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*Research article*

# Solution for fractional forced KdV equation using fractional natural decomposition method

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**Abstract:** The fractional natural decomposition method (FNDM) is employed in the present investigation to find the solution for fractional forced Korteweg-de Vries (FF-KdV) equation. Three distinct cases are chosen for each equation to validate and illustrate the effectiveness of the future technique. The behaviour for different values of Froude number ($F_r$) has been presented to assure the proficiency and reliability and of the considered method. Moreover, we captured the behaviour of the FNDM solution for distinct arbitrary order. The obtained results elucidate that, the considered method is very effective and easy to employ while analyse the behaviour of nonlinear fractional differential equations arising in connected areas of science and technology.

**Keywords:** forced KdV equation; fractional natural decomposition method; Caputo fractional derivative
**Mathematics Subject Classification:** 35A22, 35Q53, 35R11

## 1. Introduction

The concept of fractional calculus (FC) recently attracted the attention of mathematicians and scientists working in diverse areas even though it debuted over 324 years ago. From the last thirty years, it considered as a most essential tool to examine the complex as well as nonlinear phenomena, due to its auspicious properties such as nonlocality, hereditary, memory effect and analyticity. The concept of the generalized fractional has been established in order to the complexities associated to a processes to heterogeneities. More preciously, the diffusion mechanism and chaotic behaviours are effectively captured with the aid of differential and integral operators having fractional order. From last two decades, many young researchers begin to work on the fundamentals and applications of generalized fractional calculus due to the swift growth of software and mathematical algorithm.

The basic foundation for fractional calculus prescribed by many pioneers and their orientations for generalized calculus [1–6]. FC has been associated to practical projects and it has been employed to diverse areas where the interesting ans simulating consequences associated to time and hereditary properties [7–28]. The The solution for fractional differential equations describing these phenomena play a vibrant role in unfolding the physical nature of complex problems exist in real life.

Recently, the Korteweg-de Vries (KdV) equation plays an important role in describing the various phenomena [29–31]. In a two-dimensional channel flow, the impact of bottom configurations on the free-surface waves is investigated with the help of forced Korteweg-de Vries equation. The bottom topography plays a vital part in the study of shallow-water waves, and which can significantly evaluate the behaviour of wave motions [32,33]. Shallow water or long waves are the waves in water shallower than $1/20$ their actual wavelength. When the bottom configuration is more complex, the interplay between the bottom topography and solitary waves can demonstrate more stimulating dynamics of the free surface waves. When the rigid bottom of the channel has some obstacles and for an incompressible and inviscid fluid, the free surface waves of a two-dimensional channel flow have been studied [34,35]. Fluid flows over an obstacle, the forcing approximately with the KdV equation can portray the development of the free surface. The FKdV equation is very important while describing the nature sine Gordon equation as well as the nonlinear Schrödinger equation. Further, the proposed model has numerous applications in the connected branches of mathematics and physics. This equation is considered as an essential tool to study propagation of short laser pulses in optical fibers, atmosphere dynamics, geostrophic turbulence and the magnetohydrodynamic waves [36,37]. Particularly, it offers stimulating results associated with the physical problems such as acoustic waves on a crystal lattice, oceanic stratified flows encountering topographic obstacles, tsunami waves over obstacles, and shallow-water waves over rocks.

In this paper, we consider the forced KdV equation with the free water surface elevation measured $u(x,t)$ on critical flow over a hole from undisturbed water level and which introduce and nurtured by Wu in 1987 [38], and presented as follows:

$$\frac{1}{c}\frac{\partial u}{\partial t}+\left[(F_r-1)-\frac{3}{2}\frac{u(x,t)}{h}\right]\frac{\partial u}{\partial x}-\frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}=\frac{1}{2}\frac{\partial f(x)}{\partial x}, \tag{1.1}$$

where $F_r$ is Froude number and it also calls as the critical parameter, $h$ is the sea mean water depth, $f(x)$ is the external forcing term and define as $f(x)=\frac{p_a(x)}{\rho g}+b(x)$. Here, $\frac{p_a(x)}{\rho g}$ is the surface air pressure, and $b(x)$ is rigid bottom topography and is defined by $b(x)=-0.1e^{-\frac{x^n}{4}}-1$. The Froude number $(F_r)$ plays an important role in Eq (1.1), since its value elucidates the kind of critical flows over the localised obstacle. Specifically, for values greater than, equal, and less than 1 respectively represent the flow is considered as supercritical, transcritical and subcritical. In the rigid bottom topography $b(x)$, two different kinds of hole examined, namely for $n=2$ and $n=8$. These cases respectively signify the hole is expected an inverse of bell-shaped and the hole is more flattened at the bottom as well as wider. Authors in [39], considered the simplified above equation by eliminating surface air pressure and presented as follows

$$\frac{1}{c}\frac{\partial u}{\partial t}+\left[(F_r-1)-\frac{3}{2}\frac{u(x,t)}{h}\right]\frac{\partial u}{\partial x}-\frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}-\frac{1}{2}\frac{\partial b(x)}{\partial x}=0. \tag{1.2}$$

In the present investigation, we consider the fractional-order forced KdV (FF-KdV) equation by replacing the time derivative with arbitrary order derivative in order. Now, FF-KdV equation is cited as follows

$$D_t^{\alpha} u(x,t) + c\left(\left[(F_r - 1) - \frac{3}{2}\frac{u(x,t)}{h}\right]\frac{\partial u}{\partial x} - \frac{h^2}{6}\frac{\partial^3 u}{\partial x^3} - \frac{1}{2}\frac{\partial b(x)}{\partial x}\right) = 0, \quad 0 < \alpha \le 1, \tag{1.3}$$

where $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative and $\alpha$ is the arbitrary order.

Recently, many mathematicians and physicist are introduced and natured various advanced numerical as well as analytical schemes in order to find the solution and capture the its physical behaiour for diverse class of differential and integral equations having integer or fractional order describing the real world processes. Moreover, there have been numerous methods available in the literature among them Adomian decomposition method (ADM) is more magnetized method due to its efficiency and accuracy [40]. ADM has profitably and effectively employed to examine the problems arisen in science and technology without linearization and perturbation. But, for the purpose of computational work, ADM requires more time and huge computer memory. Hence, there is an certainty of the combination of this method with existing transform methods. To fulfill these requirements, Rawashdeh and Maitama introduced and nurtured the FNDM [41,42], and which is an mixture of the ADM with natural transform method (NTM). Since FNDM is a improved method of ADM, it will reduce vast computations and in addition it does not requires discretization, linearization or perturbation. Recently, due to efficacy and reliability of the projected scheme has been extremely considered by many authors to interpret results for various nonlinear problems [43–45]. The complex nonlinear differential equations can be examined with simple procedure due to the considered method offers us with extremely huge freedom to consider initial guess and equation type of linear sub-problems. The novelty of FNDM is that it provides a simple algorithm to find the solution and it defined by the Adomian polynomial, it offers the quick convergence in the achieved solution for the nonlinear portion. These polynomials are generalized to a Maclaurin series along with the arbitrary external parameter.

Due to numerous applications proposed model and also it plays an important role in describing various complex phenomena, many authors find and analysed the solution in numerically as well as analytically, for instance, authors in find the analytic solutions to proposed model [46], author in [47] present the some interesting result for the proposed model and considered model for waves generated by topography, authors in [39,48] find the approximated analytical solution by using the homotopy analysis method (HAM), authors in [49] investigated the considered problem and presented dynamics of trapped solitary waves, lines and pseudospectral solutions has been investigated by authors in [50].

## 2. Preliminaries

In this segment, we present the fundamental notion of FC and natural transform.

**Definition 1.** The Riemann-Liouville integral of a function $f(t) \in C_\delta\,(\delta \ge -1)$ having fractional order $(\alpha > 0)$ is presented as follows

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\vartheta)^{\mu-1} f(\vartheta)d\vartheta, \tag{2.1}$$

$$J^0 f(t) = f(t). \tag{2.2}$$

**Definition 2.** The Caputo fractional derivative of $f \in C_{-1}^{n}$ is presented as follows

$$D_t^{\alpha} f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\vartheta)^{n-\alpha-1} f^{(n)}(\vartheta)\, d\vartheta, & n-1 < \alpha < n, n \in \mathbb{N}. \end{cases} \tag{2.3}$$

**Definition 3.** The Mittag-Leffler type function with one-parameter is presented [51] as follows

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \ z \in \mathbb{C}. \tag{2.4}$$

**Definition 4.** The natural transform (NT) of $f(t)$ is symbolized by $\mathbb{N}[f(t)]$ for $t \in \mathbb{R}$ and presented [52] by

$$\mathbb{N}[f(t)] = R(s,\omega) = \int_{-\infty}^{\infty} e^{-st} f(\omega t)\, dt; \quad s, \omega \in (-\infty, \infty),$$

where $s$ and $\omega$ are the NT variables. Now, we present the NT as

$$\mathbb{N}[f(t)H(t)] = \mathbb{N}^{+}[f(t)] = R^{+}(s,\omega) = \int_{0}^{\infty} e^{-st} f(\omega t)\, dt; \ \ s, \omega \in (0, \infty) \ \ and \ t \in \mathbb{R}, \tag{2.5}$$

where $H(t)$ is symbolise the Heaviside function. Further, for $s = 1$, the Eq (2.5) signifies the Sumudu transform and for $\omega = 1$, Eq (2.5) is simplifies to the Laplace transform.

**Theorem 1** [52]: Let $R(s, \omega)$ be the natural transform of $f(t)$, then the NT $R_{\alpha}(s, \omega)$ of the Riemann-Liouville fractional derivative of $f(t)$ is symbolized by $D^{\alpha} f(t)$ and which is defined as

$$\mathbb{N}^{+}[D^{\alpha} f(t)] = R_{\alpha}(s, \omega) = \frac{s^{\alpha}}{\omega^{\alpha}} R(s,\omega) - \sum_{k=0}^{n-1} \frac{s^k}{\omega^{\alpha-k}} \left[D^{\alpha-k-1} f(t)\right]_{t=0}, \tag{2.6}$$

where $n$ be any positive integer and $\alpha$ is the order. Further $n-1 \le \alpha < n$.

**Theorem 2** [53]**:** Let $R(s, \omega)$ be the natural transform of $f(t)$, then the NT $R_{\alpha}(s, \omega)$ of the Caputo fractional derivative of $f(t)$ is symbolize by ${}^{c}D^{\alpha} f(t)$ and which is presented as

$$\mathbb{N}^{+}[{}^{c}D^{\alpha} f(t)] = R_{\alpha}^{c}(s, \omega) = \frac{s^{\alpha}}{\omega^{\alpha}} R(s,\omega) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{\omega^{\alpha-k}} \left[D^{k} f(t)\right]_{t=0}, \tag{2.7}$$

where $n$ be any positive integer and $\alpha$ is the order.

**Remark 1:** Some basic properties of the natural transform are defined as below:

1. $\mathbb{N}^{+}[1] = \frac{1}{s}$,
2. $\mathbb{N}^{+}[t^{\alpha}] = \frac{\Gamma(\alpha+1)\omega^{\alpha}}{s^{\alpha+1}}$,
3. $\mathbb{N}^{+}\left[f^{(n)}(t)\right] = \frac{s^n}{\omega^n} R(s, \omega) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} \frac{\Gamma(\alpha+1)\omega^{\alpha}}{s^{\alpha+1}}$.

## 3. Fundamental solution procedure of the proposed algorithm

In order to demonstrate the fundamental theory and solution procedure of FNDM [38,39], we consider

$$D_t^{\alpha} u(x,t) + Ru(x,\ t) + Fu(x,\ t) = h(x,t), \tag{3.1}$$

with initial condition

$$u(x,0) = g(x), \tag{3.2}$$

where $D^{\alpha}u(x,t)$ signifies the fractional Caputo derivative of $u(x,t)$, $R$ and $F$ respectively are the linear and nonlinear differential operator, and $h(x,t)$ is the source term. On applying NT and by the assist of Theorem 2, then Eq (3.1) provides

$$\begin{aligned} U(x,s,\omega) &= \frac{u^{\alpha}}{s^{\alpha}} \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{\omega^{\alpha-k}} \left[D^k u(x,t)\right]_{t=0} + \frac{\omega^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}[h(x,t)] \\ &- \frac{\omega^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}[R\,u(x,t) + Fu(x,t)]. \end{aligned} \tag{3.3}$$

Apply the inverse NT on Eq (3.3) to get

$$u(x,t) = G(x,t) - \mathbb{N}^{-1}\left[\frac{\omega^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}[Ru(x,\ t) + F\,u(x,t)]\right]. \tag{3.4}$$

From non-homogeneous term and given initial condition, $G(x,t)$ is exists. The infinite series solution is defined as follows

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \qquad F\,u(x,t) = \sum_{n=0}^{\infty} A_n, \tag{3.5}$$

where the $A_n$is indicating the nonlinear term of $Fu(x,\ t)$. By using the Eqs (3.4) and (3.5), we have

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) - \mathbb{N}^{-1}\left[\frac{\omega^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}\left[R \sum_{n=0}^{\infty} u_n(x,t)\right] + \sum_{n=0}^{\infty} A_n\right]. \tag{3.6}$$

By comparing both sides of Eq (3.6), we obtain

$$\begin{aligned} u_0(x,t) &= G(x,t), \\ u_1(x,t) &= -\mathbb{N}^{-1}\left[\frac{\omega^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}[Ru_0(x,t)] + A_0\right], \\ u_2(x,t) &= -\mathbb{N}^{-1}\left[\frac{\omega^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}[Ru_1(x,t)] + A_1\right], \\ &\vdots \end{aligned}$$

Similarly, we can obtain the recursive relation in general form for $n \geq 1$ and defined as

$$u_{n+1}(x,t) = -\mathbb{N}^{-1}\left[\frac{\omega^{\alpha}}{s^{\alpha}} \mathbb{N}^{+}[Ru_n(x,t)] + A_n\right]. \tag{3.7}$$

Lastly, the approximate solution is defined as follows

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

## 4. Solution for fractional forced KdV equation

Here, we consider the nonlinear fractional forced KdV equation in ordered to elucidate the applicability and efficiency of the proposed algorithm. Consider the FF-KdV equation defined in Eq (1.3) with $b(x)$ value presented in Section 1 [39]

$$D_t^{\alpha} u(x,t) + c\left(\left((F_r - 1) - \frac{3}{2}\frac{u(x,t)}{h}\right)\frac{\partial u(x,t)}{\partial t} - \frac{1}{6}h^2\frac{\partial^3 u(x,t)}{\partial t^3} + \frac{1}{2}\frac{\partial}{\partial t}\left(0.1e^{-\frac{x^n}{4}} + 1\right)\right) = 0,\ 0 < \alpha \leq 1, \tag{4.1}$$

associated with initial condition

$$u(x,0) = -\frac{2e^x}{(1+e^x)^2}. \tag{4.2}$$

By employing NT on Eq (4.1), we have

$$\mathbb{N}^+\left[D_t^{\alpha} u(x,t)\right] = -c\mathbb{N}^+\left[\left((F_r - 1) - \frac{3}{2}\frac{u}{h}\right)\frac{\partial u}{\partial t} - \frac{1}{6}h^2\frac{\partial^3 u}{\partial t^3} + \frac{1}{2}\frac{\partial}{\partial t}\left(0.1e^{-\frac{x^n}{4}} + 1\right)\right]. \tag{4.3}$$

The non-linear operator is defined as

$$\begin{aligned}\frac{s^{\alpha}}{\omega^{\alpha}}\mathbb{N}^+\left[u(x,t)\right] - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{\omega^{\alpha-k}}\left[D^k u\right]_{t=0} &= -c\mathbb{N}^+[\left((F_r - 1) - \frac{3}{2}\frac{u(x,t)}{h}\right)\frac{\partial u(x,t)}{\partial t} \\ &- \frac{1}{6}h^2\frac{\partial^3 u(x,t)}{\partial t^3} + \frac{1}{2}\frac{\partial}{\partial t}\left(0.1e^{-\frac{x^n}{4}} + 1\right)].\end{aligned} \tag{4.4}$$

By the above equation, we get

$$\begin{aligned}\mathbb{N}^+\left[u(x,t)\right] &= \frac{1}{s}\left[-\frac{2e^x}{(1+e^x)^2}\right] - \frac{c\omega^{\alpha}}{s^{\alpha}}\mathbb{N}^+[\left((F_r - 1) - \frac{3}{2}\frac{u(x,t)}{h}\right)\frac{\partial u(x,t)}{\partial t} \\ &- \frac{1}{6}h^2\frac{\partial^3 u(x,t)}{\partial t^3} + \frac{1}{2}\frac{\partial}{\partial t}\left(0.1e^{-\frac{x^n}{4}} + 1\right)].\end{aligned} \tag{4.5}$$

On employing inverse NT on Eq (4.5), we have

$$\begin{aligned}u(x,t) &= -\frac{2e^x}{(1+e^x)^2} - c\mathbb{N}^{-1}[\frac{\omega^{\alpha}}{s^{\alpha}}\mathbb{N}^+[\left((F_r - 1) - \frac{3}{2}\frac{u(x,t)}{h}\right)\frac{\partial u(x,t)}{\partial t} - \frac{1}{6}h^2\frac{\partial^3 u(x,t)}{\partial t^3} \\ &+ \frac{1}{2}\frac{\partial}{\partial t}\left(0.1e^{-\frac{x^n}{4}} + 1\right)]].\end{aligned} \tag{4.6}$$

Let us consider that, the series solution for $u(x,t)$ is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Note that, $uu_x = \sum_{n=0}^{\infty} A_n$ be the nonlinear term and it is known as the Adomian polynomial. By the help of this term, the Eq (4.6) becomes

$$\sum_{n=0}^{\infty} u_n(x,t) = -\frac{2e^x}{(1+e^x)^2} - c\mathbb{N}^{-1}[\frac{\omega^{\alpha}}{s^{\alpha}}\mathbb{N}^+[\left((F_r - 1) - \frac{3}{2}\frac{u(x,t)}{h}\right)\frac{\partial u(x,t)}{\partial t}$$

$$
- \quad \frac{1}{6}h^2\frac{\partial^3 u(x,t)}{\partial t^3} + \frac{1}{2}\frac{\partial}{\partial t}\left(0.1e^{-\frac{x^n}{4}} + 1\right)]]. \tag{4.7}
$$

We can generate the recursive relation by comparing both sides of the above equation and which are presented as follows

$$
\begin{aligned}
u_0(x,t) &= -\frac{2e^x}{(1+e^x)^2},\\
u_1(x,t) &= \frac{-t^\alpha}{\Gamma[\alpha+1]}(c(\frac{6\mathrm{e}^{2x}(-1+\mathrm{e}^x)}{(1+\mathrm{e}^x)^5 h} - \frac{\mathrm{e}^x\left(-1+11\mathrm{e}^x-11\mathrm{e}^{2x}+\mathrm{e}^{3x}\right)h^2}{3(1+\mathrm{e}^x)^5} - 0.0125\mathrm{e}^{-\frac{x^n}{4}}nx^{-1+n}\\
&+ \frac{2\mathrm{e}^x(-1+\mathrm{e}^x)(-1+F_r)}{(1+\mathrm{e}^x)^3})),\\
&\vdots
\end{aligned}
$$

Similarly, the remaining terms can be obtained with the aid of FNDM. Accordingly, we obtained the series solutions as

$$
\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots\\
&= -\frac{2e^x}{(1+e^x)^2} - \frac{t^\alpha}{\Gamma[\alpha+1]}(c(\frac{6\mathrm{e}^{2x}(-1+\mathrm{e}^x)}{(1+\mathrm{e}^x)^5 h} - \frac{\mathrm{e}^x\left(-1+11\mathrm{e}^x-11\mathrm{e}^{2x}+\mathrm{e}^{3x}\right)h^2}{3(1+\mathrm{e}^x)^5} - 0.025\mathrm{e}^{-\frac{x^2}{4}}x\\
&+ \frac{2\mathrm{e}^x(-1+\mathrm{e}^x)(-1+F_r)}{(1+\mathrm{e}^x)^3})) + \ldots
\end{aligned}
$$

## 5. Numerical results and discussions

In the present investigation, we consider three special cases ($n = 2, 4$ and 8) for the proposed model. Also, we consider constant wave speed $c \approx \sqrt{g \times h} = \sqrt{9.8}$ with a mean water depth of the sea $h = 1$. For $n = 2$, the nature of FNDM solution for FF-KdV equation with different fractional-order is captured in Figures 1 and 2. From these plots, we can see the small variation in the nature of the obtained solution for the different values of fractional order. In 3D plots, it is very difficult to see the small changes of behaviour of the obtained solution, and hence we capture and presented in the 2D plot. The behaviour of the solution for the proposed model obtained with the aid of FNDM for Froude number ($F_r$) is presented in Figure 3. In order to present more interesting consequences of the considered model, we present the behaviour of the FNDM solution for the second case ($n = 4$) in Figures 4 and 5 for different fractional order. In the same manner, for $n = 8$ surfaces for the obtained solution with distinct $F_r$ is cited in Figure 6. The response of FNDM solution for FF-KdV equation with distinct $\alpha$ is dissipated in Figure 7. We can observe from Figure 3, for distinct Froude number the obtained solution coincides at $x = 0$. From Figures 1, 2, 4, 5 and 7 we can see that by incorporating the fractional derivative in the proposed model we have some interesting behaviour. These behaviours may help the researchers in order to understand the new properties of the considered model.

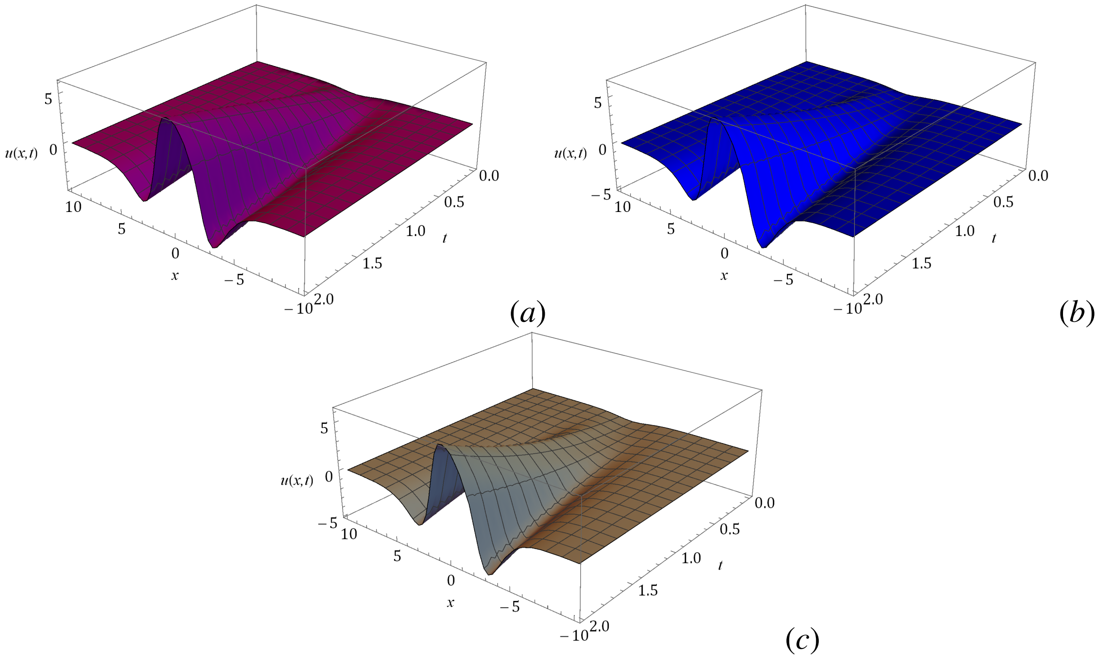


**Figure 1.** Surfaces of FNDM solution for (a) $\alpha = 0.50$, (b) $\alpha = 0.75$, (c) $\alpha = 1$ at $n = 2$ and $F_r = -1$.

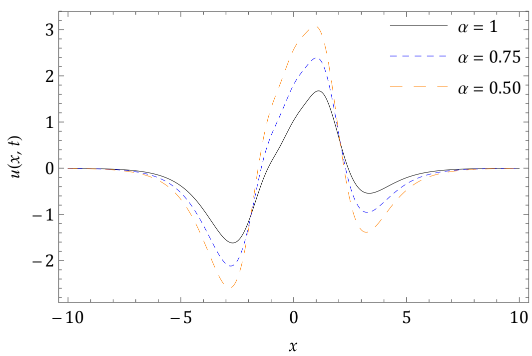


**Figure 2.** Response of obtained solution with distinct $\alpha$ at $n = 2$ and $F_r = -1$.

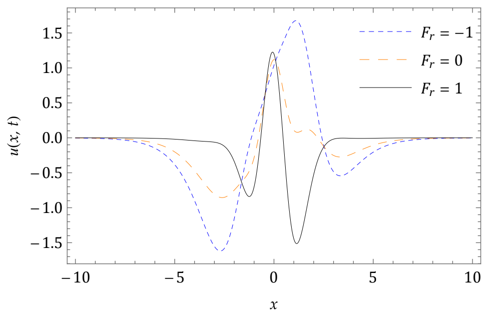


**Figure 3.** Nature of FNDM solution with distinct Froude number at $n = 2$ and $\alpha = 1$.

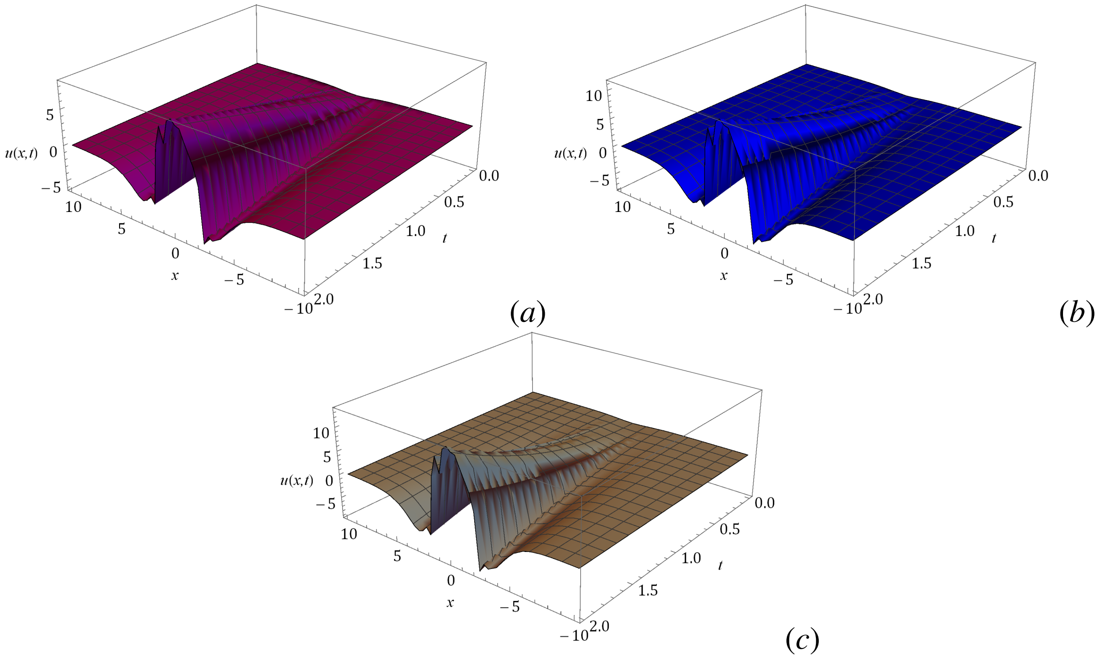


**Figure 4.** Surfaces of FNDM solution for (a) $\alpha = 0.50$, (b) $\alpha = 0.75$, (c) $\alpha = 1$ at $n = 4$ and $F_r = -1$.

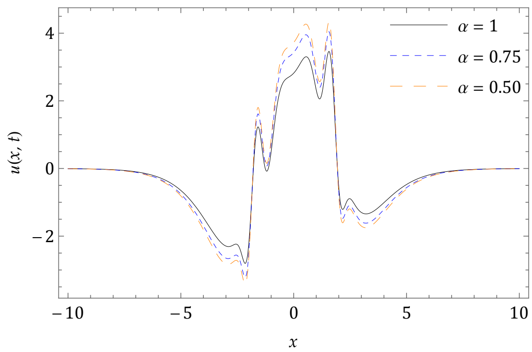


**Figure 5.** Response of obtained solution with distinct $\alpha$ at $n = 4$ and $F_r = -1$.

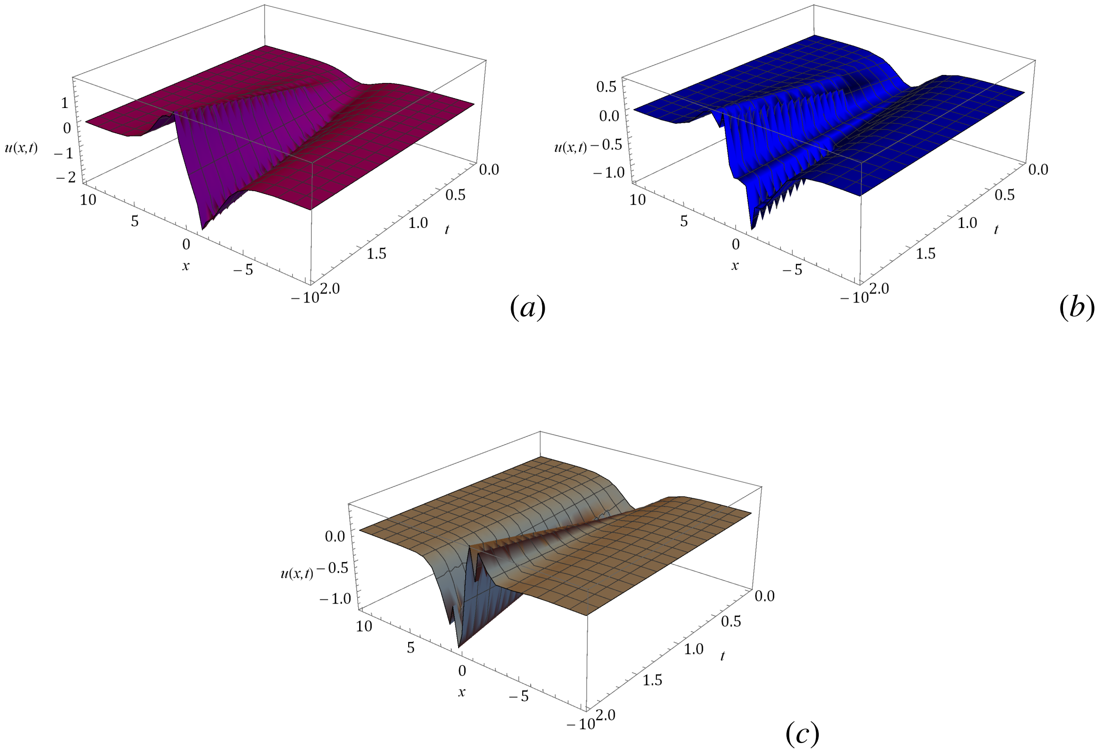


**Figure 6.** Surfaces of FNDM solution for (a)$F_r = -1$, (b) $F_r = 0$, (c) $F_r = 1$ at $n = 8$ and $\alpha = 1$.

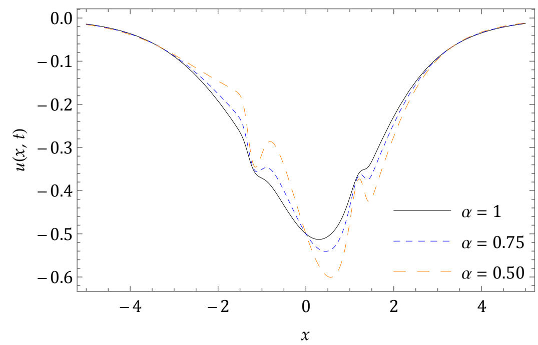


**Figure 7.** Response of obtained solution with distinct $\alpha$ at $n = 8$ and $F_r = -1$.

## 6. Conclusions

In the present framework, the FNDM is lucratively employed to find the numerical solution for fractional forced Korteweg-de Vries equation. The results achieved by the proposed scheme are interesting as compared to results achieved by traditional techniques. It is worth revealing that, in the future method, the solution for nonlinear problems can be obtained without making any discretization or transformations. The present study shows that, the projected model is highly depends on the the time instant and time history, which can effectively illustrated by the help of fractional calculus.

Finally, we can conclude the considered method is more accurate and highly effective, and it can be employed to investigate the different classes of nonlinear problems arisen in real life.

## Conflict of interest

All authors declare no conflicts of interest.

## References


1. J. Liouville, *Memoire surquelques questions de geometrieet de mecanique, et sur un nouveau genre de calcul pour resoudreces questions*, J. Ec. Polytech., **13** (1832), 1–69.
2. G. F. B. Riemann, *Auffassung der Integration und Differentiation*, Gesammelte Mathematische Werke, Leipzig, 1896.
3. M. Caputo, *Elasticita e Dissipazione*, Zanichelli, Bologna, 1969.
4. K. S. Miller, B. Ross, *An introduction to fractional calculus and fractional differential equations*, Wiley, New York, 1993.
5. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
6. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.
7. D. Baleanu, Z. B. Guvenc, J. A. T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer Dordrecht Heidelberg, London and New York, 2010.
8. N. H. Sweilam, M. M. A. Hasan, D. Baleanu, *New studies for general fractional financial models of awareness and trial advertising decisions*, Chaos, Solitons Fractals, **104** (2017), 772–784.
9. D. Baleanu, G. C. Wu, S. D. Zeng, *Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations*, Chaos, Solitons Fractals, **102** (2017), 99–105.
10. P. Veeresha, D. G. Prakasha, H. M. Baskonus, *New numerical surfaces to the mathematical model of cancer chemotherapy effect in Caputo fractional derivatives*, Chaos, **29** (2019), 013119.
11. W. Gao, P. Veeresha, D. G. Prakasha, H. M. Baskonus, *A powerful approach for fractional Drinfeld–Sokolov–Wilson equation with Mittag-Leffler law*, Alexandria Eng. J., (2019). DOI: 10.1016/j.aej.2019.11.002.
12. E. F. D. Goufo, *Application of the Caputo-Fabrizio Fractional derivative without singular kernel to Korteweg-de Vries-Burgers equation*, Math. Model. Numer. Anal., **21** (2016), 188-198.
13. P. Veeresha, D. G. Prakasha, H. M. Baskonus, *Novel simulations to the time-fractional Fisher's equation*, Math. Sci., **13** (2019), 33–42.
14. A. Prakash, P. Veeresha, D. G. Prakasha, *A homotopy technique for fractional order multi-dimensional telegraph equation via Laplace transform*, Eur. Phys. J. Plus, **134** (2019), 19.
15. B. Li, Y. Chen, H. Zhang, *Explicit exact solutions for compound KdV-type and compound KdV–Burgers-type equations with nonlinear terms of any order*, Chaos, Solitons Fractals, **15** (2003), 647–654.
16. E. F. D. Goufo, H.M. Tenkam, M. Khumalo, *A behavioral analysis of KdVB equation under the law of Mittag–Leffler function*, Chaos, Solitons Fractals, **125** (2019), 139–145.

17. P. Veeresha, D. G. Prakasha, H. M. Baskonus, *Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method*, Math. Sci., **13** (2019), 115–128.
18. A. Prakash, P. Veeresha, D. G. Prakasha, *A homotopy technique for fractional order multi-dimensional telegraph equation via Laplace transform*, Eur. Phys. J. Plus, **134** (2019), 1–18,
19. P. Dubard, P. Gaillard, C. Kleina, *On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation*, Eur. Phys. J. Special Topics, **185** (2010), 247–258.
20. P. Veeresha, D. G. Prakasha, M. A. Qurashi, *A reliable technique for fractional modified Boussinesq and approximate long wave equations*, Differ Equ, **2019** (2019), 253.
21. E. F. D. Goufo, A. Atangana, *Dynamics of traveling waves of variable order hyperbolic Liouville equation, Regulation and control*, Discrete Contin. Dyn. Syst. Ser. S, **13** (2020), 645–662.
22. P. Veeresha, D. G. Prakasha, D. Kumar, *An efficient technique for nonlinear time-fractional Klein-Fock-Gordon equation*, Appl. Math. Comput., **364** (2020), 124637.
23. D. G. Prakasha, P. Veeresha, J. Singh, *Fractional approach for equation describing the water transport in unsaturated porous media with Mittag-Leffler kernel*, Front. Phys., **7** (2019), 193.
24. E. F. D. Goufo, I. T. Toudjeu, *Around chaotic disturbance and irregularity for higher order traveling waves*, J. Math., **2018** (2018), 1–11.
25. P. Veeresha, D. G. Prakasha, *A novel technique for* (2 + 1)*-dimensional time-fractional coupled Burgers equations*, Math. Comput. Simulation, **116** (2019), 324–345.
26. E. F. D. Goufo, A. Atangana, *Extension of fragmentation process in a kinetic-diffusive-wave system*, Thermal Science, **19** (2015), 13–23.
27. P. Veeresha, D. G. Prakasha, *Solution for fractional Zakharov-Kuznetsov equations by using two reliable techniques*, Chinese J. Phys., **60** (2019), 313–330.
28. K. B. Liaskos, A. A. Pantelous, I. A. Kougioumtzoglou, *Implicit analytic solutions for the linear stochastic partial differential beam equation with fractional derivative terms*, Systems Control Lett., **121** (2018), 38–49.
29. A. Jeffrey, M. N. B. Mohamad, *Exact solutions to the KdV-Burgers' equation*, Wave Motion, **14** (1991), 369-375.
30. E. F. D. Goufo, S. Kumar, *Shallow Water Wave Models with and without Singular Kernel: Existence, Uniqueness, and Similarities*, Math. Probl. Eng., **2017** (2017), 1–9.
31. W. Zhang, Q. Chang, B. Jiang, *Explicit exact solitary-wave solutions for compound KdV-type and compound KdV–Burgers-type equations with nonlinear terms of any order*, Chaos, Solitons Fractals, **13** (2002), 311–319.
32. F. Dias, J. M. Vanden-Broeck, *Generalized critical free-surface flows*, J. Eng. Math., **42** (2002), 291–301.
33. S. S. Shen, *On the accuracy of the stationary forced Korteweg-De Vries equation as a model equation for flows over a bump*, Q. Appl. Math., **53** (1995), 701–719.
34. R. Camassa, T. Wu, *Stability of forced solitary waves*, Philos. Trans. R. Soc. Lond. A, **337** (1991), 429–466.
35. N. J. Zabuski, M. D. Kruskal, *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett., **15** (1965), 240–243.
36. D. G. Crighton, *Applications of KdV*, Acta Appl. Math., **15** (1995), 39–67.

37. W. Hereman, *Shallow Water Waves and Solitary Waves*, In: R. Meyers, Ed., *Mathematics of Complexity and Dynamical Systems*, Springer, New York, USA, 2012.
38. T. Y. T. Wu, *Generation of upstream advancing solitons by moving disturbances*, J. Fluid Mech., **184** (1987), 75–99.
39. V. D. David, Z. A. Aziz, F. Salah, *Analytical approximate solution for the forced Korteweg-de Vries (FKdV) on critical flow over a hole using homotopy analysis method*, Journal Teknologi (Sciences & Engineering), **78** (2016), 107–112.
40. G. Adomian, *A new approach to nonlinear partial differential equations*, J. Math. Anal. Appl., **102** (1984), 420–434.
41. M. S. Rawashdeh, H. Al-Jammal, *New approximate solutions to fractional nonlinear systems of partial differential equations using the FNDM*, Advances in Difference Equations, **235** (2016), 1–19,
42. M. S. Rawashdeh, H. Al-Jammal, *Numerical solutions for system of nonlinear fractional ordinary differential equations using the FNDM*, Mediterr. J. Math., **13** (2016), 4661–4677.
43. M. S. Rawashdeh, *The fractional natural decomposition method: theories and applications*, Math. Meth. Appl. Sci., **40** (2017), 2362–2376.
44. D. G. Prakasha, P. Veeresha, M. S. Rawashdeh, *Numerical solution for (2+1)-dimensional time-fractional coupled Burger equations using fractional natural decomposition method*, Math. Meth. Appl. Sci., **42** (2019), 3409–3427.
45. D. G. Prakasha, P. Veeresha, H. M. Baskonus, *Two novel computational techniques for fractional Gardner and Cahn-Hilliard equations*, Comp. and Math. Methods, **1** (2019), 1–19.
46. Z. J. Xiao, G. B. Ling, *Analytic solutions to forced KdV equation*, Commun. Theor. Phys. **52** (2009), 279–283.
47. P. A. Milewski, *The forced Korteweg-De Vries equation as a model for waves generated by topography*, Cubo, **6** (2014), 33–51.
48. V. D. David, F. Salah, M. Nazari, *Approximate analytical solution for the forced Korteweg-de Vries equation*, J. Appl. Math., **2013** (2013), 1–9.
49. S. Lee, *Dynamics of trapped solitary waves for the forced KdV equation*, Symmetry, **10** (2018), 1–13.
50. K. G. Tay, W. K. Tiong, Y. Y. Choy, *Method of lines and pseudospectral solutions of the forced Korteweg-De Vries equation with variable coefficients arises in elastic tube*, International Journal of Pure and Applied Mathematics, **116** (2017), 985–999.
51. G. M. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$ , C. R. Acad. Sci. Paris, **137** (1903), 554–558.
52. Z. H. Khan, W. A. Khan, *N-Transform - Properties and Applications*, NUST J. Engg. Sci., **1** (2008), 127–133.
53. D. Loonker, P. K. Banerji, *Solution of fractional ordinary differential equations by natural transform*, Int. J. Math. Eng. Sci., **12** (2013), 1–7.