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Research article

On a problem concerning the ring of Nash germs and the Borel mapping

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Abstract: We denote by $\mathbb{R}[[t]]$ the ring of formal power series with real coefficients. Let $\widehat{C_1} \subset \mathbb{R}[[t]]$ be a subring. We say that $\widehat{C_1}$ has the splitting property if for each $f \in \widehat{C_1}$ and $A \cup B = \mathbb{N}$ such that $A \cap B = \emptyset$, if $f = G + H$ where $G = \sum_{w \in A} a_w t^w$ and $H = \sum_{w \in B} a_w t^w$ are formal power series, then $G \in \widehat{C_1}$ and $H \in \widehat{C_1}$. It is well known that the ring of convergent power series $\mathbb{R}\{t\}$ satisfies the splitting property. In this paper, we will examine this property for a subring of $\mathbb{R}\{t\}$ and for some local rings containing strictly $\mathbb{R}\{t\}$.

Keywords: Nash germs; splitting property; Borel mapping; quasianalytic; Denjoy-Carleman classes
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1. Introduction

Let $\mathcal{E}_1$ denote the ring of germs at the origin in $\mathbb{R}$ of C^∞ functions in a neighborhood of $0 \in \mathbb{R}$ and $\mathbb{R}[[t]]$ the ring of formal series with real coefficients. If $f \in \mathcal{E}_1$, we denote by $\hat{f} \in \mathbb{R}[[t]]$ its (infinite) Taylor expansion at the origin. The mapping $\mathcal{E}_1 \ni f \mapsto \hat{f} \in \mathbb{R}[[t]]$ is called the Borel mapping. A subring $C_1 \subseteq \mathcal{E}_1$ is called quasianalytic if the restriction of the Borel mapping to C_1 is injective.
Firstly, we will introduce the notion of the splitting property and we will show that the ring of Nash germs $\mathcal{N}_1$ does not satisfy this property.
Secondly, we will investigate the Borel mapping for local quasianalytic rings. Recall that the Borel mapping takes germs at the origin in $\mathbb{R}$ of smooth functions to the sequence of the iterated partial derivatives at 0. It is a classical result due to Carleman [1, 2] that the Borel mapping restricted to the germs at 0 of functions in a quasianalytic Denjoy-Carleman classes is never onto.

2. Problem of splitting property over the ring of Nash germs.

Definition 2.1. Let $C_1 \subseteq \mathcal{E}_1$ be a quasianalytic ring. We say that $\hat{C}_1$ has the splitting property, if for each $f \in C_1$ such that $\hat{f} = \varphi_1 + \varphi_2$ where $\varphi_1 = \sum_{n\in A} a_n t^n$, $\varphi_2 = \sum_{n\in B} a_n t^n$ and $\mathbb{N} = A \cup B$, $A \cap B = \emptyset$, there exist $\psi_1, \psi_2 \in \hat{C}_1$ with $\hat{\psi_1} = \varphi_1$, $\hat{\psi_2} = \varphi_2$ and $f = \psi_1 + \psi_2$.

Example 2.2. *Let $\mathbb{R}\{t\}$ denote the ring of convergent power series. The quasianalytic ring $\mathbb{R}\{t\}$ has the splitting property.*
Indeed, if $f = \sum_{n\in\mathbb{N}} a_n t^n$ and $f = \varphi_1 + \varphi_2$ where $\varphi_1 = \sum_{n\in A} a_n t^n$ and $\varphi_2 = \sum_{n\in B} a_n t^n$ such that A and B form a partition of $\mathbb{N}$. If A or B are finite sets, the result is evident. Now let us assume that A and B both are infinite sets, we know that the radius of convergence of the power series f is equal to $\frac{1}{\limsup_{n\to\infty}(|a_n|)^{1/n}}$, but since $\frac{1}{\limsup_{n\in A, n\to\infty}(|a_n|)^{1/n}} \geq \frac{1}{\limsup_{n\to\infty}(|a_n|)^{1/n}} > 0$, we deduce that φ_1 is also a convergent power series, and so does φ_2.

Let $\mathcal{N}_1 \subseteq \mathbb{R}\{t\}$ be the ring of Nash germs. $f \in \mathcal{N}_1$, if f is analytic and algebraic over the ring of polynomials with real coefficients.

Proposition 2.3. *The ring $\mathcal{N}_1$ does not have the splitting property.*

Proof. The germ at $0 \in \mathbb{R}$ of the function $]-1, 1[\ni x \mapsto \sum_{n=0}^{\infty} x^n$ is in $\mathcal{N}_1$.

Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $\varphi(\mathbb{N})$ is strictly contained in $\mathbb{N}$ and

$$\lim_{n\to\infty} \frac{\varphi(n+1)}{\varphi(n)} = \infty. \qquad (*)$$

See the following remark for the condition $(*)$.

Consider the following function

$$]-1, 1[\ni x \mapsto s(x) = \sum_{n=0}^{\infty} x^{\varphi(n)}.$$

The result follows from the following lemma. □

Lemma 2.4. *The series $s(x) = \sum_{n=0}^{\infty} x^{\varphi(n)}$ is transcendental over $\mathbb{R}(x)$, where $\mathbb{R}(x)$ is the field of fractions of the ring $\mathbb{R}[x]$.*

Proof of the lemma 2.4. We follow the proof given in [3], page 220.
Suppose that the series $s(x)$ is a root of a polynomial $P(T)$ of degree q, so

$P(T) = a_0(x) + a_1(x)T + ... + a_q(x)T^q$, with $a_j \in \mathbb{R}[x]$. We can suppose that the polynomial $P(T)$ is irreducible over $\mathbb{R}(x)$.
Let $d \in \mathbb{N}$ be the maximum of the degrees of the polynomials $a_j(x)$.

By the condition $(*)$, there exists $n \in \mathbb{N}$ such that

$$n > d \text{ and } \varphi(n+1) \geq (q+1)\varphi(n),$$

We set

$$p(x) = 1 + x^{\varphi(1)} + x^{\varphi(2)} + ... + x^{\varphi(n)}.$$

The series $s(x) - p(x)$ is a root of the polynomial $G(T) = P(T + p(x))$. We set

$$G(T) = b_0(x) + b_1(x)T + ... + b_q(x)T^q$$

where $b_0(x) = a_0(x) + a_1(x)p(x) + ... + a_q(x)p(x)^q$.

We have the following bound of the degrees, denoted by $d^0 b_0(x)$, of the polynomial $b_0(x)$.

$d^0 b_0(x) \leq d + q\varphi(n) < n + q\varphi(n) \leq \varphi(n)(q+1) \leq \varphi(n+1)$.

We used the fact that $n \leq \varphi(n)$.

Since $G(T)$ is irreducible, the polynomial $b_0(x)$ is not the zero polynomial.
We have the following relation

$$\begin{aligned} G(s(x) - p(x)) &= b_0(x) + b_1(x)(x^{\varphi(0)} - 1 + \sum_{j=n+1}^{\infty} x^{\varphi(j)}) + ... + \\ & \quad b_q(x)(x^{\varphi(0)} - 1 + \sum_{j=n+1}^{\infty} x^{\varphi(j)})^q \\ &= 0. \end{aligned}$$

Since $d^0 b_0(x) < \varphi(n+1)$, we see that b_0 is the zero polynomial, which is a contradiction, hence the series $s(x)$ is not algebraic, which proves the lemma.

□

Here is an example of a function that can be taken for $\varphi : \mathbb{N} \to \mathbb{N}$,

$$\varphi(n) = n!.$$

The following remark will not be used later. It is given just to see that the condition $(*)$ can be improved in order to give other examples of functions $\varphi : \mathbb{N} \to \mathbb{N}$ which will enable us to construct other transcendental series from the series $\sum_{n=0}^{\infty} a_n x^n$.

Remark 2.1. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be an increasing function. The gap theorem of Fabry, see [4], states that if $s(x) = \sum_{n=0}^{\infty} x^{\varphi(n)}$ is a complex power series with radius of convergence 1 and

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = \infty, \qquad (**)$$

then the unit circle is the natural boundary of the series $s(z)$. In other words at every point on the unit circle, the series $s(z)$ fails to be analytic.

The Fabry's theorem is an improvement of Hadamard's gap theorem where the condition $(**)$ is replaced by

$$\forall n \in \mathbb{N}, \ \frac{\varphi(n+1)}{\varphi(n)} \geq \theta > 1, \text{ for some } \theta.$$

Let $\varphi : \mathbb{N} \to \mathbb{N}$ be an increasing function. The series $s(x) = \sum_{n=0}^{\infty} x^{\varphi(n)}$ has integer coefficients and its radius of convergence is 1. By a result of Fatou, see [5], if $s(x)$ is algebraic over $\mathbb{Q}(X)$, we can find a polynomial $P \in \mathbb{Q}[X], t, \sigma \in \mathbb{N}$, such that

$$s(x) = \frac{P(x)}{(1-x^{\sigma})^t}, \quad \forall x, \text{ with } |x| < 1.$$

Assume now that the function $\varphi : \mathbb{N} \to \mathbb{N}$ satisfies condition $(**)$.

By Fabry's theorem, the unit circle is the natural boundary of the series

$$s(x) = \sum_{n=0}^{\infty} x^{\varphi(n)} = \frac{P(x)}{(1-x^{\sigma})^t},$$

which is a contradiction, since the only singular points of the series $s(x)$ are the σth roots of unity. We deduce that the series $s(x)$ is not algebraic over $\mathbb{Q}(X)$. By [6], proposition **2**, the series $s(x)$ is not algebraic over $\mathbb{R}(X)$.

We give some examples of functions $\varphi : \mathbb{N} \to \mathbb{N}$ which give us transcendental series $\sum_{n=0}^{\infty} z^{\varphi(n)}$.

1. If $d \in \mathbb{N}$ such that $d > 1$, $\varphi(n) = d^n$.

2. For each $n \in \mathbb{N}$, let $\varphi(n)$ denote the nth prime number, recall that $\varphi(n) \sim n \log n$ when $n \to \infty$.

3. The Borel mapping

Recall that a subring $C_1 \subset \mathcal{E}_1$ is called quasianalytic, if the Borel mapping $\hat{} : C_1 \to \mathbb{R}[[t]]$ is injective. A famous example of a quasianalytic ring is the ring of Denjoy-Carleman class defined as follows:

Let $M = (M_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers which is logarithmically convex (i-e the sequence M_{n+1}/M_n increases). We denote by $\mathcal{D}_1(M)$ the ring of elements f of $\mathcal{E}_1$ for which there exist a neighborhood I of 0 and positive constants c and C (depending on f, but not on n) such that

$$|f^{(n)}(x)| \leq c.C^n n! M_n, \forall n \in \mathbb{N}, \forall x \in I.$$

We recall by [7, theorem 2] that each germ f in the ring $\mathcal{D}_1(M)$ is uniquely determined by the sequence $(f^{(n)}(0))_{n\in\mathbb{N}}$ if and only if

$$\sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} = \infty. \qquad (***)$$

So the fact that each element $f \in \mathcal{D}_1(M)$ is determined by the values of the sequence $(f^{(n)}(0))_{n\in\mathbb{N}}$ (i-e the quasianalyticity) is deduced just from the condition $(***)$ satisfied by the sequence M.

The interested reader will find more information about this in a very readable form in [7].

We know by a result of Carleman [2] that the Borel mapping
$\wedge : \mathcal{D}_1(M) \to \mathbb{R}[[t]]$ over a quasianalytic Denjoy-Carleman ring $\mathcal{D}_1(M)$ is not surjective. We can see this as a consequence of a result of Bang [8, corollary of theorem III] which says that if a formal power series has all its coefficients positive and if this power series is the Taylor expansion (at the origin of $\mathbb{R}$) of an infinitely differentiable function that belongs to a Denjoy-Carleman quasianalytic ring, then this power series is convergent. Whereas, the proof given in [7] is direct just by using techniques from Hilbert space.

But the problem is that this fact is difficult to study over an arbitrary quasianalytic subring of the ring of smooth germs $\mathcal{E}_1$ because we have no control over the growth of the derivatives of the functions belonging to such rings, that's why we will restrict this section to tackle this problem.

By quasianaliticity, we may assume that $\hat{C}_1 \subset \mathbb{R}[[t]]$.

Assume that these quasianalytic rings satisfy the following property called the stability under monomial division:

Let $\hat{f} \in \hat{C}_1$ and $\hat{f} = t\widehat{\varphi}$ where $\widehat{\varphi} \in \mathbb{R}[[t]]$, then $\varphi \in C_1$.

Remark 3.1. By the property of the stability under monomial division, the ring C_1 is a principal domain as it is a subring of the domain $\mathbb{R}[[t]]$.

Proposition 3.1. *If the Borel mapping* $\wedge : C_1 \to \mathbb{R}[[t]]$ *is surjective, then the ring* C_1 *satisfies the splitting property.*

Proof. Let $f \in C_1$ such that $\hat{f} = G + H$ where the support of G and H are disjoint, so by surjectivity of $\wedge$ there exists g and h in C_1 such that $\hat{g} = G$ and $\hat{h} = H$. We have then $\hat{f} = (\widehat{g+h})$ and by quasianalyticity we have $f = g + h$, consequently the ring C_1 satisfies the splitting property. □

We end this note by giving two criterions that allows us to test the non surjectivity of the Borel mapping for a quasianalytic ring.

Proposition 3.2. *Suppose that the Borel mapping* $^\wedge : C_1 \to \mathbb{R}[[t]]$ *is surjective, then* C_1 *is algebraically closed in* $\mathcal{E}_1$.

Proof. Let $f \in \mathcal{E}_1$ such that there exists a polynomial

$$P(X) = a_p(t)X^p + a_{p-1}(t)X^{p-1} + ... + a_0(t) \in C_1[X]$$

with $P(f) = 0$ and $a_0, ..., a_p \in C_1$.
By the remark 3.1, the ring $C_1[X]$ is a unitary factorization domain, we can then suppose that the discriminant $\Delta(P) \neq 0$. Recall that $\Delta(P) \in C_1$. We have

$$\hat{a}_p.(\hat{f})^p + \hat{a}_{p-1}.(\hat{f})^{p-1} + ... + \hat{a}_0 = 0. \qquad (3.1)$$

Since the Borel mapping $^\wedge : C_1 \to \mathbb{R}[|t|]$ is surjective, there exists $\varphi \in C_1$ such that $\hat{\varphi} = \hat{f}$. By quasianalyticity and (3.1), we have $P(\varphi) = 0$. Suppose that $f \neq \varphi$, so $P(X) = (X - f)(X - \varphi)Q(X)$ where $Q \in \mathcal{E}_1[X]$, then $\hat{f}$ is at least a double root of the polynomial $\hat{a}_p(t)X^p + \hat{a}_{p-1}(t)X^{p-1} + ... + \hat{a}_0(t) = 0$, so $\hat{\Delta}(P) = 0$, which is a contradiction since $\Delta(P) \in C_1$, $\Delta(P) \neq 0$, hence $\hat{\Delta}(P) \neq 0$. From this we deduce that $f = \varphi \in C_1$. □

We put,

$$\mathcal{F} = \{\mathcal{B} \subset \mathcal{E}_1 : \mathcal{B} \text{ is a quasianalytic ring, closed under derivation }\}.$$

Proposition 3.3. *Assume that the ring* C_1 *is closed under derivation such that Borel mapping* $^\wedge : C_1 \to \mathbb{R}[[t]]$ *is surjective, then* C_1 *is a maximal element (for inclusion) of* $\mathcal{F}$.

Proof. Suppose that $^\wedge : C_1 \to \mathbb{R}[[t]]$ is surjective. We must show that if $f \in \mathcal{E}_1$ such that the algebra $\mathcal{B} = C_1[f, f', f^{('')}, ...]$ is quasianalytic, then $f \in C_1$. Note that the algebra $\mathcal{B}$ is stable by derivation as each element of its is written as a polynomial in $f, f', f^{('')}, ...$ whose coefficients lie in the ring C_1. Let $g \in C_1$ such that $\hat{g} = \hat{f}$, hence $(\widehat{f - g}) = 0$, we have then $f = g$, since $f, g \in \mathcal{B}$, which proves the proposition. □

Problem : If C_1 is a maximal element of $\mathcal{F}$, does the Borel mapping be surjective?

In case we have a positive response to this problem, we will have built a quasianalytic ring strictly containing the convergent power series ring and satisfying the splitting property.

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Conflict of interest

No potential conflict of interest was reported by the author.

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