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Research article

Discrete fractional solutions to the effective mass Schrödinger equation by mean of nabla operator

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Abstract: In the current article, we investigate the second order singular differential equation namely the effective mass Schrödinger equation by means of the fractional nabla operator. We apply some classical transformations in order to reduce the governing equation, and also restrict the difference parameters involved in order to find them values. In order to achieve these important results, certain tools such as the Leibniz rule, the index law, the shift operator, and the power rule are provided in view of the discrete fractional calculus. We use all these mentioned data for two representations of the given model for homogeneous and non-homogeneous instances. The main advantage of the fractional nabla operator is to apply the singular differential equations and transform them into a fractional order model. As a result, we produce some new exact fractional solutions to the present model for a given potential.

Keywords: discrete fractional; the nabla operator; the effective mass Schrödinger equation
Mathematics Subject Classification: 97M50, 97N70

1. Introduction

Quantum mechanics determines the properties of physical systems such as introduced atoms, molecules, condensed phase materials, light, etc [1, 2]. The fractional quantum was developed using the fractional path integral method, preceded by the fractional generalization of the Schrödinger equation [3]. Quantum mechanical systems with a spatially dependent effective mass have been used widely in different branches of physics for instant progress in crystal-growth techniques for the processing of non-uniform semiconductor specimens [4–7]. Interest in this type of approach is growing today because it is an important and commonly used tool for the identification of the electronic characteristics of semiconductors [8], and quantum dots [9]. The physical descriptions of the quantum system are encapsulated in the Schrödinger equation, so that an analysis of the quantum system of physics requires an investigation and a solution of the reference model [10].

Recently, a number of authors have been interested in investigating the mass Schrödinger equation, such as Filho et al have been implemented a translation operator to represent the quantum dynamics of a position-dependent mass particle in a null or constant potential [11], Gönül et al have been investigated the mass Schrödinger equation by using a general method for obtaining its exact solutions [12], Zhang et al have been investigated the mass Schrödinger equation by implementing the appropriate coordinate transformation [13], Jana et al have been studied the mass Schrödinger equation by utilizing the Potential algebra approach [14], Sebawe Abdalla et al have been analyzed the behavior of the wave functions for scattered states in light of the parameters involved for the mass Schrödinger equation [15].

It is well known that building mathematical models that represent problems in several areas in which differentiation and integration have a prominent place makes a big contribution to science. However, it can be easily seen that the classical derivative, limited by the rate of change, falls short of describing many phenomena which could not be properly formulated by an integer order. Because of this, fractional derivatives are suggested to capture past history as in traditional integration that allows us to analyze the past and present process for more details we refer the reader to see [16–20].

Discrete fractional calculus deals with derivatives and integrals of arbitrary orders and arises in various fields of science such as engineering, applied mathematics and physics, fluid mechanics, for more details we refer the reader to see [21–27]. Several singular differential equations have been investigated by using the discrete fractional nabla operator, such as the modified hydrogen atom equation [21], the confluent hypergeometric differential equation [22], the modified Bessel equation [23], the fractional Schrödinger equation [24], Gauss equation [25], the Non-Fuchsian Differential equations [26], the generalized Laguerre differential equation [27].

The aim of this paper is to construct the fractional exact solutions for the effective mass Schrödinger equation by means of a fractional nabla operator.

2. Preliminary and properties

The differences in fractional order were first introduced by Diaz and Osler as follows [28]:

$$\Delta^{\vartheta} f(r) = \sum_{m=0}^{\infty} (-1)^m \binom{\vartheta}{m} f(r + \vartheta - m), \tag{2.1}$$

where ϑ is any real number, $f(r)$ is a special function, and $r \in \mathbb{N}_b = \{b\} + \mathbb{N}_0 = \{b, b+1, b+2, ...\}$. The concept of a fractional difference is defined in [28–31]

$$\nabla^{\vartheta} f(r) = \sum_{m=0}^{\infty} (-1)^m \binom{\vartheta}{m} f(r - m), \quad \binom{\vartheta}{m} = \frac{\Gamma(\vartheta + 1)}{\Gamma(m+1)\,\Gamma(\vartheta - m + 1)}, \tag{2.2}$$

where m is any real number.

Gray and Zhang introduced a new definition of the fractional difference through summation as follows [28–31].

Let $\vartheta \in \mathbb{R}^{+}$ such that $m - 1 \leq \vartheta < m$ where m is an integer and $\vartheta - th$ order fractional sum of g

defined by

$$\nabla_b^{-\vartheta} g(r) = \frac{1}{\Gamma(\vartheta)} \sum_{s=b}^{r} (r - \delta(s))^{\overline{\vartheta-1}} g(s), \tag{2.3}$$

where $r \in \mathbb{N}_b = \{b\} + \mathbb{N}_0 = \{b, b+1, b+2, ...\}$, $b \in \mathbb{R}$, and $\delta(s) = s - 1$ is a jump operator.
The ascending factorial is defined by [28–31]

$$\begin{aligned} r^{\overline{m}} &= \prod_{n=0}^{m-1} (r+n) \\ &= r(r+1)(r+2)...(r+m-1), \; m \in \mathbb{N}, \; r^{\overline{0}} = 1. \end{aligned} \tag{2.4}$$

Let $\vartheta \in \mathbb{R}$, then "r to the ϑ rising" given by

$$r^{\overline{\vartheta}} = \frac{\Gamma(r+\vartheta)}{\Gamma(r)}, \; r \in \mathbb{R} - \{..., -2, -1, 0\}, \; 0^{\overline{\vartheta}} = 0. \tag{2.5}$$

Let us take note of this

$$\nabla\left(r^{\overline{\vartheta}}\right) = \vartheta r^{\overline{\vartheta-1}}, \tag{2.6}$$

where $\nabla u(r) = u(r) - u(\delta(r)) = u(r) - u(r-1)$.
The fractional difference of $\vartheta - th$ is provided by

$$\begin{aligned} \nabla_b^{\vartheta} g(r) &= \nabla^k \left[\nabla^{-(k-\vartheta)} g(r)\right] \\ &= \nabla^k \left[\frac{1}{\Gamma(k-\vartheta)} \sum_{s=b}^{r} (r - \delta(s))^{\overline{k-\vartheta-1}} g(s)\right], \end{aligned} \tag{2.7}$$

where $g : \mathbb{N}_b^+ \to \mathbb{R}$, and $k \in \mathbb{N}$ [28–31].
Lemma 1. Let $f(r)$, and $g(r) : \mathbb{N}_0^+ \to \mathbb{R}$, ϑ, $\eta > 0$. The following equality holds:

$$\nabla^{-\vartheta} \nabla^{-\eta} f(r) = \nabla^{-(\vartheta+\eta)} f(r) = \nabla^{-\eta} \nabla^{-\vartheta} f(r), \tag{2.8}$$

$$\nabla^{\vartheta} [h f(r) + v g(r)] = h \nabla^{\vartheta} f(r) + v \nabla^{\vartheta} g(r), \tag{2.9}$$

where h, and v are scalars.
Lemma 2 (Power Rule). Let $m > 0$. Then the following holds [28–31].

$$\nabla_b^{-m} (r - b + 1)^{\overline{n}} = \frac{\Gamma(n+1)}{\Gamma(n+m+1)} (r - b + 1)^{\overline{n+m}}, \tag{2.10}$$

for every $r \in \mathbb{N}_b$.

Lemma 3. For any $\vartheta > 0$, the following equality holds [28–31]:

$$\nabla_{b+1}^{-\vartheta} \nabla f(r) = \nabla \nabla_b^{-\vartheta} f(r) - \frac{(r - b + 1)^{\overline{\vartheta-1}}}{\Gamma(\vartheta)} f(b). \tag{2.11}$$

Lemma 4 (Leibniz Rule): For any $\vartheta > 0$, $\vartheta - th$ order, the fractional difference of the product fg is given by [28–31].

$$\nabla_0^{\vartheta}(fg)(r) = \sum_{m=0}^{r}\binom{\vartheta}{m}\left[\nabla_0^{\vartheta-m}f(r-m)\right]\left[\nabla^m g(r)\right], \tag{2.12}$$

where $\binom{\vartheta}{m} = \frac{\Gamma(\vartheta+1)}{\Gamma(m+1)\Gamma(\vartheta-m+1)}$, and $k^m f(r) = f(r-m)$ is the standard shift operator.

Lemma 5 (Index law): Let f is analytic and single-valued, then the following equality holds [28–31]:

$$(f_m(r))_\eta = f_{m+\eta}(r) = \left(f_\eta(r)\right)_m \quad \left(f_m(r) \neq 0;\ f_\eta(r) \neq 0\right), \tag{2.13}$$

where $m,\ \eta \in \mathbb{R}$, and f_m, f_η are functions of m, η order fractional derivatives.

3. Mathematical analysis

Consider the one-dimensional effective mass Hamilton by [35]:

$$h_{eff} = -\frac{d}{d\xi}\left(\frac{1}{y(\xi)}\frac{d}{d\xi}\right) + v_{eff}(\xi), \tag{3.1}$$

where v_{eff} has the form

$$v_{eff} = v(\xi) + \frac{1}{2}(\rho+1)\frac{y''}{y^2} - \left[\lambda(\lambda+\rho+1)+\rho+1\right]\frac{y'^2}{y^3}, \tag{3.2}$$

where λ, ρ ambiguity parameters and primes are stand for the derivatives with respect to ξ. The dimensionless form $y(\xi)$ is used for mass function. When y is a constant the above module decrease to the case of constant mass SE, so v_{eff} decrease to $v(\xi)$, where v_{eff} is the potential of the mass. Then the SE takes the form

$$\left(-\frac{1}{y}\frac{d^2}{d\xi^2} + \frac{y'}{y^2}\frac{d}{d\xi} + \upsilon_{eff} - \varepsilon\right)q(\xi) = 0, \tag{3.3}$$

let us use the transformation $q = y^a(\xi)\varphi(\xi)$, then Eq. (3.3) becomes

$$\left\{-\frac{1}{y}\left[\frac{d^2}{d\xi^2} + (2a-1)\frac{y'}{y}\frac{d}{d\xi} + a\left((a-2)\left(\frac{y'}{y}\right)^2 + \frac{y''}{y}\right)\right] + \left(\upsilon_{eff} - \varepsilon\right)\right\}\varphi = 0, \tag{3.4}$$

now we assume that

$$y(\xi) = e^{-2\mu\xi}, \tag{3.5}$$

$$\upsilon(\xi) = \upsilon_0 e^{2\mu\xi} - b(2c+1)e^{\mu\xi}, \tag{3.6}$$

inserting Eq. (3.5) and Eq. (3.6) into Eq. (3.4), we obtain

$$-\left[\varphi'' - 2\mu(2a-1)\varphi' + 4a\mu^2(a-1)\varphi\right] + \left[\upsilon_0 - b(2c+1)e^{-\mu\xi} - \varepsilon e^{-2\mu\xi} + 2(\rho+1)\mu^2 - 4c^*\mu^2\right]\varphi = 0, \tag{3.7}$$

where $c^* = \lambda(\lambda+\rho+1)+\rho+1$.

The coordinate transformation $r = e^{-\mu\xi}$ leads to

$$\frac{d^2\varphi}{dr^2} + (3-4a)\frac{1}{r}\frac{d\varphi}{dr} + \frac{1}{r^2}\left(\frac{\varepsilon}{\mu^2}r^2 + \frac{1}{\mu^2}b(2c+1)r - \frac{v_0}{\mu^2} - 2(\rho+1) + 4c^* + 4a(a-1)\right)\varphi = 0, \tag{3.8}$$

for simplicity, we assume

$$\alpha = -\frac{\varepsilon}{\mu^2}, \tag{3.9}$$

$$\beta = -\frac{1}{\mu^2} b\,(2c+1)\,, \tag{3.10}$$

$$\gamma = \frac{-v_0}{\mu^2} - 2\,(\rho+1) + 4c^* + 4a\,(a-1)\,. \tag{3.11}$$

Consider the non homogeneous effective mass Schrödinger equation provided by

$$\frac{d^2\varphi}{dr^2} + (3-4a)\,\frac{1}{r}\frac{d\varphi}{dr} + \left(-\alpha - \frac{\beta}{r} + \frac{\gamma}{r^2}\right)\varphi = w\,(r)\,. \tag{3.12}$$

4. Main results

In this partition the applications of the discrete fractional nabla operator to the effective mass Schrödinger equation in homogeneous and non homogeneous instances is presented.

i) Let us consider the transformation

$$\varphi = \mathrm{e}^{r\theta} r^{\eta} \psi\,(r)\,, \tag{4.1}$$

so that

$$\frac{d\varphi}{dr} = \mathrm{e}^{r\theta} r^{-1+\eta}\,((\eta + r\theta)\,\psi\,(r) + r\psi'\,(r))\,, \tag{4.2}$$

and

$$\frac{d^2\varphi}{dr^2} = \mathrm{e}^{r\theta} r^{-2+\eta}\left(\left((-1+\eta)\,\eta + 2r\eta\theta + r^2\theta^2\right)\psi\,(r) + r\,(2\,(\eta + r\theta)\,\psi'\,(r) + r\psi''\,(r))\right). \tag{4.3}$$

Inserting Eq. (4.1), Eq. (4.2), and Eq. (4.3) into Eq. (3.12), we get

$$\begin{aligned} &r^2\psi'' + \left(2\theta r^2 + r\,(3-4a+2\eta)\right)\psi' + \left(r^2\left(\theta^2-\alpha\right) + r\,(3\theta - 4\theta a - \beta + 2\eta\theta) + \left(\gamma + 2\eta - 4a\eta + \eta^2\right)\right)\psi \\ &= r^{2-\eta}e^{\theta r} w\,(r)\,. \end{aligned} \tag{4.4}$$

Eventually, we find it to be suitable to restrict the difference parameters involved in Eq. (4.4) by means of the following equalities

$$\begin{aligned} \theta^2 - \alpha &= 0, \\ \eta^2 + (2-4a)\,\eta + \gamma &= 0, \end{aligned} \tag{4.5}$$

so that

$$\theta = \pm\sqrt{\alpha}, \tag{4.6}$$

and

$$\eta = \frac{-\,(2-4a) \pm \sqrt{(2-4a)^2 - 4\gamma}}{2}. \tag{4.7}$$

Under the parametric constrain given by Eq. (4.5), then Eq. (4.4) immediately reduce to the simpler form as

$$r^2\psi'' + \left(2\theta r^2 + r\,(3-4a+2\eta)\right)\psi' + r\,(3\theta - 4\theta a - \beta + 2\eta\theta)\,\psi = r^{2-\eta}e^{\theta r} w\,(r)\,, \tag{4.8}$$

where the values of θ and η are given by Eq. (4.6), and Eq. (4.7), respectively.

Theorem 2. Let $\psi,\ \phi \in \{\psi,\ \phi : 0 \neq |\psi_\vartheta|,\ |\phi_\vartheta| < \infty\}$, and $\vartheta \in \mathbb{R}$. Then the non homogeneous linear differential equation

$$r\psi_2 + (2\theta r + (3 - 4a + 2\eta))\,\psi_1 + (3\theta - 4\theta a - \beta + 2\eta\theta)\,\psi = \phi(r), \tag{4.9}$$

where $\phi(r) = r^{1-\eta} e^{\theta r} w\,(r)$, has a particular solution of the form

$$\psi\,(r) = \left\{\left[\phi_\vartheta \mathrm{e}^{2r\theta} r^{2-4a+2\eta+k\vartheta}\right]_{-1} \mathrm{e}^{-2r\theta} r^{-(3-4a+2\eta+k\vartheta)}\right\}_{-(\vartheta+1)}, \tag{4.10}$$

where $\psi_n\,(r) = \frac{d^n\psi}{dr^n},\ (n = 0, 1, 2),\ \psi_0 = \psi\,(r)$.

Proof: We apply the operator ∇^ϑ to both sides of Eq. (4.9), then we have

$$\nabla^\vartheta\left[\psi_2 r\right] + \nabla^\vartheta\left[\psi_1\left(2\theta r + (3 - 4a + 2\eta)\right)\right] + \nabla^\vartheta\left[\psi\left(3\theta - 4\theta a - \beta + 2\eta\theta\right)\right] = \nabla^\vartheta\phi, \tag{4.11}$$

By using Eq. (2.12) to Eq. (4.11), we get

$$r\psi_{2+\vartheta} + \left[2\theta r + (\vartheta k + 3 - 4a + 2\eta)\right]\psi_{1+\vartheta} + \left[2\theta k\vartheta + (3\theta - 4\theta a - \beta + 2\eta\theta)\right]\psi_\vartheta = \phi_\vartheta, \tag{4.12}$$

where k is a shift operator.

In order to find the value of fractional order ϑ in Eq. (4.12), we choose ϑ such that

$$2\theta k\vartheta + (3\theta - 4\theta a - \beta + 2\eta\theta) = 0, \tag{4.13}$$

from Eq. (4.13), we have

$$\vartheta = -\frac{3\theta - 4\theta a - \beta + 2\eta\theta}{2\theta k}, \tag{4.14}$$

after some process, we get

$$\psi_{2+\vartheta} + \psi_{1+\vartheta}\left[2\theta + \frac{\vartheta k + 3 - 4a + 2\eta}{r}\right] = \frac{\phi_\vartheta}{r}. \tag{4.15}$$

Therefore, setting

$$\psi_{\vartheta+1} = W = W\,(r),\ \left(\psi = W_{-(\vartheta+1)}\right). \tag{4.16}$$

From Eq. (4.15) and Eq. (4.16), one can easily get the following linear ordinary differential equation

$$W_1 + W\left[2\theta + \frac{\vartheta k + (3 - 4a + 2\eta)}{r}\right] = \frac{\phi_\vartheta}{r}, \tag{4.17}$$

a particular solution of Eq. (4.17) is given by

$$W = \left[\phi_\vartheta \mathrm{e}^{2r\theta} r^{2-4a+2\eta+k\vartheta}\right]_{-1} \mathrm{e}^{-2r\theta} r^{-(3-4a+2\eta+k\vartheta)}. \tag{4.18}$$

Thus, from relation (4.16) and Eq. (4.18), we obtain the solution of Eq. (4.9).

Theorem 3. Let $\psi \in \{\psi : 0 \neq |\psi_\vartheta| < \infty,\ \vartheta \in \mathbb{R}\}$. Then the homogeneous linear differential equation

$$r\psi_2 + (2\theta r + (3 - 4a + 2\eta))\,\psi_1 + (3\theta - 4\theta a - \beta + 2\eta\theta)\,\psi = 0, \tag{4.19}$$

has a particular solution of the form

$$\psi(r) = h[e^{2r\theta} r^{2-4a+2\eta+k\vartheta}]_{-(\vartheta+1)}, \tag{4.20}$$

where h is a constant of integration.
Proof: when $\phi(r) = 0$ in Theorem 2, we get

$$W_1 + W\left[2\theta + \frac{\vartheta k + (3 - 4a + 2\eta)}{r}\right] = 0. \tag{4.21}$$

Therefore, we get the fractional solution Eq. (4.20) for Eq. (4.21).
ii) Let us consider the transformation

$$\psi = r^{\mu}\chi(r), \tag{4.22}$$

so that

$$\frac{d\psi}{dr} = r^{-1+\mu}\left(\mu\chi(r) + r\chi'(r)\right), \tag{4.23}$$

and

$$\frac{d^2\psi}{dr^2} = r^{\mu}\left(\frac{\mu\left((-1+\mu)\chi(r) + 2r\chi'(r)\right)}{r^2} + \chi''(r)\right), \tag{4.24}$$

by substituting Eq. (4.22), Eq. (4.23) and Eq. (4.24) into Eq. (4.8), we obtain

$$\begin{aligned} &r^2\chi''(r) + \left(2\theta r^2 + r(3 - 4a + 2\eta + 2\mu)\right)\chi'(r) \\ &+ \left(\left(2\mu - 4a\mu + 2\eta\mu + \mu^2\right) + r(-\beta + 3\theta - 4a\theta + 2\eta\theta + 2\theta\mu)\right)\chi(r) = r^{2-\mu}\phi(r), \end{aligned} \tag{4.25}$$

Finally, we find it to be suitable to restrict the difference parameters involved in Eq. (4.25) by means of the following equalities:

$$2\mu - 4a\mu + 2\eta\mu + \mu^2 = 0, \tag{4.26}$$

$$\mu = -2 + 4a - 2\eta. \tag{4.27}$$

Under the parametric constrain given by Eq. (4.26), then Eq. (4.25) reduce to the simpler form as

$$r^2\chi''(r) + \left(2\theta r^2 + r(3 - 4a + 2\eta + 2\mu)\right)\chi'(r) + \left(r(-\beta + 3\theta - 4a\theta + 2\eta\theta + 2\theta\mu)\right)\chi(r) = r^{2-\mu}\phi(r). \tag{4.28}$$

Theorem 4. Let $\chi,\ Q \in \{\chi,\ Q : 0 \neq |\chi_{\vartheta}|,\ |Q_{\vartheta}| < \infty\}$, and $\vartheta \in \mathbb{R}$. Then the non homogeneous linear differential equation

$$r\chi''(r) + (2\theta r + (3 - 4a + 2\eta + 2\mu))\chi'(r) + (-\beta + 3\theta - 4a\theta + 2\eta\theta + 2\theta\mu)\chi(r) = Q(r), \tag{4.29}$$

has a solution of the form, where $Q(r) = r^{1-\mu}\phi(r)$

$$\chi(r) = \left\{e^{-2r\theta} r^{-(3-4a+2\eta+k\vartheta+2\mu)}\left[Q_{\vartheta} e^{2r\theta} r^{(2-4a+2\eta+k\vartheta+2\mu)}\right]_{-1}\right\}_{-(\vartheta+1)}, \tag{4.30}$$

where $\chi_p(r) = \frac{d^p\chi}{dr^p},\ (p = 0, 1, 2),\ \chi_0 = \chi(r)$.
Proof: For $\chi(r) \neq 0$, we apply the operator ∇^{ϑ} to both sides of Eq. (4.29), then we have

$$\nabla^{\vartheta}[\chi_2 r] + \nabla^{\vartheta}[(2\theta r + (3 - 4a + 2\eta + 2\mu))\chi_1] + \nabla^{\vartheta}[(-\beta + 3\theta - 4a\theta + 2\eta\theta + 2\theta\mu)\chi] = \nabla^{\vartheta}Q(r), \tag{4.31}$$

By using Eq. (2.12) to Eq. (4.31), we get

$$r\chi_{2+\vartheta} + \chi_{1+\vartheta}\left(2\theta r + (3 - 4a + 2\eta + 2\mu + k\vartheta)\right) + (-\beta + 3\theta - 4a\theta + 2\eta\theta + 2\theta\mu + 2\theta k\vartheta)\chi_{\vartheta} = Q_{\vartheta}, \quad (4.32)$$

where k is a shift operator.
In order to find the value of fractional order ϑ in Eq. (4.12), we choose ϑ such that

$$-\beta + 3\theta - 4a\theta + 2\eta\theta + 2\theta\mu + 2\theta k\vartheta = 0, \quad (4.33)$$

from Eq. (4.33), we have

$$\vartheta = \frac{-\beta + 3\theta - 4a\theta + 2\eta\theta + 2\theta\mu}{2\theta k}. \quad (4.34)$$

Then we have

$$\chi_{2+\vartheta} + \chi_{1+\vartheta}\left(2\theta + \frac{3 - 4a + 2\eta + 2\mu + k\vartheta}{r}\right) = \frac{1}{r}Q_{\vartheta}, \quad (4.35)$$

therefore setting

$$\chi_{\vartheta+1} = V = V(r), \ \left(\chi = V_{-(\vartheta+1)}\right). \quad (4.36)$$

From Eq. (4.35) and Eq. (4.36), we get the following linear ordinary differential equation

$$V_1 + V\left(2\theta + \frac{3 - 4a + 2\eta + 2\mu + k\vartheta}{r}\right) = \frac{1}{r}Q_{\vartheta}, \quad (4.37)$$

a particular solution of Eq. (4.37) is given by

$$V = \mathrm{e}^{-2r\theta} r^{-(3-4a+2\eta+k\vartheta+2\mu)}\left[Q_{\vartheta}\mathrm{e}^{2r\theta} r^{(2-4a+2\eta+k\vartheta+2\mu)}\right]_{-1}. \quad (4.38)$$

Thus, from relation (4.36) and Eq. (4.38), we obtain the solution of Eq. (4.30).
Theorem 5. Let $\chi \in \{\chi : 0 \neq |\chi_{\vartheta}| < \infty, \ \vartheta \in \mathbb{R}\}$. Then the homogeneous linear differential equation

$$r\chi''(r) + (2\theta r + (3 - 4a + 2\eta + 2\mu))\chi'(r) + (-\beta + 3\theta - 4a\theta + 2\eta\theta + 2\theta\mu)\chi(r) = 0, \quad (4.39)$$

has a solution of the form

$$\chi(r) = h\left\{\mathrm{e}^{-(2r\theta)} r^{-(3-4a+2\eta+k\vartheta+2\mu)}\right\}_{-(\vartheta+1)}, \quad (4.40)$$

here h is a constant of integration.
Proof: When $Q(r) = 0$ in Theorem 4, we get

$$V_1 + V\left(2\theta + \frac{3 - 4a + 2\eta + 2\mu + k\vartheta}{r}\right) = 0, \quad (4.41)$$

Therefore, we get fractional solution Eq. (4.40) for Eq. (4.41).

5. Conclusion

In the present study, the fractional nabla operator has been used to investigated the effective mass Schrödinger equation with a given potential that has arisen in quantum mechanics. We apply some classical transformations in order to reduce the given model and also used some tools in view of discrete fractional calculus. As a result, we obtained various exact fractional solutions. We believe that the solution reported will be important for the quantum system of physics.

Conflict of interest

The authors declare no conflict of interest.

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