \int |v^{(n)}|^{q-2} v^{(n)} (v^{(n)} - V_0) |u^{(n)}|^p dx + \frac{q}{Q} \int |V_0|^{q-2} V_0 (v^{(n)} - V_0) |U_0|^p dx \\
&\quad + c^{(n)} \int |\nabla v^{(n)}|^2 dx \int \nabla u^{(n)} \nabla (u^{(n)} - U_0) dx + c^{(n)} \int |\nabla u^{(n)}|^2 dx \int \nabla v^{(n)} \nabla (v^{(n)} - V_0) dx.
\end{aligned} \tag{4.1}$$

Note that as $n \to \infty$, $(b_1^{(n)})^2 + (b_2^{(n)})^2 + (c^{(n)})^2 \to 0$; both $\int |\nabla u^{(n)}|^2 dx$ and $\int |\nabla v^{(n)}|^2 dx$ are bounded; we obtain that

$$\begin{aligned}
&\int |u^{(n)}|^{p-2} u^{(n)} (u^{(n)} - U_0) |v^{(n)}|^q dx \\
&\quad \le \left(\int |u^{(n)}|^Q dx \right)^{\frac{p-1}{Q}} \left(\int |u^{(n)} - U_0)|^Q dx \right)^{\frac{1}{Q}} \left(\int |v^{(n)}|^Q dx \right)^{\frac{q}{Q}} \to 0; \\
&\int |U_0|^{p-2} U_0 (u^{(n)} - U_0) |V_0|^q dx \\
&\quad \le \left(\int |U_0|^Q dx \right)^{\frac{p-1}{Q}} \left(\int |u^{(n)} - U_0)|^Q dx \right)^{\frac{1}{Q}} \left(\int |V_0|^Q dx \right)^{\frac{q}{Q}} \to 0; \\
&\int |v^{(n)}|^{q-2} v^{(n)} (v^{(n)} - V_0) |u^{(n)}|^p dx \\
&\quad \le \left(\int |v^{(n)}|^Q dx \right)^{\frac{q-1}{Q}} \left(\int |v^{(n)} - V_0)|^Q dx \right)^{\frac{1}{Q}} \left(\int |u^{(n)}|^Q dx \right)^{\frac{p}{Q}} \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int |U_0|^p |V_0|^{q-2} V_0 (v^{(n)} - V_0) dx \\
&\quad \le \left(\int |V_0|^Q dx \right)^{\frac{q-1}{Q}} \left(\int |v^{(n)} - V_0)|^Q dx \right)^{\frac{1}{Q}} \left(\int |U_0|^Q dx \right)^{\frac{p}{Q}} \to 0.
\end{aligned}$$

Combining these with (4.1), we deduce that for n large enough,

$$0 = \mathcal{A}(u^{(n)} - U_0, v^{(n)} - V_0) + C(u^{(n)} - U_0, v^{(n)} - V_0) + o(1).$$

Hence we have proven that $(u^{(n)}, v^{(n)}) \to (U_0, V_0)$ strongly in H. And (U_0, V_0) is a positive radially symmetric solution to $(1.1)_{0,0,0}$. The proof is complete.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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