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Research article

Hermite-Hadamard type inequalities for subadditive functions

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Abstract: In this paper, we will consider subadditive functions that take an important place not only in mathematics but also in physics and many other fields of science. Subadditive functions are very important also in economics and, specifically, in financial mathematics where subadditive discount functions describe certain behaviors in intertemporal choice and its anomalies. For example, some properties and characterizations of subadditive discount functions can be found in [11]. We establish Hermite-Hadamard-like inequalities for subadditive functions. Moreover, by using an integral identity together with some well known integral inequalities, we obtain several new inequalities for subadditive functions. Moreover, using subadditive functions we give some examples for the Hermite-Hadamard type inequalities. Some applications to special means of real numbers are also given. Especially, it should be noted that the results obtained in this paper coincide with previously obtained results in the literature under certain conditions.

Keywords: subadditive function; Hermite-Hadamard inequality
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1. Introduction

Additivity, subadditivity and superadditivity are important concepts in measure theory, in several fields of mathematics and in especially inequalities. There are numerous examples of additive, subadditive and superadditive functions in various areas of mathematics, particularly norms, square roots, error function, growth rates, differential equations, integral means and distributive lattice. Inequalities and especially subadditive functions theory is one of the most extensively developing fields not only in theoretical and applied mathematics but also physics and the other applied sciences. Therefore, inequalities and convexity theory play an important role in mathematics and physics.

It is well known that inequalities and subadditive theory are an important aid in solving numerous problems of mathematical physics. Subadditivity occurs in the thermodynamic properties of nonideal solutions and mixtures like the excess molar volume and heat of mixing or excess enthalpy. In addition, inequalities and subadditive functions can be seen in electrical network, quantum relative entropy, purification, ergodic theory and dynamic systems, equilibrium and perturbations of repulsive. There has been increasing interest in the analysis of inequalities and subadditive functions. There are quite substantial literatures on such subadditive functions and inequalities. Here we mention the results of [2, 3, 5, 8, 12, 13] and the corresponding references cited therein.

Definition 1. *A real-valued function f defined on the positive real line $[0,\infty)$ is said to be convex if for every $x \geq 0, y \geq 0$, and $0 \leq t \leq 1$ satisfies the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

This definition is well known in the literature. Let $f : [a,b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality (see [6] for more information). In recent years, many new convex classes and related Hermite Hadamard type inequalities have been studied by many authors (for example see [1, 7, 9, 10, 15]).

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}$ is called additive if and only if it satisfies Cauchy's functional equation $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.*

Example 1. *The function $f(x) = ax$, $a, x \in \mathbb{R}$ is an additive function.*

Definition 3. *[14] The function $f : [0,b] \to \mathbb{R}, b > 0$ is said to be starshaped if for every $x \in [0,b]$ and $t \in [0,1]$ we have $f(tx) \leq tf(x)$.*

Definition 4. *A function $f : [0,\infty) \subset \mathbb{R} \to \mathbb{R}$ is said to be subadditive if inequality $f(x+y) \leq f(x) + f(y), \quad x, y \in [0,\infty)$ holds. If this inequality reverse, then f is called superadditive function.*

The function f is superadditive if and only if $-f$ is subadditive. If the function f is subadditive, then $f(0) \geq 0$.

According to above definitions, if a subadditive function $f : A \subset \mathbb{R} \to \mathbb{R}$ is also starshaped, then f is a convex function.

Example 2. *The square function $f(x) = \sqrt{x}$, $x \geq 0$ is a subadditive function. Really, for every $x, y \geq 0$, $f(x+y) = \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} = f(x) + f(y)$.*

Example 3. *The function $f(x) = \sin x$, $x \in [0,\pi]$ is a subadditive function. For every $x, y \in [0,\pi]$, $f(x+y) = \sin(x+y) \leq \sin x + \sin y$. More generally, the function $f(x) = |\sin(\pi x)|$ is subadditive on $\mathbb{R}$.*

Example 4. *For every $x \in \mathbb{R}$, the functions $f(x) = \frac{1}{x}, x > 0$, $f(x) = e^{-x}, x \geq 0$, $f(x) = mx + n$, $n \geq 0$, $\arctan x$, $\frac{x}{x+1}$, $\tanh x$ and x^p, $p \in (0,1]$ are subadditive functions.*

For shortness, throuhout this paper we will use the notations:

$$A \ : \ = A(a,b) = \frac{a+b}{2}, \text{ arithmetic mean}$$

$$\begin{aligned} G &:= G(a,b) = \sqrt{ab}, \text{ geometric mean} \\ L_p &:= L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \ a \neq b, p \in \mathbb{R}, p \neq -1, 0, \ p\text{-logarithmic mean} \\ I_f(u) &:= \int_0^u f(t)dt. \end{aligned}$$

Moreover, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

2. Main results

In [4], Dragomir and Agarwal used the following lemma to prove their theorems. The main purpose of this section is to establish new estimations and refinements of the Hermite-Hadamard type integral inequalities for functions whose first derivatives in absolute value are subadditive functions. Some applications to special means of real numbers are given. For this, we will use the following lemma.

Lemma 1. *[4] Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt$$

We have obtained a similarity of the left side of the inequality (1.1) for subadditive functions in the following theorem.

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval, $f : I \subset \mathbb{R} \to \mathbb{R}$ be a subadditive function, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. Then the following inequality holds for $t \in [0,1]$:*

$$f(a+b) \leq \frac{2}{b-a}\int_a^b f(x)dx.$$

Proof. By using subadditivity of the function f, we can write

$$f(a+b) = f([ta+(1-t)b] + [tb+(1-t)a]) \leq f(ta+(1-t)b) + f(tb+(1-t)a).$$

If the variable is changed as $u = ta + (1-t)b$ and $z = tb + (1-t)a$, then we get

$$f(a+b) \leq \frac{2}{b-a}\int_a^b f(x)dx.$$

This completes the proof of theorem. □

Corollary 1. *Under the conditions of Theorem 1, if $f(tx) \leq tf(x)$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f(a+b) \leq \frac{1}{b-a}\int_a^b f(x)dx.$$

So, this inequality coincides with the inequality in convex functions [4].

We have obtained a similarity of the right side of the inequality (1.1) for subadditive functions in the following theorem.

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval with $0 \in I$, $f : I \subset \mathbb{R} \to \mathbb{R}$ be a subadditive function and $a, b \in I$ with $a < b$. If $f \in L[a^-, b^+]$, $a^- = \min\{a, 0\}$, $b^+ = \max\{0, b\}$, then the following inequalities holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{a}\int_0^a f(x)dx + \frac{1}{b}\int_0^b f(x)dx.$$

Proof. By using subadditivity of the function f and by changing the variable as $u = ta + (1-t)b$, we get

$$\begin{aligned}\int_0^1 f(ta + (1-t)b)\,dt &\le \int_0^1 f(ta)dt + \int_0^1 f((1-t)b)dt\\ \frac{1}{b-a}\int_a^b f(x)dx &\le \frac{1}{a}\int_0^a f(x)dx + \frac{1}{b}\int_0^b f(x)dx.\end{aligned}$$

This completes the proof of theorem. □

Corollary 2. *Under the conditions of Theorem 2, if $f(tx) \le tf(x)$, then*

$$\begin{aligned}I_f(a) &= \int_0^a f(x)dx = a\int_0^1 f(ta)dt \le a\int_0^1 tf(a)dt = \frac{af(a)}{2}\\ I_f(b) &= \int_0^b f(x)dx = b\int_0^1 f((1-t)b)dt \le b\int_0^1 (1-t)f(b)dt = \frac{bf(b)}{2}.\end{aligned}$$

So,

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

This inequality coincides with the inequality in convex functions [4].

We have obtained the following theorem, when the function $|f'|$ is a subadditive function.

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval with $0 \in I$, $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I$ with $a < b$ and $f' \in L[a^-, b^+]$, $a^- = \min\{a, 0\}$, $b^+ = \max\{0, b\}$. If $|f'|$ is a subadditive function on $[a, b]$, then the following inequality*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{b-a}{2a^2}\left[aI_{|f'|}(a) + 2\int_0^{\frac{a}{2}} I_{|f'|}(u)\,dz - 2\int_{\frac{a}{2}}^a I_{|f'|}(u)\,dz\right]\\ &\quad + \frac{b-a}{2b^2}\left[bI_{|f'|}(b) + 2\int_0^{\frac{b}{2}} I_{|f'|}(u)\,dz - 2\int_{\frac{b}{2}}^b I_{|f'|}(u)\,dz\right]\end{aligned}$$

holds for $t \in [0, 1]$, where $\int_0^u |f'(t)|\,dt = I_{|f'|}(u)$.

Proof. By using Lemma 1 and $|f'(ta + (1-t)b)| \le |f'(ta)| + |f'((1-t)b)|$, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta + (1-t)b)\,dt\right|$$

$$
\begin{aligned}
&\leq \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt\\
&\leq \frac{b-a}{2}\int_0^1 |1-2t|\,[|f'(ta)|+|f'((1-t)b)|]\,dt\\
&= \frac{b-a}{2}\left[\int_0^1 |1-2t|\,|f'(ta)|\,dt+\int_0^1 |1-2t|\,|f'((1-t)b)|\,dt\right]\\
&= \frac{b-a}{2}\left[\int_0^{\frac{1}{2}}(1-2t)\,|f'(ta)|\,dt+\int_{\frac{1}{2}}^1(2t-1)\,|f'(ta)|\,dt\right]\\
&\quad+\frac{b-a}{2}\left[\int_0^{\frac{1}{2}}(1-2t)\,|f'((1-t)b)|\,dt+\int_{\frac{1}{2}}^1(2t-1)\,|f'((1-t)b)|\,dt\right].
\end{aligned}
$$

If we change as $y=ta$ and $z=(1-t)b$ in the last integrals, respectively, we deduce

$$
\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{2}\left[\frac{1}{a^2}(a-2z)I_{|f'|}(z)\Big|_0^{\frac{a}{2}}+\frac{2}{a^2}\int_0^{\frac{a}{2}}I_{|f'|}(u)\,dz\right.\\
&\quad\left.+\frac{1}{a^2}(2z-a)I_{|f'|}(z)\Big|_{\frac{a}{2}}^{a}-\frac{2}{a^2}\int_{\frac{a}{2}}^{a}I_{|f'|}(u)\,dz\right]\\
&\quad+\frac{b-a}{2}\left[\frac{1}{b^2}(2z-b)I_{|f'|}(z)\Big|_{\frac{b}{2}}^{b}-\frac{2}{b^2}\int_{\frac{b}{2}}^{b}I_{|f'|}(u)\,dz\right.\\
&\quad\left.+\frac{1}{b^2}(b-2z)I_{|f'|}(z)\Big|_0^{\frac{b}{2}}+\frac{2}{b^2}\int_0^{\frac{b}{2}}I_{|f'|}(u)\,dz\right]\\
&= \frac{b-a}{2a^2}\left[aI_{|f'|}(a)+2\int_0^{\frac{a}{2}}I_{|f'|}(u)\,dz-2\int_{\frac{a}{2}}^{a}I_{|f'|}(u)\,dz\right]\\
&\quad+\frac{b-a}{2b^2}\left[bI_{|f'|}(b)+2\int_0^{\frac{b}{2}}I_{|f'|}(u)\,dz-2\int_{\frac{b}{2}}^{b}I_{|f'|}(u)\,dz\right].
\end{aligned}
$$

This completes the proof of theorem. □

Corollary 3. *Under the conditions of Theorem 3, if* $|f'(tx)|\leq t\,|f'(x)|$*, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)\,[|f'(a)|+|f'(b)|]}{8}.$$

This inequality coincides with the inequality in convex functions [4].

Proposition 1. *Let* $a,b\in(0,\infty)$ *with* $a<b$*. Then the following inequality holds:*

$$\left|A\left(a^{\frac{3}{2}},b^{\frac{3}{2}}\right)-L_{\frac{3}{2}}^{\frac{3}{2}}(a,b)\right|\leq\frac{b-a)}{5}\left(\sqrt{2}+1\right)A\left(\sqrt{a},\ \sqrt{b}\right).$$

Proof. The result follows directly from Theorem 3 for $f(x)=\frac{2}{3}x^{\frac{3}{2}}$, $x>0$. □

Proposition 2. *Let $a, b \in [0, \pi]$ with $a < b$. Then the following inequality holds:*

$$\begin{aligned}\left|\frac{\sin b - \sin a}{b-a} - \frac{\cos a + \cos b}{2}\right| \quad &\leq \quad \frac{b-a}{2}\left[\frac{1}{a}(1-\cos a) + \frac{2}{a^2}\left(\sin a - \sin\frac{a}{2}\right)\right] \\ &+\frac{b-a}{2}\left[\frac{1}{b}(1-\cos b) + \frac{2}{b^2}\left(\sin b - \sin\frac{b}{2}\right)\right].\end{aligned}$$

Proof. Under the conditions of the Theorem 3, let $f(x) = -\cos x$, $x \in [0, \pi]$. Then $f'(x) = \sin x$ is subadditive on $[0, \pi]$ and the result follows directly from Theorem 3. Specially,

i. If we take $a \to \frac{\pi}{3}, b = \frac{\pi}{2}$, then we have following inequality:

$$\frac{6}{\pi}\left(2 - \sqrt{3}\right) - \frac{1}{4} \leq \frac{7}{24} + \frac{9\sqrt{3} - 4\sqrt{2} - 1}{12\pi}.$$

ii. If we take $a = \frac{\pi}{2}, b = \pi$, then we have following inequality:

$$\frac{2}{\pi} - \frac{1}{2} \leq 1 + \frac{3 - 2\sqrt{2}}{2\pi}.$$

iii. If we take $a = \frac{\pi}{3}, b = \pi$, then we have following inequality:

$$\frac{3\sqrt{3}}{4\pi} - \frac{2-\sqrt{3}}{4} \leq \frac{7}{6} + \frac{9\sqrt{3} - 11}{3\pi}.$$

□

Proposition 3. *Let $a, b \in \mathbb{R}$ with $a < b$. Then the following inequality holds:*

$$\left|A\left(a^{p+1}, b^{p+1}\right) - L_{p+1}^{p+1}(a, b)\right| \leq \frac{b-a}{p+2}\left[\frac{p2^p + 1}{2^p}\right] A\left(a^p, b^p\right).$$

Proof. Under assumption of the Theorem 3, let $f(x) = \frac{x^{p+1}}{p+1}$, $p \in (0, 1]$. Then $f'(x) = x^p$ is subadditive on $\mathbb{R}$ and the result follows from Theorem 3. □

Proposition 4. *Let $a, b \in \mathbb{R}$ with $a < b$. Then the following inequality holds:*

$$\left|\frac{m}{6}A\left(a^2, b^2\right) - mG^2(a, b)\right| \leq \frac{b-a}{4}(ma + 2n).$$

Proof. The result follows directly from Theorem 3 for the function $f(x) = \frac{m}{2}x^2 + nx$, $n > 0$. Then $f'(x) = mx + n$ is subadditive on $\mathbb{R}$ and the result follows directly from Theorem 3. □

We have obtained the following theorem by using Hölder's integral inequality, when the function $|f'|^q$ is a subadditive function.

Theorem 4. *Let $I \subset \mathbb{R}$ be an interval with $0 \in I$, $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I$ with $a < b$ and $f' \in L[a^-, b^+]$, $a^- = \min\{a, 0\}$, $b^+ = \max\{0, b\}$, and let $q > 1$. If the mapping $|f'|^q$ is a subadditive function on interval $[a, b]$, then the following inequality*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \quad \leq \quad \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{a}\int_0^a |f'(x)|^q\, dx + \frac{1}{b}\int_0^b |f'(x)|^q\, dx\right)^{\frac{1}{q}}$$

holds for $t \in [0, 1]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 1, Hölder's integral inequality and inequality

$$|f'(ta+(1-t)b)|^q \le |f'(ta)|^q + |f'((1-t)b)|^q$$

which is the subadditivity of $|f'|^q$, we obtain

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\
&\le \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
&\le \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 [|f'(ta)|^q + |f'((1-t)b)|^q]\,dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta)|^q\,dt + \int_0^1 |f'((1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{a}\int_0^a |f'(x)|^q\,dx + \frac{1}{b}\int_0^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\int_0^1 |1-2t|^p\,dt = \int_0^{\frac{1}{2}} (1-2t)^p\,dt + \int_{\frac{1}{2}}^1 (2t-1)^p\,dt = \frac{1}{p+1}.$$

This completes the proof of theorem. □

Corollary 4. *Under the conditions of Theorem 4, if* $|f'(tx)|^q \le t\,|f'(x)|^q$*, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

This inequality coincides with the inequality in convex functions [4].

Proposition 5. *Let* $a, b \in (0,\infty)$ *with* $a < b$*,* $q > 1$ *and* $s \in (-\infty, 0] \cup [1,\infty) \setminus \{-2q, q\}$*. Then the following inequality holds:*

$$\left|\frac{q}{s+q}\left[A\left(a^{\frac{s}{q}+1}, b^{\frac{s}{q}+1}\right) - L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right]\right| \le (b-a)\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(a^s, b^s).$$

Proof. Under the assumptions of Proposition, let function $f(x) = \frac{q}{s+q}x^{\frac{s}{q}+1}$, $x \in (0,\infty)$. Then $|f'(x)|^q = x^s$ is subadditive on $(0,\infty)$ and the result follows directly from Theorem 4. □

We have obtained the following theorem by using power-mean integral inequality, when the function $|f'|^q$ is a subadditive function.

Theorem 5. *Let* $I \subset \mathbb{R}$ *be an interval with* $0 \in I$*,* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function and* $a, b \in I$ *with* $a < b$ *and* $f' \in L[a^-, b^+]$*,* $a^- = \min\{a, 0\}$*,* $b^+ = \max\{0, b\}$*, and let* $q \ge 1$*. If the mapping* $|f'|^q$ *is a subadditive function on interval* $[a,b]$*, then the following inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2}{a}I_{|f'|^q}(a) + \frac{2}{a^2}\int_0^{\frac{a}{2}} I_{|f'|^q}(u)\,dz - \frac{2}{a^2}\int_{\frac{a}{2}}^a I_{|f'|^q}(u)\,dz\right.$$

$$+\frac{1}{b}I_{|f'|^q}(b)+\frac{2}{b^2}\int_0^{\frac{b}{2}}I_{|f'|^q}(u)\,dz-\frac{2}{b^2}\int_{\frac{b}{2}}^{b}I_{|f'|^q}(u)\,dz\Bigg)^{\frac{1}{q}}$$

holds for $t\in[0,1]$*, where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1, power-mean inequality and subadditivity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\le\ & \frac{b-a}{2}\int_0^1|1-2t|\,|f'(ta+(1-t)b)|\,dt\\
\le\ & \frac{b-a}{2}\left(\int_0^1|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\\
\le\ & \frac{b-a}{2}\left(\int_0^1|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\left[|f'(ta)|^q+|f'((1-t)b)|^q\right]dt\right)^{\frac{1}{q}}\\
=\ & \frac{b-a}{2}\left(\int_0^1|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\,|f'(ta)|^q\,dt+\int_0^1|1-2t|\,|f'((1-t)b)|^q\,dt\right)^{\frac{1}{q}}\\
=\ & \frac{b-a}{2}\left(\int_0^1|1-2t|\,dt\right)^{1-\frac{1}{q}}\Bigg(\int_0^{\frac{1}{2}}(1-2t)\,|f'(ta)|^q\,dt+\int_{\frac{1}{2}}^1(2t-1)\,|f'(ta)|^q\,dt\\
&+\int_0^{\frac{1}{2}}(1-2t)\,|f'((1-t)b)|^q\,dt+\int_{\frac{1}{2}}^1(2t-1)\,|f'((1-t)b)|^q\,dt\Bigg)^{\frac{1}{q}}.
\end{aligned}$$

Changing variables as $y=ta$ and $z=(1-t)b$ in the last integrals, respectively and then taking partial integrating, we obtain

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg(\frac{1}{a^2}(a-2z)I_{|f'|^q}(z)\Big|_0^{\frac{a}{2}}+\frac{2}{a^2}\int_0^{\frac{a}{2}}I_{|f'|^q}(u)\,dz\\
&+\frac{1}{a^2}(2z-a)I_{|f'|^q}(z)\Big|_{\frac{a}{2}}^{a}-\frac{2}{a^2}\int_{\frac{a}{2}}^{a}I_{|f'|^q}(u)\,dz\\
&+\frac{1}{b^2}(2z-b)I_{|f'|^q}(z)\Big|_{\frac{b}{2}}^{b}-\frac{2}{b^2}\int_{\frac{b}{2}}^{b}I_{|f'|^q}(u)\,dz\\
&+\frac{1}{b^2}(b-2z)I_{|f'|^q}(z)\Big|_0^{\frac{b}{2}}+\frac{2}{b^2}\int_0^{\frac{b}{2}}I_{|f'|^q}(u)\,dz\Bigg)^{\frac{1}{q}}\\
=\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg(\frac{2}{a}I_{|f'|^q}(a)+\frac{2}{a^2}\int_0^{\frac{a}{2}}I_{|f'|^q}(u)\,dz-\frac{2}{a^2}\int_{\frac{a}{2}}^{a}I_{|f'|^q}(u)\,dz\\
&+\frac{1}{b}I_{|f'|^q}(b)+\frac{2}{b^2}\int_0^{\frac{b}{2}}I_{|f'|^q}(u)\,dz-\frac{2}{b^2}\int_{\frac{b}{2}}^{b}I_{|f'|^q}(u)\,dz\Bigg)^{\frac{1}{q}}
\end{aligned}$$

where $\int_0^1|1-2t|\,dt=\frac{1}{2}$.

This completes the proof of theorem. □

Corollary 5. *Under the conditions of Theorem 5, if* $|f'(tx)|^q \leq t\,|f'(x)|^q$*, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).$$

Corollary 6. *Under the assumptions of Theorem 5 with* $q=1$*, we get*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq\ & \frac{b-a}{2}\left(\frac{2}{a}I_{|f'|}(a)+\frac{2}{a^2}\int_0^{\frac{a}{2}} I_{|f'|}(u)\,dz-\frac{2}{a^2}\int_{\frac{a}{2}}^a I_{|f'|}(u)\,dz\right.\\ &\left.+\frac{1}{b}I_{|f'|}(b)+\frac{2}{b^2}\int_0^{\frac{b}{2}} I_{|f'|}(u)\,dz-\frac{2}{b^2}\int_{\frac{b}{2}}^b I_{|f'|}(u)\,dz\right).\end{aligned}$$

Proposition 6. *Let* $a, b \in (0,\infty)$ *with* $a<b$*,* $q>1$ *and* $s \in (-\infty,0]\cup[1,\infty)\setminus\{-2q,q\}$*. Then the following inequality holds:*

$$\left|\frac{q}{s+q}\left[A\left(a^{\frac{s}{q}+1}, b^{\frac{s}{q}+1}\right)-L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right]\right| \leq \frac{b-a}{4}\left[\frac{1+s2^s}{(s+1)(s+2)2^s}\right]^{\frac{1}{q}} A^{\frac{1}{q}}\left(a^s, b^s\right).$$

Proof. Under the assumptions of Proposition, let function $f(x)=\frac{q}{s+q}x^{\frac{s}{q}+1}$, $x\in(0,\infty)$. Then

$$|f'(x)|^q = x^s$$

is subadditive on $(0,\infty)$ and the result follows directly from Theorem 5. □

Corollary 7. *If we take* $q=1$ *in above Proposition, we get*

$$\left|\frac{A\left(a^{s+1}, b^{s+1}\right)-L_{s+1}^{s+1}(a,b)}{s+1}\right| \leq \frac{b-a}{4}\left[\frac{1+s2^s}{(s+1)(s+2)2^s}\right]A\left(a^s, b^s\right).$$

3. Conclusion

In this paper, we studied the subadditive functions. We proved some new Hermite-Hadamard type integral inequalities for this type of functions using an identity together with Hölder and power-mean integral inequalities. Especially, some applications are also given. Different types of integral inequalities can also be obtained for subadditive functions.

Conflict of interest

The author declares no conflict of interest in this paper.

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