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Research article

Spectral properties of a fourth-order eigenvalue problem with quadratic spectral parameters in a boundary condition

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Abstract: Consider the linear eigenvalue problem of fourth-order

$$y^{(4)}(x)-(q(x)y'(x))'=\lambda y(x),\quad 0<x<l,$$

$$y(0)=y'(0)=0,$$
$$(a_0+a_1\lambda+a_2\lambda^2)y'(l)+(b_0+b_1\lambda+b_2\lambda^2)y''(l)=0,$$
$$y(l)\cos\delta-Ty(l)\sin\delta=0,$$

where λ is a spectal parameter, $\delta\in[\frac{\pi}{2},\pi]$, $Ty=y'''-qy'$, $q(x)$ is a positive absolutely continuous function on the interval $[0,l]$, δ, a_i and b_i ($i=0,1,2$) are real constants. We obtain not only the existence, simplicity and interlacing properties of the eigenvalues, the oscillation properties of the eigenfunctions, but also the asymptotic formula of the eigenvalues and the corresponding eigenfunctions for sufficiently large n. Moreover, a new inner Hilbert space and a new sufficient conditions will be given to discuss the basis properties of the system of the eigenfunctions in $L_p(0,l)$.

Keywords: fourth-order eigenvalue problem; quadratic spectral parameter; interlace; oscillation; basis properties
Mathematics Subject Classification: 34B05, 34B09

1. Introduction

For the past few years, the eigenvalue problems with eigenparameter-dependent boundary conditions have been deduced from several applied disciplines, see, for instance, [1–35]. In particular, in [11, 34–36], by using the Prüfer transformation, Binding et al. considered the Sturm-Liouville theory for the second-order eigenvalue problem with spectral parameter in the boundary conditions. They obtained the existence, simplicity and interlacing properties of the eigenvalues and the oscillation theory of the corresponding eigenfunctions. Moreover, by using the Lagrange-type identity and the classical Sturm's methods, Kerimov et al. [6, 7] also obtained the existence, simplicity and interlacing properties and the oscillation properties of the eigenfunctions for the second-order eigenvalue problems with spectral parameter in the boundary conditions. Furthermore, the basis properties of root subspaces in $L_p(0, l)$, $(1 < p < \infty)$ have been obtained under different conditions by several authors, see, for instance, [1, 7–10, 37, 38].

For the fourth-order case, it is well-known that the fourth-order differential equations are always used to describe the deformation of the elastic beam, and in some cases, the end of the beam has to been illustrated by the eigenparameter-dependent boundary conditions, see, for instance, [29, 39]. Therefore, there are also several excellent results on the linear fourth-order eigenvalue problems with spectral parameters in the boundary conditions [2–5, 9, 28–33, 39, 40]. In particular, in [33], Aliyev discussed the basis properties of systems of eigenfunctions (or root functions) for the following kind of problem:

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), \quad 0 < x < l, \tag{1.1}$$

$$y(0) = y'(0) = 0, \tag{1.2a}$$

$$(a\lambda + b)y'(l) + (c\lambda + d)y''(l) = 0, \tag{1.2b}$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0. \tag{1.2c}$$

Here, λ is a spectral parameter, $Ty = y''' - qy'$, q is a positive absolutely continuous function on the interval $[0, l]$, $\delta \in [\frac{\pi}{2}, \pi]$, where $\sigma = bc - ad > 0$. Moreover, Aliyev [2,3,33], Kerimov and Aliyev [4,5], Aliyev and Dunyamalieva [9], Aliyev and Guliyeva [29], also considered the fourth-order eigenvalue problems with linear spectral parameters in the boundary conditions. They obtained the existence, simplicity and interlacing properties of the eigenvalues, the oscillation properties of the eigenfunctions and also the basis properties of the root subspaces in L_p. However, it is noted that most of the above papers focus on the problem with linear spectral parameters in the boundary conditions. Now, the question is: if the quadratic spectral parameters appear in the boundary conditions, could we also obtain the spectral results? In fact, the problems with quadratic eigenparameter-dependent boundary conditions appeared in several applied disciplines, see, for instance, the heat conduction problems [12], the acoustics wave problems [41] and so on.

Therefore, in this paper, we try to consider the spectra of a kind of fourth-order eigenvalue problem with quadratic spectral parameters in the boundary conditions, i.e., the equation (1.1) with the boundary condition (1.2a), (1.2c) and the condition:

$$(a_0 + a_1\lambda + a_2\lambda^2)y'(l) + (b_0 + b_1\lambda + b_2\lambda^2)y''(l) = 0. \tag{1.2b'}$$

Here, for the sake of convenience, let $A(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$, $B(\lambda) = b_0 + b_1\lambda + b_2\lambda^2$, $a_2 \neq 0$, a_i and b_i $(i = 0, 1, 2)$ are real constant. It can be seen that the quadratic parameter arises in the boundary conditions. This lead us to look for a new condition to guarantee the self-adjointness and right-definiteness of the corresponding operator L, furthermore, the reality of the eigenvalues and the eigenfunctions. Therefore, we introduce a new assumption in the rest of the present paper:

(A_1) M is a positive definite matrix, where

$$M = \begin{pmatrix} k & -m \\ -m & n \end{pmatrix}, \quad n = \begin{vmatrix} b_2 & b_1 \\ a_2 & a_1 \end{vmatrix}, \quad m = \begin{vmatrix} b_0 & b_2 \\ a_0 & a_2 \end{vmatrix}, \quad k = \begin{vmatrix} b_1 & b_0 \\ a_1 & a_0 \end{vmatrix}$$

Under the assumption (A_1), we first construct a new Lagrange-type identity to prove the reality of the eigenvalues and define two fundamental functions $F(\lambda)$ and $G(\lambda)$ to look for the location of the eigenvalues on the real-axis, see section 2. Based on the properties of $F(\lambda)$ and $G(\lambda)$, the interlacing properties of the eigenvalues will be obtained, and then the oscillation properties of the eigenfunctions will be also obtained, see section 3. Furthermore, the interlacing properties of the eigenvalues help us obtain the asymptotic formulas of eigenvalues and eigenfunctions in section 4. At last, since the quadratic spectral parameters arise in the boundary condition, a new sufficient condition will be given to discuss the basis property of the system of eigenfunctions in $L_p(0,l), (1 < p < \infty)$. In particular, the system of eigenfunctions is a Riesz basis of $L_2(0,l)$. Meanwhile, for the case that $p \neq 2$, the system of eigenfunctions is just a basis of $L_p(0,l)$ and it is not complete and minimal, see section 5.

2. Preliminaries

Now, if the boundary condition $(1.2b')$ is replaced by:

$$y'(l)\cos\gamma + y''(l)\sin\gamma = 0, \quad \gamma \in [0, \pi/2], \tag{1.2b''}$$

then by Jamel Ben Amara [39](Theorem 5.1 and 5.2), the following result hold.

Theorem 2.1. The eigenvalues of boundary-value problem (1.1), $(1.2a)$, $(1.2b'')$, $(1.2c)$ form infinitely increasing sequence $\{\mu_k(\gamma)\}_{k=1}^{\infty}$ such that

$$\mu_1(\gamma) < \mu_2(\gamma) < \cdots < \mu_n(\gamma) \to \infty,$$

Moreover, the eigenfunction $v_n^{\gamma}(x)$, corresponding to the eigenvalue $\mu_n(\gamma)$, has exactly $n-1$ simple zeros in the interval $(0,l)$.

Let $\mu_n = \mu_n(0)$, $\nu_n = \mu_n(\pi/2)$, $n \in \mathbb{N}$. Set $\mu_0 = -\infty$, then, by Theorem 3 and Theorem 4 in [4], ν_n and μ_n satisfy:

$$\nu_n < \mu_n < \nu_{n+1}, \ n \geqslant 1. \tag{2.1}$$

Lemma 2.2. [4]. For each $\lambda \in \mathbb{C}$, there exists a unique (up to a factor) nontrivial solution $y(x,\lambda)$ of problem (1.1), $(1.2a)$, $(1.2c)$.

Remark 2.3. [4]. Without loss of generality, we can assume that the solution $y(x,\lambda)$ of problem (1.1), $(1.2a)$, $(1.2c)$ is an entire function of λ for each $x \in [0,l]$.

Lemma 2.4. [4]. If $\lambda \in (\mu_{n-1}, \mu_n]$, then $m(\lambda) = n-1$, where $m(\lambda)$ is the number of zeros of $y(x,\lambda)$ in the interval $(0,l)$.

Lemma 2.5. [20, 25, 28]. Suppose that (A_1) holds, then both $A(\lambda)$ and $B(\lambda)$ have two different real zeros, respectively. Moreover, $A(\lambda)$ and $B(\lambda)$ do not have the same roots.

Lemma 2.6. Suppose that $y(x,\lambda)$ is a solution of the problem (1.1), $(1.2a)$, $(1.2c)$. Then for each $x \in [0,l]$

$$(\mu - \lambda)\int_0^x y(x,\mu)y(x,\lambda)ds = y''(x,\lambda)y'(x,\mu) - y''(x,\mu)y'(x,\lambda). \tag{2.2}$$

Proof. By virtue of (1.1), we have

$$(Ty(x,\mu))'y(x,\lambda) - (Ty(x,\lambda))'y(x,\mu) = (\mu-\lambda)y(x,\mu)y(x,\lambda).$$

Integrating from 0 to x of this relation by parts and taking into account conditions (1.2a) and (1.2c), we obtain

$$(\mu-\lambda)\int_0^x y(s,\mu)y(s,\lambda)ds = y''(x,\lambda)y'(x,\mu) - y''(x,\mu)y'(x,\lambda).$$

Lemma 2.7. Suppose that $y(x,\lambda)$ is a solution of the problem (1.1), (1.2a), (1.2c). Then for each $x \in [0,l]$

$$\int_0^x y^2(s,\lambda)ds = \begin{vmatrix} y''(x,\lambda) & y'(x,\lambda) \\ \frac{\partial}{\partial\lambda}y''(x,\lambda) & \frac{\partial}{\partial\lambda}y'(x,\lambda) \end{vmatrix}. \tag{2.3}$$

Proof. Dividing both side of (2.2) by $\mu-\lambda$ we have

$$\int_0^x y(s,\mu)y(s,\lambda)ds = \frac{y''(x,\lambda)y'(x,\mu) - y''(x,\mu)y'(x,\lambda)}{\mu-\lambda}.$$

Passing to the limits as $\mu \to \lambda$, we obtain

$$\begin{aligned}\int_0^x y^2(s,\lambda)ds &= y''(x,\lambda)\frac{\partial}{\partial\lambda}y'(x,\lambda) - y'(x,\lambda)\frac{\partial}{\partial\lambda}y''(x,\lambda)\\ &= \begin{vmatrix} y''(x,\lambda) & y'(x,\lambda) \\ \frac{\partial}{\partial\lambda}y''(x,\lambda) & \frac{\partial}{\partial\lambda}y'(x,\lambda) \end{vmatrix}.\end{aligned}$$

Lemma 2.8. Suppose that (A_1) holds. Then the eigenvalues of (1.1), (1.2a), (1.2b'), (1.2c) are real, simple and form an at most countable set without finite limit points.
Proof. Let $y(x,\lambda)$ be a nontrivial solution of problem (1.1), (1.2a), (1.2c). The eigenvalues of problem (1.1),(1.2a),(1.2b'),(1.2c) are the roots of the equation

$$(a_0 + a_1\lambda + a_2\lambda^2)y'(l,\lambda) + (b_0 + b_1\lambda + b_2\lambda^2)y''(l,\lambda) = 0. \tag{2.4}$$

Firstly, let us prove that the eigenvalues of (1.1), (1.2a), (1.2b'), (1.2c) are real. Suppose on the contrary, then the problem (1.1), (1.2a), (1.2b'), (1.2c) has nonreal eigenvalues. Let λ^* be a nonreal eigenvalue of the problem (1.1), (1.2a), (1.2b'), (1.2c) and $y(x,\lambda^*)$ is the corresponding eigenfunction. Then $\overline{\lambda}^*$ is also an eigenvalue of this problem and $y(x,\overline{\lambda}^*)$ is the corresponding eigenfunction, Moreover, $y(x,\overline{\lambda}^*) = \overline{y(x,\lambda^*)}$.

Now, by Lemma 2.6 with $x = l$, $\mu = \overline{\lambda}^*$ and $\lambda = \lambda^*$, we get that

$$y''(l,\lambda^*)\overline{y'(l,\lambda^*)} - \overline{y''(l,\lambda^*)}y'(l,\lambda^*) = (\overline{\lambda}^* - \lambda^*)\int_0^l |y(x,\lambda^*)|^2dx. \tag{2.5}$$

Since λ^* is a root of (2.4), by Lemma 2.5, we have the relation

$$y''(l,\lambda^*) = -\frac{a_0 + a_1\lambda^* + a_2\lambda^{*2}}{b_0 + b_1\lambda^* + b_2\lambda^{*2}}y'(l,\lambda^*). \tag{2.6}$$

Combining this with (2.5), we obtain

$$\frac{(-n\lambda^*\overline{\lambda}^* + m(\overline{\lambda}^* + \lambda^*) - k)}{[B(\lambda^*)]^2}|y'(l,\lambda^*)|^2 = \int_0^l |y(x,\lambda^*)|^2 dx. \tag{2.7}$$

Since M is positive definite, we know that

$$n\lambda^*\overline{\lambda}^* - m(\overline{\lambda}^* + \lambda^*) + k = (1,\lambda^*)\begin{pmatrix} k & -m \\ -m & n \end{pmatrix}\begin{pmatrix} 1 \\ \overline{\lambda}^* \end{pmatrix} > 0.$$

Therefore, the left side of (2.7) is negative, a contradiction. Therefore, $\lambda^* \in \mathbb{R}$. This implies that the entire function arises from the left-hand side of (2.4) does not vanish for nonreal λ. Therefore, the roots of (2.4) form an at most countable set without finite limit points.

Secondly, we will show that (2.4) has only simple roots. Actually, if λ^* is a multiple zero of (2.4), then

$$(a_0 + a_1\lambda^* + a_2\lambda^{*2})y'(l,\lambda^*) + (b_0 + b_1\lambda^* + b_2\lambda^{*2})y''(l,\lambda^*) = 0,$$

and

$$\begin{aligned} A(\lambda^*)\frac{\partial}{\partial\lambda^*}y'(x,\lambda^*) + (a_1 + 2a_2\lambda^*)y'(x,\lambda^*) + B(\lambda^*)\frac{\partial}{\partial\lambda}y''(x,\lambda^*) \\ + (b_1 + 2b_2\lambda^*)y''(x,\lambda^*) = 0. \end{aligned} \tag{2.8}$$

By Lemma 2.7 with $x = l$, we obtain

$$\int_0^l y^2(x,\lambda^*)dx = y''(l,\lambda^*)\frac{\partial}{\partial\lambda^*}y'(l,\lambda^*) - y'(l,\lambda^*)\frac{\partial}{\partial\lambda^*}y''(l,\lambda^*). \tag{2.9}$$

By Lemma 2.5, we know that $A(\lambda)$ and $B(\lambda)$ do not equal zero together. Without loss of generality, suppose that $A(\lambda^*) \neq 0$, Then, by (2.4), (2.8) and (2.9), we have

$$-\frac{n\lambda^{*2} - 2m\lambda^* + k}{A^2(\lambda^*)}(y''(l,\lambda^*))^2 = \int_0^l y^2(x,\lambda^*)dx. \tag{2.10}$$

In view of (A_1), we get that $(1,\lambda^*)M(1,\lambda^*)^T > 0$, which implies that the left-side of (2.10) is less than zero. Therefore, we get a contradiction and all roots of (2.4) are simple. □

According to [29], we introduce the following function

$$F(\lambda) = \frac{y''(x,\lambda)}{y'(x,\lambda)}, \ \ \lambda \in K \equiv (\mathbb{C}\setminus\mathbb{R}) \cup \left(\bigcup_{n=1}^{\infty}(\mu_{n-1},\mu_n)\right).$$

using the notation $\mu_0 = -\infty$, $\mu_n(\frac{\pi}{2})$ and $\mu_n(0)$, $n \in \mathbb{N}$, are the zeros and poles of the function, respectively. Now, by Aliyev and Guliyeva ([29], Lemma 3.3 and 3.4), we have the following result.

Lemma 2.9. [29]. $F(\lambda)$ satisfies the following properties:

(a) $F(\lambda)$ is continuous and strictly decreasing on each interval (μ_{n-1},μ_n), $n \in \mathbb{N}$;

(b) $\lambda = \mu_n$ is the vertical asymptote of $F(\lambda)$ with
$\lim_{\lambda\to\mu_n^-} F(\lambda) = -\infty$ and $\lim_{\lambda\to\mu_n^+} F(\lambda) = +\infty$;

(c) $\lim_{\lambda\to-\infty} F(\lambda) = +\infty$.

Now, let us introduce another function $G(\lambda)$. By Lemma 2.5, $A(\lambda)$ has two distinct real roots ξ_1 and ξ_2, with $\xi_1 < \xi_2$, $B(\lambda)$ has two distinct real roots ζ_1 and ζ_2, with $\zeta_1 < \zeta_2$. So, we could define the function $G(\lambda)$ as

$$G(\lambda) = -\frac{a_0 + a_1\lambda + a_2\lambda^2}{b_0 + b_1\lambda + b_2\lambda^2}, \quad \lambda \neq \zeta_1, \zeta_2.$$

Therefore, $G(\xi_1) = G(\xi_2) = 0$ and the graph of $G(\lambda)$ will be divided into three branches with two vertical asymptotes $\lambda = \zeta_1$ and $\lambda = \zeta_2$. Moreover, since M is positive definite, we know that $m^2 - nk < 0$, $k > 0$ and $n > 0$.

$$G'(\lambda) = -\frac{(a_2b_1 - a_1b_2)\lambda^2 + 2(a_2b_0 - a_0b_2)\lambda + a_1b_0 - a_0b_1}{(b_0 + b_1\lambda + b_2\lambda^2)^2} = \frac{n\lambda^2 - 2m\lambda + k}{(b_0 + b_1\lambda + b_2\lambda^2)^2}.$$

Combining this with the fact that M is a positive definite matrix, we know that $G'(\lambda) > 0$ for $\lambda \in (-\infty, \zeta_1)$, $\lambda \in (\zeta_1, \zeta_2)$ and $\lambda \in (\zeta_2, +\infty)$. Therefore, $G'(\lambda)$ is increasing on each of its branches, and

$$\lim_{s \to \zeta_i^-} G(s) = +\infty, \qquad \lim_{s \to \zeta_i^+} G(s) = -\infty, \qquad \lim_{s \to \infty} G(s) = -\frac{a_2}{b_2}.$$

The followings are the graphs of $G(\lambda)$ in two cases: $\frac{a_2}{b_2} < 0$ and $\frac{a_2}{b_2} > 0$ (Figure 1 and 2).

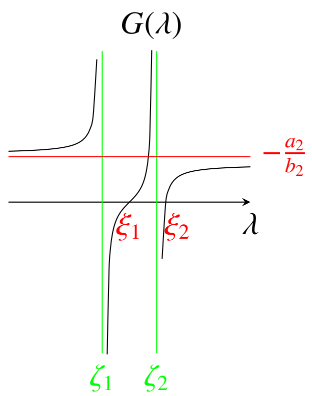

Figure 1. $\frac{a_2}{b_2} < 0$

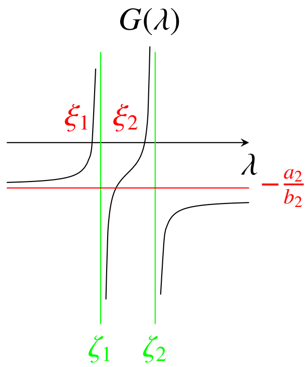

Figure 2. $\frac{a_2}{b_2} > 0$

3. Interlacing and oscillation results

From Lemma 2.9, we know that the graph of $F(\lambda)$ is composed of countable strictly decreasing branches, let us use $\mathscr{L}_1, \mathscr{L}_2, \cdots \mathscr{L}_n, \cdots$ to denote these branches. By the properties of $G(\lambda)$, we denote these three branches by $\mathscr{D}_0$, $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively. In addition, μ_n is the eigenvalue of boundary value problem (1.1), (1.2a), (1.2b'') and (1.2c) for $\gamma = 0$, ν_n is the eigenvalue of boundary value problem (1.1), (1.2a), (1.2b'') and (1.2c) for $\gamma = \frac{\pi}{2}$.

Suppose that ζ_1 intersects the T_1-th branches $\mathscr{L}_{T_1}$ and ζ_2 intersects the T_2-th branch $\mathscr{L}_{T_2}$ of $F(\lambda)$ or their right hand asymptotes, i.e., we select two nonnegative integers T_1 and T_2, such that

$$\mu_{T_1-1} < \zeta_1 \leqslant \mu_{T_1}, \quad \mu_{T_2-1} < \zeta_2 \leqslant \mu_{T_2}. \tag{3.1}$$

Define two nonnegative integers L_1 and L_2, such that

$$\nu_{L_1-1} < \xi_1 \leqslant \nu_{L_1}, \quad \nu_{L_2-1} < \xi_2 \leqslant \nu_{L_2}. \tag{3.2}$$

Without loss of generality, suppose that $\zeta_1 < \zeta_2$ and $\xi_1 < \xi_2$. Then $T_1 \leqslant T_2$ and $L_1 \leqslant L_2$. Furthermore, the properties of $G(\lambda)$ imply that

$$T_1 \leqslant L_1 \leqslant T_2 \leqslant L_2 \text{ and } \zeta_1 < \xi_1 < \zeta_2 < \xi_2, \text{ for } \frac{a_2}{b_2} < 0 \tag{3.3}$$

and

$$L_1 \leqslant T_1 \leqslant L_2 \leqslant T_2 \text{ and } \xi_1 < \zeta_1 < \xi_2 < \zeta_2, \text{ for } \frac{a_2}{b_2} > 0. \tag{3.4}$$

Theorem 3.1 (*Interlacing results*). Suppose that (A_1) holds. If $\frac{a_2}{b_2} < 0$ and $T_1 = T_2$, then the eigenvalues of problems (1.1), (1.2a), (1.2b'), (1.2c) form an infinitely increasing sequence λ_1, λ_2, $\cdots$ λ_n, without finite accumulative point. Furthermore, these eigenvalues satisfy the following interlacing properties:

(a) If $\mu_{T_1-1} \leqslant \zeta_1 < \nu_{T_1} < \zeta_2 \leqslant \mu_{T_1}$, then the eigenvalues $\{\lambda_n\}$ satisfy the following interlacing inequalities:

$$\lambda_n < \nu_n, \text{for } n = 1, 2, \cdots T_1 - 1, \tag{3.5}$$

$$\nu_n < \lambda_{n+2} < \mu_n, \text{for } n = T_1 + 1, T_2, \cdots L_2 - 1, \tag{3.6}$$

$$\lambda_{n+2} < \nu_n < \mu_n, \text{for } n = L_2 + 1, L_2 + 2, \cdots \tag{3.7}$$

Here, $\lambda_{L_2+2} = \nu_{L_2}$ if and only if $\nu_{L_2} = \xi_2$, otherwise, $\xi_2 < \lambda_{L_2+2} < \nu_{L_2} < \mu_{L_2}$ or $\nu_{L_2} < \lambda_{L_2+2} < \xi_2 < \mu_{L_2}$. Furthermore, if $\nu_{T_1} \leqslant \xi_1$, then the $T_1 - 1$-th, T_1-th and $T_1 + 1$-th eigenvalues satisfy the following inequalities:

$$\lambda_{T_1} < \mu_{T_1-1} \leqslant \zeta_1 < \nu_{T_1} < \lambda_{T_1+1} < \xi_1 < \zeta_2 < \lambda_{T_1+2} \leqslant \mu_{T_1}. \tag{3.8}$$

If $\nu_{T_1} > \xi_1$, then the $T_1 - 1$-th, T_1-th and $T_1 + 1$-th eigenvalues satisfy the following inequalities:

$$\lambda_{T_1} < \mu_{T_1-1} \leqslant \zeta_1 < \xi_1 < \lambda_{T_1+1} < \nu_{T_1} < \zeta_2 < \lambda_{T_1+2} \leqslant \mu_{T_1}. \tag{3.9}$$

Here, $\lambda_{T_1+1} = \nu_{T_1}$ if and only if $\xi_1 = \nu_{T_1}$, $\lambda_{T_1+2} = \mu_{T_1}$ if and only if $\zeta_1 = \mu_{T_1}$.

(b) If $\zeta_1 < \zeta_2 \leqslant \nu_{T_1}$, then (3.5), (3.6) and (3.7) also hold, and the $T_1 - 1$-th, T_1-th and $T_1 + 1$-th eigenvalues satisfy

$$\lambda_{T_1} \leqslant \zeta_1 < \xi_1 < \lambda_{T_1+1} < \zeta_2 \leqslant \nu_{T_1} < \lambda_{T_1+2} < \mu_{T_1}. \tag{3.10}$$

Here, $\lambda_{L_2+2} = \nu_{L_2}$ if and only if $\nu_{L_2} = \xi_2$, otherwise, $\xi_2 < \lambda_{L_2+2} < \nu_{L_2} < \mu_{L_2}$ or $\nu_{L_2} < \lambda_{L_2+2} < \xi_2 < \mu_{L_2}$.

(c) If $\nu_{T_1} \leqslant \zeta_1 < \zeta_2 \leqslant \mu_{T_1}$, then (3.5), (3.6) and (3.7) also hold, and the $T_1 - 1$-th, T_1-th and $T_1 + 1$-th eigenvalues satisfy

$$\lambda_{T_1} < \nu_{T_1} \leqslant \zeta_1 < \lambda_{T_1+1} < \xi_1 < \zeta_2 \leqslant \lambda_{T_1+2} \leqslant \mu_{T_1}. \tag{3.11}$$

Here, $\lambda_{L_2+1} = \nu_{L_2}$ if and only if $\nu_{L_2} = \xi_2$, otherwise, $\xi_2 < \lambda_{L_2+1} < \nu_{L_2} < \mu_{L_2}$ or $\nu_{L_2} < \lambda_{L_2+1} < \xi_2 < \mu_{L_2}$.

Proof. Firstly, we will show that how $\mathscr{D}_0$ intersects the branches of $F(\lambda)$. Actually, $\mathscr{D}_0$ intersects the branches of $F(\lambda)$ from $\mathscr{L}_1$ to $\mathscr{L}_{T_1-1}$. More precisely, if $\mu_{T_1-1} < \zeta_1 < \nu_{T_1} < \zeta_2 < \mu_{T_1}$, then $\mathscr{D}_0$ intersects the upper parts of $\mathscr{L}_i$ for $i = 1, 2 \cdots T_1 - 1$. Therefore, for $\mu_{T_1-1} < \zeta_1 < \nu_{T_1} < \zeta_2 < \mu_{T_1}$, we have found the first $T_1 - 1$ intersection points of $F(\lambda)$ and $G(\lambda)$. The projection of these points on λ-axis are the eigenvalues of the problem (1.1), (1.2a), (1.2b'), (1.2c) and these eigenvalues satisfy (3.5).

Secondly, let us look for the intersection points of $\mathscr{D}_2$ and $\mathscr{L}_i$. When $\nu_{L_2-1} < \xi_2 \leqslant \nu_{L_2}$, $\mathscr{D}_2$ intersects the lower parts of $\mathscr{L}_i$ for $i = T_1+1, T_1+2\cdots L_2-1$. Meanwhile, $\mathscr{D}_2$ intersects the upper parts of $\mathscr{L}_i$ for $i = L_2+1, T_2+2\cdots$ This implies that (3.6) and (3.7) also hold for this case.

Finally, we consider the intersection point of $\mathscr{D}_1$ and the branches of $F(\lambda)$. Since $\mu_{T_1-1} < \zeta_i \leq \mu_{T_1}$, $\{i = 1, 2\}$, we obtain that $\mathscr{D}_1$ only intersect the branch $\mathscr{L}_{T_1}$.

Case I. If $\nu_{T_1} \leqslant \xi_1$, then $\mathscr{D}_1$ intersects the lower of the branch $\mathscr{L}_{T_1}$, we obtain the T_1+1-th eigenvalue, let's denote it by λ_{T_1+1}. Obviously, it satisfies $\zeta_1 < \nu_{T_1} < \lambda_{T_1+1} \leqslant \xi_1 < \zeta_2$. This implies that (3.8) also holds for this case.

Case II. If $\nu_{T_1} > \xi_1$, then $\mathscr{D}_1$ intersects the upper of the branch $\mathscr{L}_{T_1}$, we obtain the T_1+1-th eigenvalue, let's denote it by λ_{T_1+1}. Obviously, it satisfies $\zeta_1 < \xi_1 < \lambda_{T_1+1} < \nu_{T_1} < \zeta_2$. This implies that (3.9) also holds for this case.

Case III. $\zeta_1 < \zeta_2 \leqslant \nu_{T_1}$. In this case, $\xi_2 < \nu_{T_1}$ and $\mathscr{D}_1$ only intersects the upper parts of branch $\mathscr{L}_{T_1}$, Then, by the monotonicity of $G(\lambda)$ and $F(\lambda)$, we could get the T_1+1-th eigenvalue λ_{T_1+1} of problem (1.1), (1.2*a*), (1.2*b*′), (1.2*c*). Obviously, it satisfies $\zeta_1 < \xi_1 < \lambda_{T_1+1} < \zeta_2 \leqslant \nu_{T_1}$. Therefore, (3.10) holds.

Case IV. $\nu_{T_1} \leqslant \zeta_1 < \zeta_2 \leqslant \mu_{T_1}$. In this case, $\xi_2 > \nu_{T_1}$ and $\mathscr{D}_1$ only intersects the lower parts of branch $\mathscr{L}_{T_1}$, Then, by the monotonicity of $G(\lambda)$ and $F(\lambda)$, we could get the T_1+1-th eigenvalue λ_{T_1+1} of problem (1.1), (1.2*a*), (1.2*b*′), (1.2*c*). Obviously, it satisfies $\nu_{T_1} \leqslant \zeta_1 < \lambda_{T_1+1} < \xi_1 < \zeta_2 \leqslant \mu_{T_1}$. Therefore, (3.11) holds. □

Theorem 3.2. (*Oscillation result*) Under the assumptions and the notation of Theorem 3.1, the eigenfunction $y_k(x)$, corresponding to λ_k, satisfies the following oscillation properties: (*i*) for $k \leqslant T_1$, the eigenfunction $y_k(x)$ has exactly $k-1$ simple zeros; (*ii*) for $k = T_1+1$, the eigenfunction $y_k(x)$ has exactly $k-2$ simple zeros; (*iii*) for $k \geqslant T_1+2$, the eigenfunction $y_k(x)$ has exactly $k-3$ simple zeros.

Proof. First, from (3.5) and (3.8), we get that the first T_1 eigenvalues satisfy: $\mu_{k-1} < \lambda_k < \mu_k$, $k = 1, 2, \cdots T_1$. By Lemma 2.4, we get that the eigenfunction $y_k(x)$ has exactly $k-1$ simple zeros in $(0, l)$; Second, by (3.5)–(3.9) and $\lambda_{T_1+1} = \nu_{T_1}$ if and only if $\nu_{T_1} = \xi_1$, we obtain that $\mu_{T_1-1} < \lambda_{T_1+1} < \mu_{T_1}$. Therefore, Lemma 2.4 implies that the T_1+1-th eigenfunction $y_{T_1+1}(x)$ has exactly T_1-1 simple zeros in $(0, l)$; Third, from (3.6)–(3.11), we know that $\mu_{T_1-1} < \lambda_{T_1+2} < \mu_{T_1}$ and $\mu_{k-1} < \lambda_{k+2} < \mu_k$, $k = T_1+1, T_1+2\cdots$. Therefore, by Lemma 2.4, the eigenfunction $y_k(x)$ corresponding to λ_k has exactly $k-3$ simple zeros in $(0, l)$. □

The following interlacing results and the oscillation properties will obtained by using the similar method to Theorem 3.1 and Theorem 3.2. So, we only state the results and omit the proof here.

Theorem 3.3. (*Interlacing results*) Suppose that (A_1) holds. If $\frac{a_2}{b_2} < 0$ and $T_1 < T_2$, then the eigenvalues of problems (1.1), (1.2*a*), (1.2*b*′), (1.2*c*) form an infinitely increasing sequence $\lambda_1, \lambda_2, \cdots \lambda_n$, without finite accumulative point. Furthermore, the following cases will hold:

Case (a). If $\mu_{T_1-1} < \zeta_1 < \mu_{T_1}$ and $\mu_{T_2-1} < \zeta_2 < \mu_{T_2}$, then

$$\lambda_n < \nu_n < \mu_n, \quad \text{for} \quad n = 1, 2, \cdots T_1 - 1, \tag{3.12}$$

$$\nu_n < \lambda_{n+1} < \mu_n, \text{for } n = T_1+1\cdots L_1-1; \lambda_{n+1} < \nu_n < \mu_n, \text{for } n = L_1+1\cdots T_2-1, \tag{3.13}$$

$$\nu_n < \lambda_{n+2} < \mu_n, \text{for } n = T_2+1, T_2+2\cdots L_2-1, \lambda_{n+2} < \nu_n < \mu_n, \text{for } n = L_2+1, L_2+2\cdots \tag{3.14}$$

Here, $\mu_{T_1-1} < \lambda_{T_1} < \nu_{T_1} < \mu_{T_1}$, $\zeta_1 \leqslant \lambda_{T_1+1} < \mu_{T_1}$; $\lambda_{L_1+1} = \nu_{L_1}$ if and only if $\nu_{L_1} = \xi_1$, otherwise, $\nu_{L_1} < \lambda_{L_1+1} < \xi_1 < \mu_{L_1}$; $\mu_{T_2-1} < \lambda_{T_2+1} < \nu_{T_2}$, $\zeta_2 \leqslant \lambda_{T_2+2} < \mu_{T_2}$; $\lambda_{L_2+2} = \nu_{L_2}$ if and only if $\nu_{L_2} = \xi_2$, otherwise, $\nu_{L_2} < \lambda_{L_2+2} < \xi_2 < \mu_{L_2}$.
Case (b). If $\zeta_1 = \mu_{T_1}$ and $\zeta_2 = \mu_{T_2}$, then

$$\lambda_n < \nu_n < \mu_n, \text{for } n = 1, 2, \cdots T_1, \tag{3.15}$$

$$\nu_n < \lambda_n < \mu_n, \text{for } n = T_1 + 1 \cdots L_1 - 1; \lambda_n < \nu_n < \mu_n, \text{for } n = L_1 + 1 \cdots T_2 - 1, \tag{3.16}$$

$$\nu_n < \lambda_n < \mu_n, \text{for } n = T_2, T_2 + 1 \cdots L_2 - 1; \lambda_n < \nu_n < \mu_n, \text{for } n = L_2 + 1, L_2 + 2 \cdots \tag{3.17}$$

Here, $\lambda_{L_1} = \nu_{L_1}$ if and only if $\nu_{L_1} = \xi_1$, otherwise, $\nu_{L_1} < \lambda_{L_1} < \xi_1 < \mu_{L_1}$; $\lambda_{L_2} = \nu_{L_2}$ if and only if $\nu_{L_2} = \xi_2$, otherwise, $\nu_{L_2} < \lambda_{L_2} < \xi_2 < \mu_{L_2}$.
Case (c). If $\zeta_1 = \mu_{T_1}, \mu_{T_2-1} < \zeta_2 < \mu_{T_2}$, then

$$\lambda_n < \nu_n < \mu_n, \text{for } n = 1, 2, \cdots T_1, \tag{3.18}$$

$$\nu_n < \lambda_n < \mu_n, \text{for } n = T_1 + 1 \cdots L_1 - 1; \lambda_n < \nu_n < \mu_n, \text{for } n = L_1 + 1 \cdots T_2 - 1, \tag{3.19}$$

$$\nu_n < \lambda_{n+1} < \mu_n, \text{for } n = T_2 + 1, T_2 + 2 \cdots L_2 - 1; \lambda_{n+1} < \nu_n < \mu_n, \text{for } i = L_2 + 1, L_2 + 2 \cdots \tag{3.20}$$

Here, $\lambda_{L_1} = \nu_{L_1}$ if and only if $\nu_{L_1} = \xi_1$, otherwise, $\nu_{L_1} < \lambda_{L_1} < \xi_1 < \mu_{L_1}$; $\mu_{T_2-1} < \lambda_{T_2} < \nu_{T_2}$, $\zeta_2 \leqslant \lambda_{T_2+1} < \mu_{T_2}$; $\lambda_{L_2+1} = \nu_{L_2}$ if and only if $\nu_{L_2} = \xi_2$, otherwise, $\nu_{L_2} < \lambda_{L_2+1} < \xi_2 < \mu_{L_2}$.
Case (d). If $\mu_{T_1-1} < \zeta_1 < \mu_{T_1}$ $\zeta_2 = \mu_{T_2}$, then

$$\lambda_n < \nu_n < \mu_n, \text{for } n = 1, 2, \cdots T_1 - 1, \tag{3.21}$$

$$\nu_n < \lambda_{n+1} < \mu_n, \text{for } n = T_1 + 1 \cdots L_1 - 1; \lambda_{n+1} < \nu_n < \mu_n, \text{for } n = L_1 + 1 \cdots T_2, \tag{3.22}$$

$$\nu_n < \lambda_{n+1} < \mu_n, \text{for } n = T_2 + 1 \cdots L_2 - 1; \lambda_{n+1} < \nu_n < \mu_n, \text{for } n = L_2 + 1, L_2 + 2 \cdots \tag{3.23}$$

Here, $\lambda_{T_1} < \nu_{T_1} < \mu_{T_1}$, $\zeta_1 \leqslant \lambda_{T_1+1} < \mu_{T_1}$; $\lambda_{L_1+1} = \nu_{L_1}$ if and only if $\nu_{L_1} = \xi_1$, otherwise, $\nu_{L_1} < \lambda_{L_1+1} < \xi_1 < \mu_{L_1}$; $\lambda_{L_2+1} = \nu_{L_2}$ if and only if $\nu_{L_2} = \xi_2$, otherwise, $\nu_{L_2} < \lambda_{L_2+1} < \xi_2 < \mu_{L_2}$.

Theorem 3.4. (*Oscillation result*) Under the assumptions and the notation of Theorem 3.3, the eigenfunction $y_k(x)$, corresponding to λ_k, satisfies the following oscillation properties: (i) for $k \leqslant T_1$, the eigenfunction $y_k(x)$ has exactly $k - 1$ simple zeros; (ii) for $T_1 + 1 \leqslant k \leqslant T_2$, the eigenfunction $y_k(x)$ has exactly $k - 2$ simple zeros; (iii) for $k \geqslant T_2 + 1$, the eigenfunction $y_k(x)$ has exactly $k - 3$ simple zeros.

Remark. For the case that $\frac{a_2}{b_2} > 0$, we could get the interlacing and oscillation results by the same methods with obvious changes. The biggest difference between this case and the case that $\frac{a_2}{b_2} < 0$ is the interlacing results of the eigenvalues, see the following corollary.
Corollary 3.5. The following relations hold for sufficiently large $n \in \mathbb{N}$;

$$\mu_{n-2} < \lambda_n < \nu_{n-1} < \mu_{n-1}, \quad \text{for } b_2 \neq 0, \ \frac{a_2}{b_2} < 0; \tag{3.24}$$

$$\mu_{n-2} < \nu_{n-1} < \lambda_n < \mu_{n-1}, \quad \text{for } b_2 \neq 0, \ \frac{a_2}{b_2} > 0; \tag{3.25}$$

$$\mu_{n-1} < \lambda_{n+1} < \nu_n < \mu_n, \quad \text{for } b_2 = 0. \tag{3.26}$$

4. The asymptotic formulas for the eigenvalues and eigenfunctions

First, let us introduce the notation $s(\delta_1, \delta_2) \equiv \operatorname{sgn}\delta_1 + \operatorname{sgn}\delta_2$. Define numbers α, β, $\alpha_n, \beta_n, n \in \mathbb{N}$ and functions $\psi(x,t)$ and $\varphi(x,t)$, $x \in [0,l]$, $t \in \mathbb{R}$, as follows:

$$\alpha = \frac{1}{4}(1 + \operatorname{sgn}\gamma), \ \ \beta = \frac{5 + s(a_2, b_2)}{4}, \tag{4.1}$$

$$\alpha_n = \frac{(n-\alpha)\pi}{l}, \ \ \beta_n = \frac{(n-\beta)\pi}{l}, \tag{4.2}$$

$$\psi(x,t) = \sin tx - \cos tx + e^{-tx} + (-1)^{1-\operatorname{sgn}\gamma}\sqrt{2}\sin(tl + (-1)^{\operatorname{sgn}\gamma}\frac{\pi}{4})e^{-t(l-x)}, \tag{4.3}$$

$$\varphi(x,t) = \sin tx - \cos tx + e^{-tx} + (-1)^{1-s(a_2,b_2)}\sqrt{2}\sin(tl + (-1)^{s(a_2,b_2)}\frac{\pi}{4})e^{-t(l-x)}. \tag{4.4}$$

Theorem 4.1. Suppose that (A_1) holds. Then the eigenvalues and eigenfunctions of boundary value problem (1.1), (1.2a), (1.2b') and (1.2b''), (1.2c) have the following asymptotic formulas:

$$\sqrt[4]{\mu_n(\gamma)} = \alpha_n + O\left(\frac{1}{n}\right), \tag{4.5}$$

$$v_n^{(\gamma)}(x) = \psi(x, \alpha_n) + O\left(\frac{1}{n}\right), \tag{4.6}$$

$$\sqrt[4]{\lambda_n} = \beta_n + O\left(\frac{1}{n}\right), \tag{4.7}$$

$$y_n(x,t) = \varphi(x, \beta_n) + O\left(\frac{1}{n}\right). \tag{4.8}$$

Proof. Set $\lambda = \rho^4$, in view of ([42], P_{49}), the Eq.(1.1) has four linearly independent solutions $y_k(x,\rho)(k = 1 \cdots 4)$, which are regular with respect to ρ. If ρ is sufficiently large, then these solutions satisfy the relations

$$y_k^{(s)}(x,\rho) = (\rho\omega_k)^s\left[1 + O\left(\frac{1}{\rho}\right)\right] \ \ k = 1, \cdots 4, \ \ s = 0, \cdots 3, \tag{4.9}$$

where $\omega_k(k = 1, \cdots 4)$ are four different roots of 1.

From ([42], P_{56}), the boundary conditions (1.2a), (1.2b'), (1.2c) are regular. So, if we set

$$\tau_n = \sqrt[4]{\mu_n(0)} = \sqrt[4]{\mu_n}, \ \ \sigma_n = \sqrt[4]{\mu_n(\gamma)} \ \ \left(\gamma \in (0, \frac{\pi}{2}]\right), \ \ \rho_n = \sqrt[4]{\lambda_n},$$

then by Theorem 2 in ([42], P_{61}) for $\delta \in (0, \frac{\pi}{2}]$, for sufficiently large indices k, we have

$$\tau_{k+n_0} = (k - \frac{1}{4})\frac{\pi}{l} + O\left(\frac{1}{k}\right), \tag{4.10}$$

$$\sigma_{k+n_1} = (k - \frac{1}{2})\frac{\pi}{l} + O\left(\frac{1}{k}\right), \tag{4.11}$$

where n_0 and n_1 are some integers. According to the relations (4.10), (4.11), Theorem 2.1 and Property 1 in ([43], Sec.4), $n_1 = n_0 + 1$.

By taking into account relations (4.9) in the boundary conditions (1.2) and Corollary 3.5, we obtain for sufficiently large k that

$$\rho_{k+n_2} = (k-\frac{1}{4})\frac{\pi}{l} + O\left(\frac{1}{k}\right) \quad \text{if } b_2 = 0,\ a_2 \neq 0, \tag{4.12}$$

$$\rho_{k+n_3} = (k-\frac{1}{2})\frac{\pi}{l} + O\left(\frac{1}{k}\right) \quad \text{if } b_2 \neq 0, \tag{4.13}$$

where n_2 and n_3 are some integers, From Theorem 3.1, Theorem 3.3, Corollary 3.5 and (4.10)-(4.13), $n_2 = n_0 + 1$ and $n_3 = n_0 + 2$.

Therefore, for sufficiently large k,we have

$$\rho_{k+n_0+1} = \begin{cases} (k-\frac{1}{4})\frac{\pi}{l} + O\left(\frac{1}{k}\right) & \text{if } b_2 = 0,\ a_2 \neq 0, \\ (k-\frac{1}{2})\frac{\pi}{l} + O\left(\frac{1}{k}\right) & \text{if } b_2 \neq 0. \end{cases} \tag{4.14}$$

Next, considering the relations (4.9), (4.10), (4.12) and (4.13), we obtain the asymptotic formulas

$$v_{k+n_0}^{(0)}(x) = \sin(k-\frac{1}{4})\frac{\pi}{l}x - \cos(k-\frac{1}{4})\frac{\pi}{l}x + e^{-(k-\frac{1}{4})\frac{\pi}{l}x} + O\left(\frac{1}{k}\right), \tag{4.15}$$

$$\begin{aligned} v_{k+n_0+1}^{(\gamma)}(x) = {} & \sin(k-\frac{1}{2})\frac{\pi}{l}x - \cos(k-\frac{1}{2})\frac{\pi}{l}x + e^{-(k-\frac{1}{2})\frac{\pi}{l}x} \\ & +(-1)^{k+n_0+1}e^{-(k-\frac{1}{2})\frac{\pi}{l}(l-x)} + O\left(\frac{1}{k}\right), \end{aligned} \tag{4.16}$$

$$y_{k+n_0+1} = \begin{cases} \sin(k-\frac{1}{4})\frac{\pi}{l}x - \cos(k-\frac{1}{4})\frac{\pi}{l}x + e^{-(k-\frac{1}{4})\frac{\pi}{l}x} + O\left(\frac{1}{k}\right), \text{if } b_2 = 0,\ a_2 \neq 0, \\ \sin(k-\frac{1}{2})\frac{\pi}{l}x - \cos(k-\frac{1}{2})\frac{\pi}{l}x + e^{-(k-\frac{1}{2})\frac{\pi}{l}x} \\ \qquad + (-1)^{k+n_0+1}e^{-(k-\frac{1}{2})\frac{\pi}{l}(l-x)} + O\left(\frac{1}{k}\right), \quad \text{if } b_2 \neq 0. \end{cases} \tag{4.17}$$

Let $k = 2n$, where n is a sufficiently large positive integer. Consider the formula

$$v_{2n+n_0}^{(0)}(x) = \sin(2n-\frac{1}{4})\frac{\pi}{l}x - \cos(2n-\frac{1}{4})\frac{\pi}{l}x + e^{-(2n-\frac{1}{4})\frac{\pi}{l}x} + O\left(\frac{1}{k}\right). \tag{4.18}$$

If we use t to substitute $(2n-\frac{1}{4})\frac{\pi}{l}x, x \in [0, l]$, then

$$x = \frac{tl}{(2n-\frac{1}{4})\pi}, \quad t \in \left[0, (2n-\frac{1}{4})\pi\right].$$

Set

$$\psi(t) = v^{(0)}_{2n+n_0}\left(\frac{tl}{(2n-\frac{1}{4})\pi}\right), \tag{4.19}$$

then

$$\psi(t) = \sin t - \cos t + e^{-t} + O\left(\frac{1}{n}\right), \tag{4.20}$$

$$\psi'(t) = \cos t + \sin t - e^{-t} + O\left(\frac{1}{n}\right). \tag{4.21}$$

For any fixed $r \in \{1, 2, \cdots n-1\}$ and $t \in [2\pi r, 2\pi(r+1)]$, let $\xi = t - 2\pi r$, then $t = \xi + 2\pi r, 0 \le \xi \le 2\pi$, and

$$\phi(\xi) := \psi(\xi + 2\pi r) = \sin\xi - \cos\xi + e^{-(\xi+2\pi r)} + O\left(\frac{1}{n}\right). \tag{4.22}$$

It follows from (4.21) and (4.22) that

$$\phi'(\xi) = \cos\xi + \sin\xi - e^{-(\xi+2\pi r)} + O\left(\frac{1}{n}\right). \tag{4.23}$$

By (4.22) and (4.23), we have

$$\phi(0) < 0,\ \phi(\frac{\pi}{2}) > 0,\ \phi(\pi) > 0,\ \phi(\frac{3\pi}{2}) < 0,\ \phi(2\pi) < 0$$

and

$$\phi'(\xi) > 0 \text{ for } \xi \in (0, \frac{3\pi}{4}), \phi'(\xi) < 0 \text{ for } \xi \in (\frac{3\pi}{4}, \frac{7\pi}{4}),\ \phi'(\xi) > 0 \ \text{ for } \xi \in (\frac{7\pi}{4}, 2\pi).$$

So, $\phi(\xi)$ has only two zeros in the interval $(0, 2\pi)$. Therefore, the function $\psi(t)$ has exactly $2n$ zeros in the interval $(0, 2n\pi)$.

Let $t \in [(2n-\frac{1}{4})\pi, 2n\pi]$. Then, by (4.20) and (4.21), we know that $\psi((2n-\frac{1}{4})\pi) < 0, \psi(2n\pi) < 0$ and $\psi'(t) > 0$. Consequently, the function $\psi(t)$ has no zeros in the interval $[(2n-\frac{1}{4})\pi, 2n\pi]$.

Because of the function $\psi(t)$ has exactly $2n$ zeros in the interval $(0, 2n\pi)$ and no zeros in the interval $[(2n-\frac{1}{4})\pi, 2n\pi]$, we find that the function $\psi(t)$ has exactly $2n$ zeros in the interval $(0, 2n\pi)$. Hence, the function $v^{(0)}_{2n+n_0}(x)$ has exactly $2n$ zeros in the interval $(0, l)$. Then, by Lemma 2.4, the function $v^{(0)}_{2n+n_0}(x)$ corresponds to the eigenvalue μ_{2n+1} of the boundary value problem (1.1), (1.2a), (1.2b'), (1.2c) for $\gamma = 0$. This implies that $n_0 = 1$.

Setting $n_0 = 1$ in (4.9), (4.14)-(4.17), we obtain the formula for $\delta \in (0, \frac{\pi}{2}]$. The remaining cases can be considered similarly. □

5. Basis properties of the system of eigenfunctions in $L_p(0, l)$

In this section, we consider the basis properties of the eigenfunction. In fact, the boundary value problem (1.1), (1.2a), (1.2b') and (1.2c) is equivalent to the eigenvalue problem of a self-adjoint and right-definiteness operator L under the assumption (A_1), i.e. $L\widehat{y} = \lambda\widehat{y}$. To be specific, let us consider the boundary condition (1.2b'). To wit, let $y_0 = a_0 y'(l) + b_0 y''(l)$, $y_1 = a_1 y'(l) + b_1 y''(l)$, $y_2 = a_2 y'(l) + b_2 y''(l)$. Then, the boundary condition (1.2b') can be rewritten as

$$y_0 + \lambda y_1 + \lambda^2 y_2 = 0. \tag{5.1}$$

Let $H = L_2(0,l) \bigoplus \mathbb{C}^2$ be a Hilbert space with the inner product

$$(\widehat{y}, \widehat{v}) = (\{y, y_2, y_1 + \lambda y_2\}, \{v, v_2, v_1 + \lambda v_2\}) = (y, v) + \frac{1}{|M|}(y_2, y_1 + \lambda y_2) M \begin{pmatrix} \overline{v_2} \\ \overline{v_1 + \lambda v_2} \end{pmatrix}, \tag{5.2}$$

where $(\cdot,\cdot)$ is the inner product of $L_2(0,l)$ and $(y,v) = \int_0^l y\bar{v}dx$. Define an operator L: $D(L) \to$ range $(Ty)' \bigoplus \mathbb{C}^2$,

$$L\widehat{y} = L\{y, y_2, y_1 + \lambda y_2\} = \{(Ty)', \lambda y_2, -y_0\}, \tag{5.3}$$

where

$$D(L) = \{\{y, y_2, y_1 + \lambda y_2\} \in H : y(x) \in W_2^4(0,l), (Ty)' \in L_2(0,l),$$
$$y \text{ satisfis (1.2a) and (1.2c)}\}. \tag{5.4}$$

It is not difficult to see that L is a self-adjoint operator in H. Then, the eigenvalue problem (1.1), $(1.2a)$, $(1.2b')$, $(1.2c)$ is equivalent to the eigenvalue problem of operator equation $L\widehat{y} = \lambda\widehat{y}$. By direct calculation, we will find that L is a right-definiteness operator in H. Therefore, with the assumption (A_1), the eigenvalues of L are real, and the corresponding eigenfunctions are real.

Since L is self-adjoint, then the system of eigenvectors $\{y_i(x)\}_{i=1}^{\infty}$ is orthogonal in H, where $y_i(x)$, $i \in \mathbb{N}$, is the eigenfunction of problem (1.1), $(1.2a)$, $(1.2c)$ and $y_0 + \lambda y_1 + \lambda^2 y_2 = 0$. Next, we set $\tau_i = \int_0^l y_i^2(x)dx + \frac{1}{|M|}(y_{2i}, y_{1_i} + \lambda y_{2_i}) M \begin{pmatrix} y_{2i} \\ y_{1_i} + \lambda y_{2_i} \end{pmatrix}$, obviously, M is positive definite, then $\tau_i \neq 0$.

Lemma 5.1. Suppose that (A_1) holds. Then there exists a system $\{\widehat{u_i}\}_{i=1}^{\infty}$ with $\widehat{u_i} = \{u_i, u_{2_i}, u_{1_i} + \lambda u_{2_i}\}$ and $\widehat{u_i} = \tau_i^{-1}\widehat{y_i}$. Moreover, this system is biorthogonal to the system $\{\widehat{y_i}\}_{i=1}^{\infty}$, for $\widehat{y_i} = \{y_i, y_{2_i}, y_{1_i} + \lambda y_{2_i}\}$, and is a Riesz basis of H.

Proof. Because of L is self-adjoint, then the system of eigenvectors $\{y_i(x)\}_{i=1}^{\infty}$ is orthogonal in H. So, we can obtain

$$(\widehat{y_i}, \widehat{u_j}) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

that is to say, the system of $\{\widehat{u_i}\}_{i=1}^{\infty}$ is biorthogonal to the system $\{\widehat{y_i}\}_{i=1}^{\infty}$, by ([44], Lemma 3.3.1), we know that the system $\{\widehat{y_i}\}_{i=1}^{\infty}$ is complete in H. Then, according to [44], Theorem 3.6.6, we get that $\{\widehat{y_i}\}_{i=1}^{\infty}$ is a Riesz basis. □

Let r and l be two arbitrary fixed nonnegative integers, and

$$\Delta_{r,l} = \begin{vmatrix} u_{2_r} & u_{2_l} \\ u_{1_r} & u_{1_l} \end{vmatrix}. \tag{5.5}$$

Theorem 5.2. If $\Delta_{r,l} \neq 0$, then the eigenfunctions $\{y_i(x)\}_{i=1, i\neq r,l}^{\infty}$ of problem (1.1), (1.2) form a basis in the space $L_p(0,l)$, $(1 < p < \infty)$. Moreover, if $p = 2$, this basis is a Riesz basis; If $\Delta_{r,l} = 0$, then the system of $\{y_i(x)\}_{i=1,i\neq r,l}^{\infty}$ is not complete and not minimal in the space $L_p(0,l)$, $(1 < p < \infty)$.

Proof. The Fourier expansion of any vector $\widehat{f} = \{f, f_2, f_1 + \lambda f_2\} \in H$ is

$$\widehat{f} = \sum_{i=1}^{\infty} (\widehat{f}, \widehat{y_i})\widehat{u_i} = \sum_{i=1}^{\infty} (\{f, f_2, f_1 + \lambda f_2\}, \{y_i, y_{2_i}, y_{1_i} + \lambda y_{2_i}\})\{u_i, u_{2_i}, u_{1_i} + \lambda u_{2_i}\}$$

$$= \sum_{i=1}^{\infty}\left((f,y_i)+\frac{1}{|M|}(f_2,f_1+\lambda f_2)M\left(\frac{\overline{y_{2i}}}{y_{1_i}+\lambda y_{2_i}}\right)\right)\{u_i,u_{2_i},u_{1_i}+\lambda u_{2_i}\}. \tag{5.6}$$

If $\widehat{f}=\{f,0,0\}$, we have

$$\widehat{f}=\{f,0,0\}=\sum_{i=1}^{\infty}(\widehat{f},\widehat{y_i})\widehat{u_i}=\sum_{i=1}^{\infty}(f,y_i)\{u_i,u_{2_i},u_{1_i}+\lambda u_{2_i}\}. \tag{5.7}$$

In other words,

$$f=\sum_{i=1}^{\infty}(f,y_i)u_i, \tag{5.8}$$

$$0=\sum_{i=1}^{\infty}(f,y_i)u_{2_i},\quad 0=\sum_{i=1}^{\infty}(f,y_i)(u_{1_i}+\lambda u_{2_i}). \tag{5.9}$$

Let $\Delta_{r,l}\neq 0$. Then, according to (5.8) and (5.9), we have

$$(f,y_r)=-\Delta_{r,l}^{-1}\sum_{\substack{i=1\\ i\neq r,l}}^{\infty}(f,y_i)\Delta_{i,l},\quad (f,y_l)=-\Delta_{r,l}^{-1}\sum_{\substack{i=1\\ i\neq r,l}}^{\infty}(f,y_i)\Delta_{r,i}. \tag{5.10}$$

Taking into account (5.9) and (5.10), we get

$$f=\sum_{\substack{i=1\\ i\neq r,l}}^{\infty}(f,y_i)\{u_i-\Delta_{r,l}^{-1}\left(\Delta_{i,l}u_r+\Delta_{r,i}u_l\right)\}. \tag{5.11}$$

In addition, we can obtain

$$\begin{aligned}
&\left(y_i,\{u_k-\Delta_{r,l}^{-1}\left(\Delta_{k,l}u_r+\Delta_{r,k}u_l\right)\}\right)=\Delta_{r,l}^{-1}(y_i,\Delta_{r,l}u_k-\Delta_{k,l}u_r-\Delta_{r,k}u_l)\\
&=\Delta_{r,l}^{-1}\left\{\Delta_{r,l}\left\{(\widehat{y_i},\widehat{u_k})-\frac{1}{|M|}(y_{2_i},y_{1_i}+\lambda y_{2_i})M\left(\frac{\overline{u}_{2_k}}{u_{1_k}+\lambda u_{2_k}}\right)\right\}\right.\\
&-\Delta_{k,l}\left\{(\widehat{y_i},\widehat{u_r})-\frac{1}{|M|}(y_{2_i},y_{1_i}+\lambda y_{2_i})M\left(\frac{\overline{u_2}_r}{u_{1_r}+\lambda u_{2_r}}\right)\right\}\\
&\left.-\Delta_{r,k}\left\{(\widehat{y_i},\widehat{u_l})-\frac{1}{|M|}(y_{2_i},y_{1_i}+\lambda y_{2_i})M\left(\frac{\overline{u}_{2_l}}{u_{1_l}+\lambda u_{2_l}}\right)\right\}\right\}\\
&=(\widehat{y_i},\widehat{u_k})=\delta_{i,k},\quad i\neq r,l,
\end{aligned} \tag{5.12}$$

From relation (5.12), we know that the system $\{u_i-\Delta_{r,l}^{-1}\left(\Delta_{i,l}u_r+\Delta_{r,i}u_l\right)\}_{i=1,i\neq r,l}^{\infty}$ is a Riesz basis in $L_2(0,l)$. Then, from [45], the system $\{y_i(x)\}_{i=1,i\neq r,l}^{\infty}$ is a Riesz basis in $L_2(0,l)$ as well. Next, the property of the system $\{y_i(x)\}_{i=1,i\neq r,l}^{\infty}$ in $L_p(0,l)$, $p\in(1,+\infty)\setminus\{2\}$ can be proved similarly by Theorem 5.1 in [7].

Let $\Delta_{r,l}=0$, which implies the system of linear homogeneous equations

$$c_1u_{2_r}-c_2u_{2_l}=0,\quad c_1u_{1_r}-c_2u_{1_l}=0 \tag{5.13}$$

has a nontrivial solution $\{c_1, c_2\}$. Then, by (5.2), (5.12) and (5.13),we obtain

$$(y_i, c_1 u_r + c_2 u_l) = \left\{ c_1 (\widehat{y_i}, \widehat{u_r}) - \frac{c_1}{|M|}(y_{2_i}, y_{1_i} + \lambda y_{2_i}) M \left(\frac{\overline{u}_{2_r}}{u_{1_r} + \lambda u_{2_r}} \right) \right\}$$

$$+ \left\{ c_2 (\widehat{y_i}, \widehat{u_l}) - \frac{c_2}{|M|}(y_{2_i}, y_{1_i} + \lambda y_{2_i}) M \left(\frac{\overline{u}_{2_l}}{u_{1_l} + \lambda u_{2_l}} \right) \right\}$$

$$= -\frac{1}{|M|}(c_1 u_{2_r} - c_2 u_{2_l})(n\lambda^2 y_{2_i} - 2m\lambda y_{2_i} + k y_{2_i} + n\lambda y_{2_i} - m y_{1_i})$$

$$- \frac{1}{|M|}(c_1 u_{1_r} - c_2 u_{1_l})(n\lambda y_{2_i} + n y_{1_i} - m y_{2_i}) = 0, \ i \in \mathbb{N}, \ \ i \neq r, l. \tag{5.14}$$

Thus, the function $c_1 u_r(x) + c_2 u_l(x)$ is orthogonal to all functions of the system $\{y_i(x)\}_{i=1, i\neq r,l}^{\infty}$; i.e. the system $\{y_i(x)\}_{i=1, i\neq r,l}^{\infty}$ is incomplete in $L_2(0,l)$.

Let $P : H \to L_2(0,l)$ be the projective of H into $L_2(0,l)$ such that $P\widehat{y} = y$. The operator P is a Φ_+-operator (recall that a linear bounded operator A acting from one Banach space E_1 into the other one E_2 is referred to as a Φ_+-operator if the operator A is normally solvable, and the dimension $\alpha(A)$ of the subspace Ker$A$ of all solutions of the equation $A\varphi = 0 (\varphi \in E_1)$ is finite (see [46]), and $\alpha(P) = 2$. Then, by Lemma 3 in [46], the system $\{y_i(x)\}_{i=1}^{\infty} = \{P\widehat{y_i}\}_{i=1}^{\infty}$ is a deficiency basis in $L_2(0,l)$ with deficiency number $\alpha(P) = 2$.) Therefore, there exist positive integers η and ι such that $\eta < \iota$, and the system $\{y_i(x)\}_{i=1, i\neq r,l}^{\infty}$ is a basis in $L_2(0,l)$. Consequently, $\Delta_{\eta,\iota} \neq 0$.

For any function $g \in L_2(0,l)$, the Fourier expansion of g is as follows

$$g = \sum_{\substack{k=1 \\ k\neq\eta,\iota}}^{\infty} (g, u_k - \Delta_{\eta,\iota}^{-1} \left(\Delta_{k,\iota} u_r + \Delta_{\eta,k} u_l \right)) y_k. \tag{5.15}$$

According to the relations (5.15), (5.2) and (5.12), we have

$$y_\eta = \sum_{\substack{k=1 \\ k\neq\eta,\iota}}^{\infty} (y_\eta, u_k - \Delta_{\eta,\iota}^{-1} \left(\Delta_{k,\iota} u_r + \Delta_{\eta,k} u_l \right)) y_k = -\Delta_{\eta,\iota}^{-1} \sum_{\substack{k=1 \\ k\neq\eta,\iota}}^{\infty} \Delta_{k,\iota} y_k$$

$$= -\Delta_{\eta,\iota}^{-1} \sum_{\substack{k=1 \\ k\neq\eta,\iota,r,l}}^{\infty} \Delta_{k,\iota} y_k - \Delta_{\eta,\iota}^{-1} \{\Delta_{r,\iota} y_r + \Delta_{l,\iota} y_l\}, \tag{5.16}$$

$$y_\iota = \sum_{\substack{k=1 \\ k\neq\eta,\iota}}^{\infty} (y_\iota, u_k - \Delta_{\eta,\iota}^{-1} \left(\Delta_{k,\iota} u_r + \Delta_{\eta,k} u_l \right)) y_k = -\Delta_{\eta,\iota}^{-1} \sum_{\substack{k=1 \\ k\neq\eta,\iota}}^{\infty} \Delta_{\eta,k} y_k$$

$$= -\Delta_{\eta,\iota}^{-1} \sum_{\substack{k=1 \\ k\neq\eta,\iota,r,l}}^{\infty} \Delta_{\eta,k} y_k - \Delta_{\eta,\iota}^{-1} \{\Delta_{\eta,r} y_r + \Delta_{\eta,l} y_l\}. \tag{5.17}$$

Since $\Delta_{r,l} = 0$, we know that the linear homogeneous equations

$$\widetilde{c_1} u_{2_r} + \widetilde{c_2} u_{2_l} = 0, \quad \widetilde{c_1} u_{1_r} + \widetilde{c_2} u_{1_l} = 0 \tag{5.18}$$

has a nonzero solution $\{\widetilde{c_1}, \widetilde{c_2}\}$.

Next, we consider the linear inhomogeneous equations

$$d_1 u_{1_\iota} - d_2 u_{1_\eta} = \widetilde{c_1}\Delta_{\eta,\iota}, \quad -d_1 u_{2_\iota} + d_2 u_{2_\eta} = \widetilde{c_2}\Delta_{\eta,\iota}. \tag{5.19}$$

The coefficient matrix of the linear inhomogeneous equation above is not equal to zero, and

$$\begin{vmatrix} u_{1_\iota} & -u_{1_\eta} \\ -u_{2_\iota} & u_{2_\eta} \end{vmatrix} = u_{1_\iota}u_{2_\eta} - u_{1_\eta}u_{2_\iota} = \Delta_{\eta,\iota} \neq 0.$$

So, the linear inhomogeneous equations has a unique nonzero solution $\{d_1, d_2\}$.

In view of (5.16), (5.17), (5.18) and (5.19), we obtain

$$\begin{aligned} d_1 y_\eta + d_2 y_\iota = & -\Delta_{\eta,\iota}^{-1} \sum_{\substack{k=1 \\ k\neq \eta,\iota,r,l}}^{\infty} (d_1\Delta_{k,\iota} + d_2\Delta_{\eta,k}) y_k - \Delta_{\eta,\iota}^{-1}\{(d_1\Delta_{r,\iota} + d_2\Delta_{\eta,r}) y_r \\ & +(d_1\Delta_{l,\iota} + d_2\Delta_{\eta,l}) y_l\} \\ = & -\Delta_{\eta,\iota}^{-1} \sum_{\substack{k=1 \\ k\neq \eta,\iota,r,l}}^{\infty} (d_1\Delta_{k,\iota} + d_2\Delta_{\eta,k}) y_k. \end{aligned} \tag{5.20}$$

By (5.20), we know that the relation implies the system $\{y_k(x)\}_{k=1,k\neq r,l}^{\infty}$ in nonminimal in $L_2(0,l)$.

Therefore, if $\Delta_{r,l} = 0$, then the system $\{y_k(x)\}_{k=1,k\neq r,l}^{\infty}$ is incomplete and nonminimal in $L_2(0,l)$. Obviously, that system is incomplete and nonminimal in $L_p(0,l)$, $1 < p < \infty$. □

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Conflict of interest

The authors declare no conflict of interest in this paper.

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