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Research article

On the oscillation of differential equations in frame of generalized proportional fractional derivatives

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Abstract: In this paper, sufficient conditions are established for the oscillation of all solutions of generalized proportional fractional differential equations of the form

$$\begin{cases} {}_aD^{\alpha,\rho}x(t) + \xi_1(t, x(t)) = \mu(t) + \xi_2(t, x(t)), & t > a \geq 0, \\ \lim_{t\to a^+} {}_aI^{j-\alpha,\rho}x(t) = b_j, & j = 1, 2, \dots, n, \end{cases}$$

where $n = \lceil\alpha\rceil$, ${}_aD^{\alpha,\rho}$ is the generalized proportional fractional derivative operator of order $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, $0 < \rho \leq 1$ in the Riemann-Liouville setting and ${}_aI^{\alpha,\rho}$ is the generalized proportional fractional integral operator. The results are also obtained for the generalized proportional fractional differential equations in the Caputo setting. Numerical examples are provided to illustrate the applicability of the main results.

Keywords: proportional fractional integral; proportional fractional derivative; proportional fractional operator; fractional differential equation; oscillation theory
Mathematics Subject Classification: 26A33, 34A08, 34C10

1. Introduction

Fractional calculus (FC), dealing with an arbitrary (non-integer) order differential and integral operators, has played an important role in the development of many fields in science and engineering. It has been realized that fractional differential equations (FDEs) can provide practical tools when they are applied to mathematical models that describe natural real world problems. Due to their nonlocal character and dependence on history, FDEs have been involved in many topics in physics, biology, electroanalytical chemistry, medical sciences and economy. For more details and explanations, we refer the reader to the books [1–5] and references cited therein.

Recently, Jarad *et al.* [6] introduced a new type of fractional derivative so called generalized proportional fractional (GPF) derivative generated by local derivatives [7, 8] which are considered as modification of the conformable derivatives [9, 10]. The peculiarity of the new derivative is that it involves two fractional order, preserves the semigroup property, possesses nonlocal character and upon limiting cases it converges to the original function and its derivative. The GPF derivative is well behaved and has a substantial advantageous over the classical derivatives in the sense that it generalizes previously defined derivatives in the literature. We list here some recent results which have been elaborated in frame of GPF derivative [11–13] and other related works (see [14–16]).

The oscillation theory for fractional differential and difference equations has been studied by some works (see for examples [17–31]). In 2012, Grace *et al.* [17] first studied the oscillation theory of solutions for fractional initial value problem of the form

$$\begin{cases} {}_aD^{\alpha}x(t) + f_1(t, x(t)) = v(t) + f_2(t, x(t)), \quad t > a, \\ \lim_{t\to a^+} {}_aI^{1-\alpha}x(t) = b, \end{cases} \tag{1.1}$$

where ${}_aD^{\alpha}$ denotes the Riemann-Liouville fractional derivative starting at a point a, of order α with $\alpha \in (0, 1]$ and ${}_aI^{1-\alpha}$ is the Riemann-Liouville fractional integral starting at a point a, of order $1 - \alpha$, $b \in \mathbb{R}$, $f_i \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$, $(i = 1, 2)$ and $v \in C([a, \infty), \mathbb{R})$.

In 2013, Chen *et al.* [18] improved and extended some work in [17] by considering the forced fractional differential equation with initial conditions of the form

$$\begin{cases} {}_aD^{\alpha}x(t) + f_1(t, x(t)) = v(t) + f_2(t, x(t)), \quad t > a \ge 0, \\ {}_aD^{\alpha-k}x(t) = b_k,\ (k = 1, 2, \dots, m-1), \quad \lim_{t\to a^+} {}_aI^{m-\alpha}x(t) = b_m, \end{cases} \tag{1.2}$$

where ${}_aD^{\alpha}$ is the Riemann-Liouville fractional derivative starting at a point a, of order α of x, $\alpha \in (m-1, m]$, $m \ge 1$ is an integer, ${}_aI^{1-\alpha}$ is the Riemann-Liouville fractional integral starting at a point a, of order $m - \alpha$ of x, $b_k(k = 1, 2, \dots, m-1)$ are/is constants/constant, $f_i \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$, $(i = 1, 2)$ and $v \in C([a, \infty), \mathbb{R})$.

Recently, Abdalla [24] studied the oscillation of a conformable initial value problem of the form

$$\begin{cases} {}_aD^{\alpha,\rho}x(t) + f_1(t, x(t)) = r(t) + f_2(t, x(t)), \quad t > a, \\ \lim_{t\to a^+} {}_aI^{j-\alpha,\rho}x(t) = b_j,\ (j = 1, 2, \dots, m), \end{cases} \tag{1.3}$$

where $m = \lceil\alpha\rceil = \min\{m \in \mathbb{Z} | m \ge \alpha\}$, ${}_aD^{\alpha,\rho}$ is the left conformable derivative of order $\alpha \in \mathbb{C}$, $Re(\alpha) \ge 0$ in Riemann-Liouville stting and ${}_aI^{\alpha,\rho}$ is the left conformable integral operator.

Motivated by the above papers, the objective of this paper is to establish several oscillation criteria of solutions for the generalized proportional fractional differential equation with initial conditions of the form

$$\begin{cases} {}_aD^{\alpha,\rho}x(t) + \xi_1(t, x(t)) = \mu(t) + \xi_2(t, x(t)), \quad t > a \ge 0, \\ \lim_{t\to a^+} {}_aI^{j-\alpha,\rho}x(t) = b_j, \quad j = 1, 2, \dots, n, \end{cases} \tag{1.4}$$

where $n = \lceil \alpha \rceil$, ${}_aD^{\alpha,\rho}$ denotes the proportional Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$ of x, $Re(\alpha) \geq 0$, $\rho \in (0,1]$ and ${}_aI^{j-\alpha,\rho}$ denotes the generalized proportional fractional integral of order $j-\alpha \in \mathbb{C}$, $b_j \in \mathbb{R}$, $j = 1,2,\ldots,n$, $\xi_i \in C([a,\infty)\times\mathbb{R},\mathbb{R})$ and $\mu \in C([a,\infty),\mathbb{R})$.

Moreover, we study the oscillation theory of solutions for the generalized proportional fractional differential equation with initial conditions in the Caputo setting of the form

$$\begin{cases} {}_a^CD^{\alpha,\rho}x(t) + \xi_1(t,x(t)) = \mu(t) + \xi_2(t,x(t)), \quad t > a \geq 0, \\ (D^{k,\rho}x)(a) = b_k, \quad k = 0,1,\ldots,n-1, \end{cases} \tag{1.5}$$

where $n = \lceil \alpha \rceil$, ${}_a^CD^{\alpha,\rho}$ denotes the generalized proportional Caputo fractional derivative of order $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, $\rho \in (0,1]$ and $D^{k,\rho} = \underbrace{D^\rho D^\rho \cdots D^\rho}_{\text{k times}}$, and D^ρ is the proportional derivative.

This paper is organized as follows: Section 2 devoted to providing essential preliminaries on the GPF derivatives and integrals as well as stating some basic properties and fundamental lemmas that will be used in the proofs of the main results. In Section 3, the main osillation results are presented. Finally, numerical examples are provided in Section 4 to explain the applicability of the proven main results.

2. Preliminaries and background materials

In this section, we introduce some standard definitions and essential lemmas that will be needed to prove the main results in the remaining part of this paper. For their justifications and proofs, the reader can consult [6,33].

In control theory, a proportional derivative controller (PDC) for controller output x at time t with two tuning parameters has the algorithm

$$x(t) = \kappa_p E(t) + \kappa_d \frac{d}{dt}E(t),$$

where κ_p and κ_d are the proportional control parameter and the derivative control parameter, respectively. The function E is the input deviation or the error between the state variable and the process variable. The recent examinations have represented that PDC has straightway incorporation in the control of complex networks models; see [32].

For $\rho \in [0,1]$, let the functions $\kappa_0, \kappa_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have

$$\lim_{\rho\to0^+} \kappa_1(\rho,t) = 1, \quad \lim_{\rho\to0^+} \kappa_0(\rho,t) = 0, \lim_{\rho\to1^-} \kappa_1(\rho,t) = 0, \quad \lim_{\rho\to1^-} \kappa_0(\rho,t) = 1,$$

and $\kappa_1(\rho,t) \neq 0$, $\rho \in [0,1)$, $\kappa_0(\rho,t) \neq 0$, $\rho \in (0,1]$. Then, Anderson *et al.* in [7] defined the proportional derivative of order ρ by

$$D^\rho\xi(t) = \kappa_1(\rho,t)\xi(t) + \kappa_0(\rho,t)\xi'(t). \tag{2.1}$$

provided that the right-hand side exists at $t \in \mathbb{R}$ and $\xi' := \frac{d}{dt}\xi(t)$. κ_1 is a type of proportional gain κ_p, κ_0 is a type of derivative gain κ_d, ξ is the error and $x = D^\rho\xi$ is the controller output. The reader can study the paper [8] for more information about the control theory of the proportional derivative and its component functions. We shall restrict ourselves to the case when $\kappa_1(\rho,t) = 1-\rho$ and $\kappa_0(\rho,t) = \rho$. Thus, (2.1) becomes

$$D^\rho\xi(t) = (1-\rho)\xi(t) + \rho\xi'(t). \tag{2.2}$$

It is easily to see that $\lim_{\rho\to 0^+} D^\rho\xi(t) = \xi(t)$ and $\lim_{\rho\to 1^-} D^\rho\xi(t) = \xi'(t)$. Therefore, (2.2) is analyzed to be more general than the conformable derivative which evidently does not tend to the original functions as ρ tends to 0.

Firstly, we give the definition of GPF integral as the following.

Definition 2.1. [6] For $\rho \in (0, 1]$, $\alpha \in \mathbb{C}$ and $Re(\alpha) > 0$, the GPF integral of ξ of order α is

$$ {}_aI^{\alpha,\rho}\xi(t) = \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\xi(s)ds = \rho^{-\alpha}e^{\frac{\rho-1}{\rho}t}\,{}_aI^\alpha\Big(e^{\frac{1-\rho}{\rho}t}\xi(t)\Big), \tag{2.3}$$

where ${}_aI^\alpha$ is Riemann–Liouville fractional integral.

We define the GPF derivatives of Riemann–Liouville and Caputo types as follows.

Definition 2.2. [6] For $\rho \in (0, 1]$, $\alpha \in \mathbb{C}$ with $Re(\alpha) \geq 0$ and $n = [Re(\alpha)] + 1$, then the GPF derivative of Riemann–Liouville type of ξ of order α is

$$ {}_aD^{\alpha,\rho}\xi(t) = D^{n,\rho}\,{}_aI^{n-\alpha,\rho}\xi(t) = \frac{D_t^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{n-\alpha-1}\xi(s)ds, \tag{2.4}$$

where $[Re(\alpha)]$ represents the integer part of the real number α.

Definition 2.3. [6] For $\rho \in (0, 1]$, $\alpha \in \mathbb{C}$ with $Re(\alpha) \geq 0$, the GPF derivative of Caputo type of ξ of order α is

$$ {}_a^CD^{\alpha,\rho}\xi(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{n-\alpha-1}D^{n,\rho}\xi(s)ds, \tag{2.5}$$

where $n = [Re(\alpha)] + 1$ and $[Re(\alpha)]$ represents the integer part of the real number α.

Lemma 2.4. [6] *Let $Re(\alpha) > 0$, $n = [Re(\alpha)]$, $\xi \in L_1(a, b)$, ${}_aI^{\alpha,\rho}f(t) \in AC^n[a, b]$ and $\rho \in (0, 1]$. Then*

$$ {}_aI^{\alpha,\rho}\,{}_aD^{\alpha,\rho}\xi(t) = \xi(t) - e^{\frac{\rho-1}{\rho}(t-a)}\sum_{j=1}^{n}\frac{{}_aI^{j-\alpha,\rho}\xi(a^+)}{\rho^{\alpha-j}\Gamma(\alpha+1-j)}(t-a)^{\alpha-j}. \tag{2.6}$$

Lemma 2.5. [6] *For $\rho \in (0, 1]$ and $n = [Re(\alpha)] + 1$, we have $({}_a^CD^{\alpha,\rho}\,{}_aI^{\alpha,\rho}\xi)(t) = \xi(t)$, and*

$$ {}_aI^{\alpha,\rho}\,{}_a^CD^{\alpha,\rho}\xi(t) = \xi(t) - \sum_{k=0}^{n-1}\frac{(D^{k,\rho}\xi)(a)}{\rho^k k!}(t-a)^k e^{\frac{\rho-1}{\rho}(t-a)}. \tag{2.7}$$

Proposition 2.6. [6] Let $\alpha,\beta \in \mathbb{C}$ be such that $Re(\alpha) \geq 0$ and $Re(\beta) > 0$. Then, for any $\rho \in (0, 1]$ and $n = [Re(\alpha)] + 1$, we have

(i) $({}_aI^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\rho^\alpha}e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta+\alpha-1}, \quad Re(\alpha) > 0.$

(ii) $({}_a^CD^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1})(x) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta-\alpha-1}, \quad Re(\alpha) > n.$

(iii) $({}_a^CD^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}(t-a)^k)(x) = 0, \quad Re(\alpha) > n, \quad k = 0, 1, \ldots, n-1.$

Lemma 2.7. [33] *The inequality of Young.*

(i) Let A, $B \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$, where the equality holds if and only if $B = A^{p-1}$.

(ii) *Let* $A \geq 0$, $B > 0$, $0 < p < 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *then* $AB \geq \frac{1}{p}A^p + \frac{1}{q}B^q$, *where the equality holds if and only if* $B = A^{p-1}$.

The solution x of the problem (1.4) (the problem (1.5)) is siad to be oscillatory if it has arbitrarily large zeros on $(0, \infty)$; otherwise, it is called nonoscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

3. Oscillation criteria

To obtain oscillation criteria of the main results, we list the following assumptions to prove our results:

(H_1) $x\xi_i(t,x) > 0, \quad i = 1,2, \quad x \neq 0, \quad t > 0,$
(H_2) $|\xi_1(t,x)| \geq \sigma_1(t)|x|^\beta$ and $|\xi_2(t,x)| \leq \sigma_2(t)|x|^\gamma, \quad x \neq 0, \quad t \geq 0,$
(H_3) $|\xi_1(t,x)| \leq \sigma_1(t)|x|^\beta$ and $|\xi_2(t,x)| \geq \sigma_2(t)|x|^\gamma, \quad x \neq 0, \quad t \geq 0,$

where $\sigma_1, \sigma_2 \in C([0,\infty),(0,\infty))$ and β, γ are positive real numbers.

For the sake of computational convenience, we define

$$\phi(t) = \Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)}\sum_{j=1}^{n}\frac{\rho^j{}_aI^{j-\alpha,\rho}x(a^+)}{\Gamma(\alpha+1-j)}(t-a)^{\alpha-j}, \tag{3.1}$$

$$\psi(t,\tau_1) = \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}H(s,x(s))ds, \tag{3.2}$$

$$\lambda(t) = \rho^\alpha\Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)}\sum_{k=0}^{n-1}\frac{(D^{k,\rho}f)(a)}{\rho^k k!}(t-a)^k. \tag{3.3}$$

3.1. Oscillation criterias of the GPF differential equations in Riemann type

In this section, we study the oscillation theory for the problem (1.4). By using Lemma 2.4, the solution of the problem (1.4) can be represented by

$$x(t) = e^{\frac{\rho-1}{\rho}(t-a)}\sum_{j=1}^{n}\frac{{}_aI^{j-\alpha,\rho}x(a^+)}{\rho^{\alpha-j}\Gamma(\alpha+1-j)}(t-a)^{\alpha-j} + {}_aI^{\alpha,\rho}H(t,x(t)), \tag{3.4}$$

where $H(t,x(t)) = \mu(t) - \xi_1(t,x(t)) + \xi_2(t,x(t))$ and $H(a,x(a)) = 0$.

Theorem 3.1. *Let* $\xi_2 = 0$ *in the problem* (1.4) *and the assumption* (H_1) *hold. If*

$$\liminf_{t\to\infty} t^{1-\alpha}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds = -\infty, \tag{3.5}$$

and

$$\limsup_{t\to\infty} t^{1-\alpha}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds = \infty, \tag{3.6}$$

for every sufficiently large τ, *then every solution of the problem* (1.4) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of the problem (1.4) with $\xi_2 = 0$. Assume that $x(t)$ is an eventually positive solution of the problem (1.4). Then there exists a point $\tau_1 > a$ is large enough such that $x(t) > 0$ for $t \geq \tau_1$. Hence, the assumption (H_1) implies that $\xi_1(t,x) > 0$ for $t \geq \tau_1$. From (3.4), we have

$$\begin{aligned}\rho^\alpha \Gamma(\alpha)x(t) &\leq \Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)}\sum_{j=1}^{n}\frac{\rho^j {}_aI^{j-\alpha,\rho}x(a^+)}{\Gamma(\alpha+1-j)}(t-a)^{\alpha-j}\\ &\quad+\int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}H(s,x(s))ds+\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds,\end{aligned}$$

where $H(t,x(t)) = \mu(t) - \xi_1(t,x(t)) + \xi_2(t,x(t))$. Using (3.1) and (3.2), we get

$$\rho^\alpha\Gamma(\alpha)x(t) \leq \phi(t) + \psi(t,\tau_1) + \int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds. \tag{3.7}$$

Multiplying $t^{1-\alpha}$ into both sides of (3.7), we obtain

$$0 < t^{1-\alpha}\rho^\alpha\Gamma(\alpha)x(t) \leq t^{1-\alpha}\phi(t) + t^{1-\alpha}\psi(t,\tau_1) + t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds. \tag{3.8}$$

Let $\tau_2 > \tau_1$. We divided the proof into two cases as follows:

Case I. Let $0 < \alpha \leq 1$. Then $n = 1$ and $t^{1-\alpha}\phi(t) = \rho b_1 e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{\alpha-1}$. Since the function $r_1(t) = e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{\alpha-1}$ is decreasing for $0 < \rho, \alpha \leq 1$, we observe that for $t \geq \tau_2$,

$$\left|t^{1-\alpha}\phi(t)\right| = \left|\rho b_1 e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{\alpha-1}\right| \leq \rho|b_1|e^{\frac{\rho-1}{\rho}(\tau_2-a)}\left(\frac{\tau_2-a}{\tau_2}\right)^{\alpha-1} := \omega_1(\tau_2). \tag{3.9}$$

The function $r_2(t) = e^{\frac{\rho-1}{\rho}(t-s)}\left(\frac{t-a}{t}\right)^{\alpha-1}$ is decreasing for $0 < \rho, \alpha \leq 1$, we get, for $t \geq \tau_2$,

$$\begin{aligned}\left|t^{1-\alpha}\psi(t,\tau_1)\right| &= \left|t^{1-\alpha}\int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)-\xi_1(s,x(s))+\xi_2(s,x(s))]ds\right|\\ &\leq \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(t-s)}\left(\frac{t-s}{t}\right)^{\alpha-1}|\mu(s)-\xi_1(s,x(s))+\xi_2(s,x(s))|ds\\ &\leq \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(\tau_2-s)}\left(\frac{\tau_2-s}{\tau_2}\right)^{\alpha-1}|\mu(s)-\xi_1(s,x(s))+\xi_2(s,x(s))|ds\\ &:= \omega_2(\tau_1,\tau_2).\end{aligned} \tag{3.10}$$

From (3.8), (3.9) and (3.10), we get, for $t \geq \tau_2$,

$$t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds \geq -[\omega_1(\tau_2)+\omega_2(\tau_1,\tau_2)].$$

Since, the right hand side of the above inequality is a negative constant, we conclude that

$$\liminf_{t\to\infty} t^{1-\alpha}\int_{T^*}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds \geq -[\omega_1(\tau_2)+\omega_2(\tau_1,\tau_2)] > -\infty,$$

which leads to a contradiction with the assumption (3.5).

Case II. Let $\alpha > 1$. Then $n \geq 2$. Also $e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{\alpha-1} \leq 1$ for $\alpha > 1$ and $0 < \rho \leq 1$. The function $r_3(t) = (t-a)^{1-j}$ is decreasing for $j > 1$. Thus, for $t \geq \tau_2$, we get

$$\begin{aligned}
\left|t^{1-\alpha}\phi(t)\right| &= \left|t^{1-\alpha}\Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)}\sum_{j=1}^{n}\frac{\rho^j{}_aI^{j-\alpha,\rho}x(a^+)}{\Gamma(\alpha+1-j)}(t-a)^{\alpha-j}\right| \\
&= \left|e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{\alpha-1}\Gamma(\alpha)\sum_{j=1}^{n}\frac{\rho^j b_j(t-a)^{1-j}}{\Gamma(\alpha+1-j)}\right| \\
&\leq \Gamma(\alpha)\sum_{j=1}^{n}\frac{\rho^j|b_j|(t-a)^{1-j}}{\Gamma(\alpha+1-j)} \\
&\leq \Gamma(\alpha)\sum_{j=1}^{n}\frac{\rho^j|b_j|(\tau_2-a)^{1-j}}{\Gamma(\alpha+1-j)} := \omega_3(\tau_2). \qquad (3.11)
\end{aligned}$$

Since $e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{\alpha-1} \leq 1$ for $\alpha > 1$ and $0 < \rho \leq 1$, we also have

$$\begin{aligned}
\left|t^{1-\alpha}\psi(t,\tau_1)\right| &= \left|\int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(t-s)}\left(\frac{t-s}{t}\right)^{\alpha-1}[\mu(s)-\xi_1(s,x(s))+\xi_2(s,x(s))]ds\right| \\
&\leq \int_a^{\tau_1}|\mu(s)-\xi_1(s,x(s))+\xi_2(s,x(s))|ds := \omega_4(\tau_1). \qquad (3.12)
\end{aligned}$$

From (3.8), (3.11) and (3.12), we have, for $t \geq \tau_2$,

$$t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds \geq -[\omega_3(\tau_2)+\omega_4(\tau_1)],$$

Since, the right hand side of the above inequality is a negative constant, we conclude that

$$\liminf_{t\to\infty} t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds \geq -[\omega_3(\tau_2)+\omega_4(\tau_1)] > -\infty,$$

which contradicts the assumption (3.5).

Hence, we can conclude that the solution $x(t)$ is oscillatory. In case $x(t)$ is an eventually negative solution of the problem (1.4), similar arguments may by applied to a contradiction with the assumption (3.6). □

Theorem 3.2. *Let the assumptions* (H_1) *and* (H_2) *hold with* $\beta > \gamma$. *If*

$$\liminf_{t\to\infty} t^{1-\alpha}\int_{\tau}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)+K(s)]ds = -\infty, \qquad (3.13)$$

and

$$\limsup_{t\to\infty} t^{1-\alpha}\int_{\tau}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)-K(s)]ds = \infty, \qquad (3.14)$$

for every sufficiently large τ, where

$$K(t) = \frac{\beta-\gamma}{\gamma}[\sigma_1(t)]^{\frac{\gamma}{\gamma-\beta}}\left[\frac{\gamma\sigma_2(t)}{\beta}\right]^{\frac{\beta}{\beta-\gamma}}, \tag{3.15}$$

then every solution of the problem (1.4) *is oscillatory.*

Proof. we prove by contradiction process. Let $x(t)$ be a non-oscillatory solution of the problem (1.4) with $x(t) > 0$ for $t \geq \tau_1 > a$. Using the assumptions (H_2) and (H_3), we have

$$\xi_2(s,x) - \xi_1(s,x) \leq \sigma_2(s)x^\gamma(s) - \sigma_1(s)x^\beta(s).$$

Let $A = x^\gamma(s)$ and $B = \frac{\gamma\sigma_2(s)}{\beta\sigma_1(s)}$, $p = \frac{\beta}{\gamma}$ and $q = \frac{\beta}{\beta-\gamma}$, then from the part (i) of Lemma 2.7 we get

$$\begin{aligned} \sigma_2(s)x^\gamma(s) - \sigma_1(s)x^\beta(s) &= \frac{\beta\sigma_1(s)}{\gamma}\left[x^\gamma(s)\frac{\gamma\sigma_2(s)}{\beta\sigma_1(s)} - \frac{\gamma}{\beta}(x^\gamma(s))^{\frac{\beta}{\gamma}}\right] \\ &= \frac{\beta\sigma_1(s)}{\gamma}\left[AB - \frac{1}{p}A^p\right] \\ &\leq \frac{\beta\sigma_1(s)}{\gamma}\cdot\frac{1}{q}B^q \\ &= \frac{\beta-\gamma}{\gamma}[\sigma_1(s)]^{\frac{\gamma}{\gamma-\beta}}\left[\frac{\gamma\sigma_2(s)}{\beta}\right]^{\frac{\beta}{\beta-\gamma}} \\ &:= K(s), \end{aligned} \tag{3.16}$$

where K is defined by (3.15). Then from the Eq. (3.4), we obtain

$$\begin{aligned} \rho^\alpha\Gamma(\alpha)x(t) &= \phi(t) + \psi(t,\tau_1) + \int_{\tau_1}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s) - \xi_1(s,x(s)) + \xi_2(s,x(s))]ds \\ &\leq \phi(t) + \psi(t,\tau_1) + \int_{\tau_1}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\left[\mu(s) + \sigma_2(s)x^\gamma(s) - \sigma_1(s)x^\beta(s)\right]ds \\ &\leq \phi(t) + \psi(t,\tau_1) + \int_{\tau_1}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\left[\mu(s) + K(s)\right]ds. \end{aligned} \tag{3.17}$$

The remaining part of the proof is the same as that of Theorem 3.1 and hence is omitted. □

Theorem 3.3. *Let $\alpha \geq 1$ and suppose that the assumptions (H_1) and (H_3) hold with $\beta < \gamma$. If*

$$\limsup_{t\to\infty} t^{1-\alpha}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s) + K(s)]ds = -\infty, \tag{3.18}$$

and

$$\liminf_{t\to\infty} t^{1-\alpha}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s) - K(s)]ds = \infty, \tag{3.19}$$

for every sufficiently large τ, where K is defined by (3.15)*, then every bounded solution of the problem* (1.4) *is oscillatory.*

Proof. Let $x(t)$ be a bounded non-oscillatory solution of the problem (1.4). Then there exist positive constants m and M such that

$$m \leq x(t) \leq M, \quad \text{and} \quad t \geq a. \tag{3.20}$$

Assume that $x(t)$ is a bounded eventually positive solution of the problem (1.4). Then there exists $\tau_1 > a$ such that $x(t) > 0$ for $t \geq \tau_1 > a$. Using the assumptions (H_1) and (H_3), we get $\xi_2(s, x) - \xi_1(s, x) \geq \sigma_2(s)x^{\gamma}(s) - \sigma_1(s)x^{\beta}(s)$. Applying Lemma 2.7 (*ii*) and similar to the proof of (3.16), we find that

$$\sigma_2(s)x^{\gamma}(t) - \sigma_1(s)x^{\beta}(s) \geq K(s), \quad s \geq \tau_1.$$

From (3.4) and similar to (3.17), we obtain

$$\rho^{\alpha}\Gamma(\alpha)x(t) \geq \phi(t) + \psi(t, \tau_1) + \int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\left[\mu(s) + K(s)\right] ds.$$

Multiplying by $t^{1-\alpha}$, we get

$$t^{1-\alpha}\rho^{\alpha}\Gamma(\alpha)x(t) \geq t^{1-\alpha}\phi(t) + t^{1-\alpha}\psi(t, \tau_1) + t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\left[\mu(s) + K(s)\right] ds. \tag{3.21}$$

Take $\tau_2 > \tau_1$. We consider two cases as follows.

Case I. Let $0 < \alpha \leq 1$. Then (3.9) and (3.10) are still correct. Thus, from (3.21) and using (3.20), we compute that

$$\begin{aligned} M\rho^{\alpha}\Gamma(\alpha) &\geq t^{1-\alpha}\rho^{\alpha}\Gamma(\alpha)x(t) \\ &\geq -\omega_1(\tau_2) - \omega_2(\tau_1, \tau_2) + t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\left[\mu(s) + K(s)\right] ds \end{aligned}$$

for $t \geq \tau_2$. Then, we get

$$\limsup_{t\to\infty} t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\left[\mu(s) + K(s)\right] ds \leq \omega_1(\tau_2) + \omega_2(\tau_1, \tau_2) + M\rho^{\alpha}\Gamma(\alpha) < \infty,$$

which leads to a contradiction with the assumption (3.18).

Case II. Let $\alpha > 1$. Then (3.11) and (3.12) are still correct. Hence, from (3.21) and using (3.20), we calculate that

$$M\rho^{\alpha}\Gamma(\alpha)t^{1-\alpha} \geq -\omega_3(\tau_2) - \omega_4(\tau_1) + t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\left[\mu(s) + K(s)\right] ds,$$

for $t \geq \tau_2$. Since $\lim_{t\to\infty} t^{1-\alpha} = 0$ for $\alpha > 1$, hence, we conclude that

$$\limsup_{t\to\infty} t^{1-\alpha}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\left[\mu(s) + K(s)\right] ds \leq +\omega_3(\tau_2) + \omega_4(\tau_1) < \infty,$$

which gives a contradiction with the assumption (3.18). Therefore, we conclude that $x(t)$ is oscillatory. In case $x(t)$ is eventually bounded negative, similar arguments lead to a contradiction with the assumption (3.19). □

3.2. Oscillation criterias of the GPF differential equations in Caputo type

In this section, we study the oscillation theory for the problem (1.5). By using Lemma 2.5, the solution of the problem (1.5) can be written as

$$x(t) = e^{\frac{\rho-1}{\rho}(t-a)} \sum_{k=0}^{n-1} \frac{(D^{k,\rho}x)(a)}{\rho^k k!}(t-a)^k + {}_aI^{\alpha,\rho}H(t,x(t)), \tag{3.22}$$

where $H(t,x(t)) = \mu(t) - \xi_1(t,x(t)) + \xi_2(t,x(t))$ and $H(a,x(a)) = 0$.

Theorem 3.4. *Let $\xi_2 = 0$ in* (1.5) *and the assumption* (H_1) *hold. If*

$$\liminf_{t\to\infty} t^{1-n} \int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds = -\infty, \tag{3.23}$$

and

$$\limsup_{t\to\infty} t^{1-n} \int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds = \infty, \tag{3.24}$$

for every sufficiently large τ, then every solution of the problem (1.5) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of the problem (1.5). Assume that $x(t)$ is an eventually positive solution of the problem (1.5). Then there exists a point $\tau_1 > a$ such that $x(t) > 0$ for $t \geq \tau_1$. Hence, the assumption (H_1) implies that $\xi_1(t,x(t)) > 0$ for $t \geq \tau_1$. From (3.22), we have

$$\begin{aligned} \rho^\alpha\Gamma(\alpha)x(t) \quad &\leq \quad \rho^\alpha\Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)} \sum_{k=0}^{n-1} \frac{(D^{k,\rho}x)(a)}{\rho^k k!}(t-a)^k \\ &\quad + \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}H(s,x(s))ds + \int_{\tau_1}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds, \end{aligned}$$

where $H(t,x(t)) = \mu(t) - \xi_1(t,x(t)) + \xi_2(t,x(t))$. Using (3.2) and (3.3), we get that

$$\rho^\alpha\Gamma(\alpha)x(t) \leq \lambda(t) + \psi(t,\tau_1) + \int_{\tau_1}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds, \tag{3.25}$$

where ψ and λ are defined by (3.2) and (3.3), respectively.

Multiplying t^{1-n} into both sides of (3.25), we obtain

$$0 < t^{1-n}\rho^\alpha\Gamma(\alpha)x(t) \leq t^{1-n}\lambda(t) + t^{1-n}\psi(t,\tau_1) + t^{1-n}\int_{\tau_1}^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds. \tag{3.26}$$

Let $\tau_2 > \tau_1$. We divided the proof into two cases as follows:

Case I. Let $0 < \alpha \leq 1$. Then $n = 1$ and $t^{1-n}\lambda(t) = b_0\rho^\alpha\Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)}$. Since the function $q_1(t) = e^{\frac{\rho-1}{\rho}(t-a)}$ is decreasing for $0 < \rho \leq 1$, we observe that, for $t \geq \tau_2 > s$,

$$\left|t^{1-n}\lambda(t)\right| = \left|b_0\rho^\alpha\Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)}\right| \leq |b_0|\rho^\alpha e^{\frac{\rho-1}{\rho}(\tau_2-a)} := \theta_1(\tau_2). \tag{3.27}$$

The function $q_2(t) = e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}$ is decreasing for $0 < \rho \leq 1$ and $0 < \alpha \leq 1$, we get, for $t \geq \tau_2 > s$,

$$\begin{aligned}
\left|t^{1-n}\psi(t,\tau_1)\right| &= \left|t^{1-n}\int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)-\xi_1(s,x(s))+\xi_2(s,x(s))]ds\right| \\
&\leq \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}|\mu(s)-\xi_1(s,x(s))+\xi_2(s,x(s))|ds \\
&\leq \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{\alpha-1}|\mu(s)-\xi_1(s,x(s))+\xi_2(s,x(s))|ds \\
&:= \theta_2(\tau_1,\tau_2). \qquad (3.28)
\end{aligned}$$

From (3.26), (3.27) and (3.28), we get, for $t \geq \tau_2$,

$$t^{1-n}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds \geq -[\theta_1(\tau_2)+\theta_2(\tau_1,\tau_2)],$$

then,

$$\liminf_{t\to\infty} t^{1-n}\int_{\tau_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds \geq -[\theta_1(\tau_2)+\theta_2(\tau_1,\tau_2)] > -\infty,$$

which leads to a contradiction with the assumption (3.23).

Case II. Let $\alpha > 1$. Then $n \geq 2$. Also $e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{n-1} \leq 1$ for $\alpha > 1$ and $0 < \rho \leq 1$. The function $q_3(t) = (t-a)^{k-n+1}$ is decreasing for $k > n-1$ and $0 < \rho \leq 1$. Thus, for $t \geq \tau_2$, we have

$$\begin{aligned}
\left|t^{1-n}\lambda(t)\right| &= \left|t^{1-n}\rho^{\alpha}\Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)}\sum_{k=0}^{n-1}\frac{(D^{k,\rho}x)(a)}{\rho^k k!}(t-a)^k\right| \\
&= \left|\rho^{\alpha}\Gamma(\alpha)e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{n-1}\sum_{k=0}^{n-1}\frac{(D^{k,\rho}x)(a)}{\rho^k k!}(t-a)^{k-n+1}\right| \\
&\leq \rho^{\alpha}\Gamma(\alpha)\sum_{k=0}^{n-1}\frac{\left|(D^{k,\rho}x)(a)\right|}{\rho^k k!}(t-a)^{k-n+1} \\
&\leq \rho^{\alpha}\Gamma(\alpha)\sum_{k=0}^{n-1}\frac{\left|(D^{k,\rho}x)(a)\right|}{\rho^k k!}(\tau_2-a)^{k-n+1} \\
&:= \theta_3(\tau_2). \qquad (3.29)
\end{aligned}$$

Also, since $e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{t}\right)^{\alpha-1} \leq 1$ for $\alpha > 1$ and $0 < \rho \leq 1$, we get

$$\left|t^{1-n}\psi(t,\tau_1)\right| \leq \omega_4(\tau_1).$$

From (3.26) and (3.29), we get a contradiction the assumption (3.23). Therefore, we can conclude that the solution $x(t)$ is oscillatory. In case $x(t)$ is an eventually negative solution of the problem (1.5), similar arguments may by applied to achieve a contradiction with the assumption (3.24). □

By employing similar arguments to proofs of Theorem 3.5 and Theorem 3.6, we can prove the following results. Hence, we stated the following two theorems without proofs.

Theorem 3.5. *Let the assumptions* (H_1) *and* (H_2) *hold with* $\beta > \gamma$*. If*

$$\liminf_{t\to\infty} t^{1-n}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)+K(s)]ds = -\infty, \tag{3.30}$$

and

$$\limsup_{t\to\infty} t^{1-n}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)-K(s)]ds = \infty, \tag{3.31}$$

for every sufficiently large τ*, where* K *is defined by* (3.15)*, then every solution of the problem* (1.5) *is oscillatory.*

Theorem 3.6. *Let* $\alpha \geq 1$ *and suppose that the assumptions* (H_1) *and* (H_3) *hold with* $\beta < \gamma$*. If*

$$\limsup_{t\to\infty} t^{1-n}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)+K(s)]ds = -\infty, \tag{3.32}$$

and

$$\liminf_{t\to\infty} t^{1-n}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)-K(s)]ds = \infty, \tag{3.33}$$

for every sufficiently large τ*, where* K *is defined by* (3.15)*, then every bounded solution of the problem* (1.5) *is oscillatory.*

4. Numerical examples

In this section, we build three numerical examples for the sake of understanding the applicability of our theoretical results.

Example 4.1. Consider the following GPF differential equation in the Riemann–Liouville setting

$$\begin{cases} {}_aD^{\alpha,\rho}x(t) + x^5(t) = \rho^\alpha \sin t, \\ \lim_{t\to a^+} {}_aI^{1-\alpha,\rho}x(t) = 0, \quad 0<\alpha<1, \quad \rho>0, \end{cases} \tag{4.1}$$

where $\xi_1(t,x) = x^5(t)$, $\xi_2(t,x) = 0$ and $\mu(t) = \rho^\alpha \sin t$. Then the assumption (H_1) holds. Moreover, it is easily to check that

$$\liminf_{t\to\infty} t^{1-\alpha}\rho^\alpha\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\sin(s)ds = -\infty,$$

and

$$\limsup_{t\to\infty} t^{1-\alpha}\rho^\alpha\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\sin(s)ds = \infty,$$

This shows that the conditions (3.13) and (3.14) of Theorem 3.1 hold. Therefore, we can conclude that every solution of the problem (4.1) is oscillatory.

Example 4.2. Consider the following GPF differential equation in the Riemann–Liouville setting

$$\begin{cases} {}_aD^{\alpha,\rho}x(t) + x^3(t)e^{\frac{\rho-1}{3\rho}t}\ln(t+e) = \frac{2\rho^\alpha e^{\frac{\rho-1}{\rho}t}(t-a)^{2-\alpha}}{\Gamma(3-\alpha)} \\ \qquad\qquad + \Big[(t-a)^6 - (t-a)^{0.5}\Big]\ln(t+e) + x^{0.25}(t)e^{\frac{4(\rho-1)}{\rho}t}\ln(t+e), \\ \lim_{t\to a^+} {}_aI^{1-\alpha,\rho}x(t) = b_1, \end{cases} \tag{4.2}$$

where $n = 1$, $\xi_1(t,x) = x^3(t)e^{\frac{\rho-1}{3\rho}t}\ln(t+e)$, $\xi_2(t,x) = x^{0.25}(t)e^{\frac{4(\rho-1)}{\rho}t}\ln(t+e)$ and $\mu(t) = 2\rho^\alpha e^{\frac{\rho-1}{\rho}t}(t-a)^{2-\alpha}/\Gamma(3-\alpha) + [(t-a)^6 - (t-a)^{0.5}]\ln(t+e)$. It is easy to verify that the assumptions (H_1) and (H_2) are satisfied for $\beta = 3$, $\gamma = 0.25$ and $\sigma_1(t) = \sigma_2(t) = \ln(t+e)$. However, we show in the following that assumption (3.18) does not hold. For every sufficiently large $\tau \geq 1$ and all $t \geq \tau$, we have $\mu(t) > 0$. Calculating $K(t)$ is defined by (3.15), we find that $K(t) = (11)(12)^{-\frac{12}{11}}\ln(t+e) \geq 0.74$. Then, using Proposition 2.6 (i) with $b = 1$, we get

$$\begin{aligned} & \liminf_{t\to\infty} t^{1-\alpha}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}[\mu(s)+K(s)]ds \\ \geq\ & \liminf_{t\to\infty} t^{1-\alpha}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}K(s)ds \\ \geq\ & \liminf_{t\to\infty} 0.74\, t^{1-\alpha}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}e^{\frac{\rho-1}{\rho}t}(s-\tau)^0 ds \\ =\ & \liminf_{t\to\infty} 0.74\, t^{1-\alpha}\rho^\alpha\Gamma(\alpha)({}_{T^*}I^{\alpha,\rho}e^{\frac{\rho-1}{\rho}s}(s-\tau)^0)(t) \\ =\ & \liminf_{t\to\infty} \frac{0.74\, t}{\alpha}e^{\frac{\rho-1}{\rho}t}\left(\frac{t-\tau}{t}\right)^\alpha \\ =\ & \infty. \end{aligned}$$

However, using Proposition 2.6 (i) with $\beta = 1$, one can easily verify that $x(t) = e^{\frac{\rho-1}{\rho}t}(t-a)^2$ is a nonoscillatiory solution of (4.2). The initial condition is also satisfied bacause

$$({}_aI^{1-\alpha,\rho}e^{\frac{\rho-1}{\rho}s}(s-a)^2)(t) = \frac{2\rho^{\alpha-1}e^{\frac{\rho-1}{\rho}t}(t-a)^{3-\alpha}}{\Gamma(4-\alpha)}$$

and

$$({}_aD^{\alpha,\rho}e^{\frac{\rho-1}{\rho}s}(s-a)^2)(t) = \frac{2\rho^{\alpha}e^{\frac{\rho-1}{\rho}t}(t-a)^{2-\alpha}}{\Gamma(3-\alpha)}.$$

Example 4.3. Consider the following GPF differential equation in the Caputo setting

$$\begin{cases} {}_a^CD^{\alpha,\rho}x(t) + \sqrt{2}e^t x^5(t) = \frac{3\sqrt{\pi}\rho^\alpha e^{\frac{\rho-1}{\rho}t}(t-a)^{\frac{3}{2}-\alpha}}{4\Gamma(\frac{5}{2}-\alpha)} + \sqrt{2}e^t x^3(t)(t-a)^{15}, \\ x(a) = 0, \quad 0 < \alpha < 1, \quad \rho > 0, \end{cases} \tag{4.3}$$

where $\xi_1(t,x) = \sqrt{2}e^t x^5(t)$, $\xi_2(t,x) = \sqrt{2}e^t x^3(t)(t-a)^{15}$ and $\mu(t) = \frac{3\sqrt{\pi}\rho^\alpha e^{\frac{\rho-1}{\rho}t}(t-a)^{\frac{3}{2}-\alpha}}{4\Gamma(\frac{5}{2}-\alpha)}$. The assumption (H_1) is satisfied. However, the condition (3.23) does not hold since

$$\liminf_{t\to\infty} t^{1-n}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\mu(s)ds \;\geq\; \liminf_{t\to\infty}\int_\tau^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}(s-\tau)^0 ds$$

$$
\begin{aligned}
&= \liminf_{t\to\infty} \rho^{\alpha}\Gamma(\alpha)e^{\frac{\rho-1}{\rho}t}(t-\tau)^{\alpha} \\
&= \infty.
\end{aligned}
$$

Using Proposition 2.6 (ii) with $\beta = \frac{5}{2}$, and the fact that

$$
{}_{a}^{C}D^{\alpha,\rho}x(t) = \frac{3\sqrt{\pi}\rho^{\alpha}e^{\frac{\rho-1}{\rho}t}(t-a)^{\frac{3}{2}-\alpha}}{4\Gamma(\frac{5}{2}-\alpha)},
$$

one can eaily check that $x(t) = e^{\frac{\rho-1}{\rho}t}(t-a)^{\frac{3}{2}}$ is a nonoscillatory solution of (4.3).

5. Conclusion

In this paper, some oscillation criteria for a type of fractional differential equations have been established. The main results are obtained in frame of the new defined GPF derivative and within the Riemann and Caputo type settings. The GPF derivative has many advantages over the existing derivatives in the literature and thus it operates as an extension and a complement to previously defined fractional derivatives. Numerical examples are presented to demonstrate the effectiveness of theoretical findings. Moreover, counter examples are constructed to show the existence of a nonoscillatory solution in case the proposed assumptions do not hold. It would be of great interest to investigate in future the oscillation of GPF differential equations within Hadamard type derivative or other types of new fractional derivatives.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, Wiley, New York, 1993.
2. I. Podlubny, *Fractional Differential Equations*, Acedemic Press, San Diego, 1999.
3. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
4. A. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.
5. V. Lakshmikanthan, S. Leela, J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

6. F. Jarad, T. Abdeljawad, J. Alzabut, *Generalized fractional derivatives generated by a class of local proportional derivatives*, Eur. Phys. J. Spec. Top., **226** (2017), 3457–3471.
7. D. R. Anderson, D. J. Ulness, *Newly defined conformable derivatives*, Adv. Dyn. Syst. Appl. **10** (2015), 109–137.
8. D. R. Anderson, *Second-order self-adjoint differential equations using a proportional-derivative controller*, Commun. Appl. Nonlinear Anal., **24** (2017), 17–48.
9. T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2013), 57–66.
10. R. Khalil, M. Al Horani, A.Yousef, et al. *A new Definition Of Fractional Derivative*, J. Comput. Appl. Math., **264** (2014), 65–70.
11. J. Alzabut, T. Abdeljawad, F. Jarad, et al. *A Gronwall inequality via the generalized proportional fractional derivative with applications*, J. Inequal. Appl., **2019** (2019), 101.
12. T. Abdeljawad, F. Jarad, S. Mallak, et al. *Lyapunov type inequalities via fractional proportional derivatives and application on the free zero disc of Kilbas-Saigo generalized Mittag-Leffler functions*, Eur. Phys. J. Plus, **134** (2019), 247.
13. J. Alzabut, W. Sudsutad, Z. Kayar, et al. *A new Gronwall–Bellman inequality in a frame of generalized proportional fractional proportional derivative*, Mathematics, **7** (2019), 747.
14. N. Sene, J. F. Gómez Aguilar, *Fractional mass-spring-damper system described by generalized fractional order derivatives*, Fractal Fract., **3** (2019), 39.
15. A. Fleitas, J. F. Gómez Aguilar, J. E. Nápoles Valdés, et al. *Analysis of the local Drude model involving the generalized fractional derivative*, Optik, **193** (2019), 163008.
16. N. Sene, J. F. Gómez Aguilar, *Analytical solutions of electrical circuits considering certain generalized fractional derivatives*, Eur. Phys. J. Plus, **134** (2019), 260.
17. S. R. Grace, R. P. Agarwal, P. J. Y. Wong, et al. *On the oscillation of fractional differential equations*, Fract. Calc. Appl. Anal., **15** (2012), 222–231.
18. D. X. Chen, P. X. Qu, Y. H. Lan, *Forced oscillation of certain fractional differential equations*, Adv. Differ. Equ., **2013** (2013), 125.
19. J. Alzabut, T. Abdeljawad, *Sufficient conditions for the oscillation of nonlinear fractional difference equations*, J. Fract. Calc. Appl., **5** (2014), 177–178.
20. B. Abdalla, K. Abodayeh, T. Abdeljawad, et al. *New oscillation criteria for forced nonlinear fractional difference equations*, Vietnam J. Math., **45** (2017), 609-618.
21. J. Yang, A. Liu, T. Liu, *Forced oscillation of nonlinear fractional differential equations with damping term*, Adv. Differ. Equ., **2015** (2015), 1.
22. E. Tunc, O. Tunc, *On the oscillation of a class of damped fractional difference equations*, Miskolc Math. Notes, **17** (2016), 647–656.
23. B. Abdalla, *On the oscillation of q-fractional differential equations*, Adv. Differ. Equ., **2017** (2017), 254.
24. B. Abdalla, *Oscillation of differential equations in the frame of nonlocal fractional derivatives generated by conformable derivatives*, Adv. Differ. Equ., **2018** (2018), 107.

25. B. Abdalla, J. Alzabut, T. Abdeljawad, *On the oscillation of higher order fractional difference equations with mixed nonlinearities*, Hacettepe J. Math. Stat., **47** (2018), 207–217.
26. J. Alzabut, T. Abdeljawad, H. Aleabaiah, *Oscillation criteria for forced and damped nable fractional difference equations*, J. Comput. Anal. Appl., **24** (2018), 1387–1394.
27. S. L. Marian, M. R. Sagayaraj, A. G. M. Selvam, et al. *Oscillation of caputo like discrete fractional equations*, Int. J. Pure Appl. Math., **89** (2013), 667–677.
28. A. Kilicman, V. Sadhasivamy, M. Deepaz, et al. *Oscillatory behavior of three dimensional α-fractional delay differential systems*, Symmetry, **10** (2018), 769.
29. J. Shao, Z. Zheng, F. Meng, *Oscillation criteria for fractional differential equations with mixed nonlinearlities*, Adv. Differ. Equ., **2013** (2013), 323.
30. Q. H. Feng, F. W. Meng, *Oscillation of solutions to nonlinear forced fractional differential equations*, Electron. J. Differ. Equ., **2013** (2013), 169.
31. Z. I. Han, Y. G. Zhao, Y. Sun, et al. *Oscillation for a class of fractional differential equation*, Discrete Dyn. Nat. Soc., **2013** (2013), 390282.
32. D. Dawei, Z. Xiaoyun, C. Jinde, et al. *Bifurcation control of complex networks model via PD controller*, Neurocomputing, **175** (2016), 1–9.
33. G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, 2 Eds., Cambridge University Press, Cambridge, 1988.

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