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Research article

Blow-up criterion for incompressible nematic type liquid crystal equations in three-dimensional space

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Abstract: In this paper, we consider two incompressible nematic type liquid crystal models in three-dimensional space. Blow-up criterions for weak and smooth solutions are established in homogenous and nonhomogenous Besov space with negative regular index, respectively. As a result, we improve some previous results in Besov space.

Keywords: blow-up criterion; incompressible; liquid crystal equations; Besov space
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1. Introduction

Liquid crystal is classified as an intermediate state of matter between the crystalline solid and the isotropic fluid state. It flows like a liquid and their molecules are oriented in a crystal like structure, showing kinematic behavior. Generally, liquid crystals are categorized as nematic, smectic and cholesteric forms. The nematic form is the most common in hydrodynamic theory of liquid crystal. The simplest model to study nematic liquid crystals equilibrium phenomena was introduced by Oseen [34] in 1933 and Frank [11] in 1958. A few years later, Ericksen and Leslie proposed conservation laws and the hydrodynamics theory of nematic liquid crystals, see [7, 8, 19].

The evolution of liquid crystal material flow under the influence of velocity field and director field is governed by the set of following partial differential equations:

$$\begin{cases} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = \nabla\cdot(\nabla d \odot \nabla d), \\ d_t - \Delta d + u\cdot\nabla d = |\nabla d|^2 d, \\ \nabla\cdot u = 0, \quad |d|^2 = 1, \\ u(x,0) = u_0(x), \quad d(x,0) = d_0(x), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^3\times[0,\infty) \to \mathbb{R}^3$ denotes the velocity vector field, $p : \mathbb{R}^3\times[0,\infty) \to \mathbb{R}$ is the scalar pressure field, $d : \mathbb{R}^3\times[0,\infty) \to \mathbb{S}^2$ represents kinematic variable or director field, ν is kinematic viscosity respectively. u_0, d_0 are the initial data and u_0 satisfies divergence free condition $\nabla\cdot u_0 = 0$. The notation $\nabla d\odot\nabla d$ is 3×3 matrix with (i, j)-*th* entries, given by $\partial_i d\cdot\partial_j d\,(1 \le i, j \le 3)$.

The system (1.1) is the reduced form of general Ericksen-Leslie system, proposed by Lin [21] in 1989. The local existence of solutions can be found in [24, 40]. For the global well-posedness, Wang [38] proved the global-in-time existence of strong solutions in the whole space provided that the initial data are suitably small in BMO space. In 2012, the initial-boundary value problem is considered by Li and Wang [20] in a bounded smooth domain, where the existence and uniqueness are established for both the local strong solution with large initial data and the global strong solution with small initial data. In 2013, Hineman and Wang [14] studied the Cauchy problem and showed that there exists a unique, global smooth solution with small initial data $\|(u_0, \nabla d_0)\|_{L^3(\mathbb{R}^3)}$. Moreover, the global solution has monotone decreasing L^3-energy for $t \ge 0$. In 2015, Liu and Xu [32], by using Fourier splitting technique and energy method, established the global well-posedness and time decay rates of the classical solutions with smooth initial data which are of small energy. In the same year, Liu et al. [28] established the global well-posedness with initial data in critical homogeneous Besov space $B^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)\times B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)$ and the vertical component of the initial velocity u_0^3 may be large, which are further discussed in [26] and the temporal decay estimates in Besov space are also proved. Recently, Huang et al. [16] obtained optimal time-decay rates in $L^r(\mathbb{R}^3_+)$ for $r \ge 1$ of global strong solutions to the nematic liquid crystal flows in $\mathbb{R}^3_+$, provided the initial data has small $L^3(\mathbb{R}^3_+)$-norm. For more results on the asymptotic behavior of solutions, we refer to [5, 6] and the references therein.

However, the system (1.1) can be viewed as Navier-Stokes equations coupling the heat flow of a harmonic map and the strong solutions of a harmonic map must be blow up at finite time [3], we cannot expect that (1.1) has a global strong solution with general initial data. Therefore, it is important to study the mechanism of blow-up for strong or smooth solutions. First, we review some previous results which related to our main results in this paper. In 2012, Huang and Wang [17] proved the following BKM criterion

$$\nabla\times u \in L^1\left(0,T; L^\infty(\mathbb{R}^3)\right), \quad \nabla d \in L^2\left(0,T; L^\infty(\mathbb{R}^3)\right). \tag{1.2}$$

In 2013, Liu and Zhao [29] established the blow-up criterion in Besov space

$$u \in L^\infty\left(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right), \quad \nabla d \in L^\infty\left(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right), \tag{1.3}$$

which is improved in [30] by giving the logarithmically blow-up criterion

$$\int_0^T \frac{\|\nabla u\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^{0}_{\infty,\infty}}}{\ln\left(e + \|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^{0}_{\infty,\infty}}\right)} dt < +\infty, \tag{1.4}$$

In 2017, BKM criterion in Besov spaces of negative regular index is given by Yuan and Wei in [42], the authors proved that if

$$\omega \in L^{\frac{2}{2-r}}\left(0, T; \dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)\right), \quad \nabla d \in L^2\left(0, T; \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3)\right), \tag{1.5}$$

then the solution remains smooth after time T, where $\omega = \nabla \times u$ and $r \in (0, 2)$. In the same year, Zhao [46] proved the following blow-up criterion in terms of the horizontal gradient of two horizontal velocity components and the gradient of liquid crystal molecular orientation field

$$\int_0^T \left(\|\nabla_h u^h\|^q_{\dot{B}^0_{p,\frac{2p}{3}}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}\right) dt < +\infty, \quad \text{with} \quad \frac{3}{p} + \frac{2}{q} = 2, \quad \frac{3}{2} < p \leq \infty. \tag{1.6}$$

where $u^h = (u_1, u_2)$ and $\nabla_h = (\partial_1, \partial_2)$. For the blow-up criterion in terms of the pressure, we refer to [25, 27], and the references therein.

Because of nonlinearly, the term $|\nabla d|^2 d$ in Eq. $(1.1)_2$ makes the system more complex. For the simplification of the model, the term $|\nabla d|^2 d$ can be replaced by Ginzburg-Landau function $f(d)$ in Ericksen terminology, see [7, 8]. In particular, Dirichlet energy for director field, $d : \mathbb{R}^3 \times (0, \infty) \to \mathbb{S}^2$

$$E(d) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla d|^2 dx,$$

is replaced by Ginzburg-Landau energy for $d : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$

$$E_\epsilon(d) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla d|^2 + \frac{1}{4\epsilon^2}(1 - |d|^2)^2 dx, \quad \epsilon > 0.$$

Then, the reduced system with Ginzburg-Landau approximation is written for $(u, d) : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3 \times \mathbb{R}^3$ as follows:

$$\begin{cases} u_t - \nu\Delta u + u \cdot \nabla u + \nabla p = \nabla \cdot (\nabla d \odot \nabla d), \\ d_t - \Delta d + u \cdot \nabla d = -f(d), \\ \nabla \cdot u = 0, \quad |d| \leq 1, \\ u(x, 0) = u_0(x), \quad d(x, 0) = d_0(x). \end{cases} \tag{1.7}$$

whereas $f(d)$ is defined as $f(d) = \frac{1}{\epsilon^2}(|d|^2 - 1)d$ for some positive constant $\epsilon > 0$.

Next, let us recall some well-posedness results about Ginzburg-Landau approximation system (1.7). The mathematical analysis was first initiated by Lin and Liu [22] in 1995, they proved the global existence of weak solution and local in time smooth solution. Moreover, they also proved the existence and uniqueness of global classical solution when viscosity is large. Later on, Lin and Liu [23] established the partial regularity of suitable weak solutions, which is a natural generalization of an earlier work of Caffarelli-Kohn-Nirenberg on the Navier-Stokes system. In 2001, Coutand and Shkoller [4] showed the local well-posedness of initial-boundary value problem for any regular initial data. Moreover, they also gave sufficient conditions for the global existence of the solution and some stability conditions additionally. In 2010, Hu and Wang [15] proved the existence and uniqueness of global strong solution with smallness assumption on initial data in bounded domain. In 2013, Zhao et al. [47] proved that when initial data belongs to the critical Besov spaces with negative order, there exists a unique local solution, and the solution is globally in time when initial data is small enough. For more results on the global well-posedness of system (1.7), we refer to [36, 37] and the references therein.

At last, we review some known results about blow-up criterion of the system (1.7). In 2009, Guillén-González et al. [12] proved the following two kinds of blow-up criterion

$$u \in L^{\frac{2p}{p-3}}\left(0, T; L^p(\mathbb{R}^3)\right), \quad 3 < p \leq \infty, \quad \text{or} \quad \nabla u \in L^{\frac{2q}{q-3}}\left(0, T; L^q(\mathbb{R}^3)\right), \quad \frac{3}{2} < q \leq \infty. \tag{1.8}$$

and blow-up criterion for d

$$\nabla d \in L^{\frac{2p}{p-3}}\left(0, T; L^p(\mathbb{R}^3)\right), \quad 3 < p \leq \infty, \quad \text{or} \quad \Delta d \in L^{\frac{2q}{q-3}}\left(0, T; L^q(\mathbb{R}^3)\right), \quad \frac{3}{2} < q \leq \infty. \tag{1.9}$$

Later, Fan and Ozawa [9] improved these results in homogenous Besov spaces by showing that smoothness of solution beyond T implies

$$u \in L^2\left(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)\right), \quad \text{or} \quad \nabla u \in L^1\left(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)\right), \tag{1.10}$$

which are further refined by

$$\int_0^T \frac{\|u\|^{\frac{2}{1-s}}_{\dot{B}^{-s}_{\infty,\infty}}}{1 + \ln\left(e + \|u\|_{\dot{B}^{-s}_{\infty,\infty}}\right)} dt < +\infty, \quad 0 < s < 1, \tag{1.11}$$

in paper [10]. In 2014, Zhang [44] gave the following Osgood type regularity criterion for liquid crystal flow if

$$\sup_{2 \leq q < \infty} \int_0^T \frac{\|\bar{S_q} \nabla u(t)\|_{L^\infty}}{q \ln q} dt < +\infty,$$

then the smooth solution can be extended beyond time T, where $\bar{S_q} = \sum_{k=-q}^{q} \dot{\Delta}_k$, $\dot{\Delta}_k$ denotes the frequency localization operator. Recently, Zhang [43] also proved that if

$$u \in L^{\frac{2}{1+s}}\left(0, T; \dot{B}^s_{\infty,\infty}(\mathbb{R}^3)\right), \quad 0 < s < 1, \tag{1.12}$$

then the solution remains smooth after time T. For the blow-up criterion in terms of one direction of the velocity, we refer to [31, 39, 48, 49] and references therein.

Motivated by the above mentioned results, in this paper we study the blow-up criterion for Ericksen-Leslie system (1.1) and Ericksen-Leslie system with Ginzburg-Landau approximation system (1.7) in both homogenous and nonhomogenous Besov space. In [29] and [42], the authors proved the blow-up criterions in Besov space with negative regular index. However, there is no evidence shows that the inequality (2.4) in Lemma 2.3 is still valid in nonhomogenous Besov space at this moment. Hence, the methods in the references [29, 42] can not be used in nonhomogenous Besov space directly when we consider the regular index is negative. Inspired by work [33], in this paper, we first introduce the Besov type space V_Θ with $\|\cdot\|_{V_\Theta}$-norm to establish the blow-up criterion of approximation system (1.7) in nonhomogenous Besov space. It is worth mentioning that when α belongs to $(0, 1)$, the norm $\|\cdot\|_{V_\Theta}$

is weaker than $B^{\alpha-1}_{\infty,\infty}$-norm. Moreover, $\|\cdot\|_{V_\Theta}$-norm can also be extended to homogenous Besov space, then the result in [44] can be improved. On the other hand, we consider Ericksen-Leslie system (1.1) and establish the blow-up criterion in terms of velocity field and director field, which extend the result in [45] to liquid crystal equations and improve the blow-up criterion (1.5) in [42].

The rest of this paper is organized as follows. In section 2, we introduce some preliminaries and state our main results for Ginzburg-Landau approximation equations. In section 3, we give the proofs of Theorem 2.1 in nonhomogeneous Besov space and Theorem 2.2 in homogenous Besov space, respectively. Finally, in section 4, we establish the blow-up criterion for Ericksen-Leslie model which extend some previous results.

2. Preliminaries and main results

Before presenting our results, we introduce some function spaces and some notations, see [1] and [35]. First, we are going to recall some basic facts on Littlewood-Paley theory. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(\xi)dx.$$

Choose two nonnegative radial functions χ and φ ,valued in the interval [0,1], supported in $B = \{\xi \in \mathbb{R}^3, |\xi| \leqslant \frac{4}{3}\}$, $C = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$, respectively, such that

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3,$$

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$. The nonhomogeneous dyadic blocks Δ_j are defined by

$$\Delta_j = 0 \quad \text{if} \quad j \leqslant -2, \quad \Delta_{-1}u = \chi(D)u = \int_{\mathbb{R}^3} \widetilde{h}(y)u(x-y)dy,$$

and

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} h(2^j y)u(x-y)dy \quad \text{if} \quad j \geqslant 0.$$

The nonhomogeneous low-frequency cut-off operator S_j is defined by

$$S_j u = \chi(2^{-j}D)u = \sum_{k\leqslant j-1} \Delta_k u.$$

The homogenous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} h(2^j y)u(x-y)dy,$$

$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} \widetilde{h}(2^j y)u(x-y)dy.$$

Formally, Δ_j is a frequency projection to the annulus$\{|\xi| \approx 2^j\}$, and S_j is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. Then, from Littlewood-Paley's decomposition implies that

$$u = \Delta_{-1}u + \sum_{j=0}^{\infty} \Delta_j u \quad \text{and} \quad u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u. \tag{2.1}$$

Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. The nonhomogenous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \{u \in \mathcal{S}'(\mathbb{R}^3); \|u\|_{B^s_{p,q}} < +\infty\},$$

where

$$\|u\|_{B^s_{p,q}} = \begin{cases} \Big(\sum_{j=-1}^{\infty} 2^{jsq}\|\Delta_j u\|^q_{L^p}\Big)^{\frac{1}{q}}, & \text{for} \quad q < +\infty, \\ \sup_{j\geqslant -1} 2^{js}\|\Delta_j u\|_{L^p}, & \text{for} \quad q = +\infty. \end{cases}$$

The homogenous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{u \in \mathcal{S}'_h(\mathbb{R}^3); \|u\|_{\dot{B}^s_{p,q}} < +\infty\},$$

where

$$\|u\|_{\dot{B}^s_{p,q}} = \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\dot{\Delta}_j u\|^q_{L^p}\Big)^{\frac{1}{q}}, & \text{for} \quad q < +\infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j u\|_{L^p}, & \text{for} \quad q = +\infty. \end{cases}$$

Next we introduce the modified nonhomogenous space of Besov type which is derived from the reference [33].

Definition 2.1. *We denote the space by V_Θ, it consists of all tempered distributions u such that $\{u \in \mathcal{S}'(\mathbb{R}^3); \|u\|_{V_\Theta} < \infty\}$ and it's norm defined by*

$$\|u\|_{V_\Theta} = \sup_{N\geq 2} \frac{\left\|\sum_{j=-1}^{N} \Delta_j u\right\|_\infty}{\Theta(N)},$$

where $\|\cdot\|_\infty$ denotes the $L^\infty(\mathbb{R}^3)$-norm, and Θ is a nondecreasing function on $[1, \infty)$.

Next, we present the well-known commutator estimate which will be used in the energy estimate. The details can be found in [18] for example.

Lemma 2.1. *Suppose that $s > 0$ and $p \in (1, \infty)$. Let f, g be two smooth functions such that $\nabla f \in L^{p_1}$, $\Lambda^s f \in L^{p_3}$, $\Lambda^{s-1}g \in L^{p_2}$ and $g \in L^{p_4}$, then there exist a constant C independent of f and g such that*

$$\|[\Lambda^s, f]g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \tag{2.2}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$, $p_2, p_3 \in (1, \infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

here $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$.

In order to prove our results, we will use the following fractional version of the Gagliardo-Nirenberg inequality which is due to Brezis-Mironescu [2] and Hajaiej-Molinet-Ozawa-Wang [13].

Lemma 2.2. *Let* $1 < p, q, r < \infty$, $0 \leqslant \theta \leqslant 1$ *and* $s, s_1 \in \mathbb{R}$. *Assume that* $u \in C_c^\infty(\mathbb{R}^3)$, *Then*

$$\|\Lambda^s u\|_{L^p} \leqslant C\|u\|_{L^q}^{1-\theta}\|\Lambda^{s_1} u\|_{L^r}^{\theta}, \tag{2.3}$$

where

$$\frac{1}{p} - \frac{s}{n} = \frac{1-\theta}{q} + \theta\left(\frac{1}{r} - \frac{s_1}{n}\right), \quad s \leqslant \theta s_1.$$

We now present a generalization of the refined Sobolev embedding stated in [1].

Lemma 2.3. *Let* $1 \leq p < \infty$ *and* r *be a positive real number. Then there exists a constant* C *such that*

$$\|f\|_{L^p} \leq C\|f\|_{\dot{B}^{-r}_{\infty,\infty}}^{1-\theta}\|f\|_{\dot{B}^{\beta}_{q,q}}^{\theta}, \quad \beta = r\left(\frac{p}{q} - 1\right) \quad and \quad \theta = \frac{q}{p}. \tag{2.4}$$

In particular, for $q = 2$ *and* $p = 3$, *we have*

$$\|f\|_{L^3}^3 \leq C\|f\|_{\dot{B}^{-r}_{\infty,\infty}}^{1-\theta}\|f\|_{\dot{H}^{\frac{r}{2}}}^2, \quad with \quad r > 0.$$

The following logarithmic Sobolev inequality which plays an important role in the control of the L^∞-norm of velocity u.

Lemma 2.4. *Let* $m > \frac{3}{2}$, *then there exists* C *depending only on* m, p *and* Θ *such that*

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leqslant C(1 + \|u\|_{V_\Theta}\Theta(\log(\|u\|_{H^m} + e))), \tag{2.5}$$

for all $u \in H^m(\mathbb{R}^3)$.

Proof. First, by using Littlewood-Paley theory, we decompose the function into low and high frequencies. More precisely, we write

$$u(x) = u_l(x) + u_h(x), \tag{2.6}$$

where

$$u_l(x) = \sum_{j=-1}^{N} \Delta_j u \quad and \quad u_h(x) = \sum_{j>N} \Delta_j u,$$

and the integer N will be determined later.

For the high frequency part $u_h(x)$, we can show that

$$\|u_h(x)\|_\infty \leqslant \sum_{j>N} \|\Delta_j u\|_\infty \leqslant C \sum_{j>N} 2^{-(m-3/2)j}\|u\|_{B^m_{2,\infty}} \leqslant C2^{-(m-3/2)N}\|u\|_{H^m}, \tag{2.7}$$

for $j \geqslant 0$ and $m > \frac{3}{2}$, where we have used the following Bernstein estimate

$$\|\Delta_j u\|_{L^{p_2}} \leqslant C2^{jd(\frac{1}{p_1} - \frac{1}{p_2})}\|\Delta_j u\|_{L^{p_1}} \quad \text{for} \quad j \geqslant 0, \quad 1 \leqslant p_1 \leqslant p_2 \leqslant \infty,$$

and the space embedding relationship $W^{s,p} \hookrightarrow B^s_{p,max(p,2)} \hookrightarrow B^m_{p,\infty}$, see [41].

From definition 2.1, we have

$$\|u_l(x)\|_\infty \leqslant \Theta(N)\|u\|_{V_\Theta}. \tag{2.8}$$

Taking (2.6), (2.7) and (2.8) into consideration, we get

$$\|u(x)\|_\infty \leqslant C\left(2^{-(m-3/2)N}\|u\|_{H^m} + \Theta(N)\|u\|_{V_\Theta}\right). \tag{2.9}$$

If we take $N = \left[\frac{\log(\|u\|_{H^m}+e)}{(m-3/2)}\right] + 1$, where $[\,\cdot\,]$ denotes Gauss symbol, then we have the desired estimate (2.5). □

Remark 2.1. *In this paper, we consider the case* $\Theta(N) = 2^{(1-\alpha)N}$, $0 < \alpha < 1$, $m = 2$, *then from inequalities (2.5) and (2.9), we have*

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leqslant C + C\|u\|_{V_\Theta}\left(\|u\|_{H^2(\mathbb{R}^3)} + e\right)^{2(1-\alpha)}. \tag{2.10}$$

Now, we state our main results in the framework of nonhomogenous Besov space.

Theorem 2.1. *Let the initial data* $(u_0, d_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *and the pair* (u, d) *be the weak solution to the nematic liquid crystal flows (1.7) on time* $[0, T^*)$ *for some* $0 < T^* < +\infty$. *If there exists a positive constant M such that*

$$\int_0^{T^*} \|u(t)\|_{V_\Theta}^{\frac{4}{4\alpha-3}} dt \leq M < +\infty, \quad \frac{3}{4} < \alpha < 1, \tag{2.11}$$

then (u, d) *can be extended beyond* T^*.

Remark 2.2. *Here the norm* $\|\cdot\|_{V_\Theta}$ *is weaker than* $B^{\alpha-1}_{\infty,\infty}$*-norm, for* $0 < \alpha < 1$, *since the following equivalent norm*

$$C^{-|s|}\|u\|_{B^s_{p,r}} \leq \left\|\left(2^{js}\|S_j u\|_{L^p}\right)_j\right\|_{l^r} \leq C\left(1 + \frac{1}{|s|}\right)\|u\|_{B^s_{p,r}}, \tag{2.12}$$

holds for some constant C provided $s < 0$, *see [1].*

Remark 2.3. *When we take* $\Theta(N) = 2^{(1-\alpha)N}$, $0 < \alpha < 1$, *then the following inequality holds*

$$\frac{\|\sum_{j=-1}^N \Delta_j u\|_\infty}{2^{(1-\alpha)N}} \leq \frac{C\|\sum_{j=-1}^N \Delta_j u\|_\infty}{N\log N} \leq \frac{C\|\sum_{j=-1}^N \Delta_j u\|_\infty}{N+1} \leqslant C\|u\|_{B^0_{\infty,\infty}},$$

for $N \geq 2$. *We should point out that Theorem 2.1 can also be applied to homogenous Besov space, as a consequence, we improve the result given by Zhang et.al. in reference [44].*

For the homogenous case, we have the following result.

Theorem 2.2. *Let the initial data* $(u_0, d_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *and the pair* (u, d) *be the weak solution to the nematic liquid crystal flows (1.7) on time* $[0, T^*)$ *for some* $0 < T^* < +\infty$. *If there exists a positive constant M such that*

$$\int_0^{T^*} \|\nabla u(t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}} dt \leq M < +\infty, \quad 0 < r < 2, \tag{2.13}$$

then (u, d) *is smooth up to time* $t = T^*$.

Remark 2.4. *When the macroscopic average of the nematic liquid crystal orientation d is a constant, the nematic liquid crystal flow reduces to the incompressible Navier-Stokes equations, the result proved in [45] is a straightforward consequence of Theorem 2.2.*

Noticing the fact $\|\nabla u(t)\|_{\dot{B}^{-r}_{\infty,\infty}} \leq C\|u(t)\|_{\dot{B}^{1-r}_{\infty,\infty}}$, we have the following corollary.

Corollary 2.1. *Assume that the initial data $(u_0, d_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. Let (u, d) be a local weak solution of the system (1.7). Suppose that*

$$\int_0^{T^*} \|u(t)\|_{\dot{B}^{1-r}_{\infty,\infty}}^{\frac{2}{2-r}} dt \leq M < +\infty, \quad 0 < r < 2, \tag{2.14}$$

then the solution (u, d) can be extended past time T^*.

Remark 2.5. *If we take $s = 1 - r$ in (2.14), then the regular index of blow-up criterion (1.12) in [43] can be extended to $-1 < s < 1$.*

3. Blow-up criterion for Ginzburg-Landau approximation system

In this section we prove Theorem 2.1 and Theorem 2.2. Suppose T^* is the maximal time of the existence for the local solution (see [22]), then global in time weak solution exists under assumptions of (2.11) and (2.13), respectively.

3.1. Proof of Theorem 2.1.

Proof. First, in order to get L^2 energy estimate, multiplying the first equation of (1.7) by u, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^3} \nabla \cdot (\nabla d \odot \nabla d) \cdot u \, dx \\ &= -\int_{\mathbb{R}^3} \left(\frac{1}{2}\nabla(|\nabla d|^2) + \Delta d \cdot \nabla d\right) \cdot u \, dx \\ &= -\int_{\mathbb{R}^3} (\Delta d \cdot \nabla d) \cdot u \, dx, \end{aligned} \tag{3.1}$$

similarly, multiplying the second equation of (1.7) by $(-\Delta d + f(d))$, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla d\|_{L^2}^2 + \frac{1}{4}\|d\|_{L^4}^4\right) + \|(\Delta d - f(d))\|_{L^2}^2 = \int_{\mathbb{R}^3} (u \cdot \nabla) d \cdot \Delta d \, dx, \tag{3.2}$$

adding (3.1) and (3.2) together, we obtain

$$\|u(t)\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \int_0^T \left(\|\nabla u(t)\|_{L^2}^2 + \|(\Delta d - f(d))\|_{L^2}^2\right) dt \leq C. \tag{3.3}$$

Next, applying ∇ on the first equation of (1.7), multiplying the resulting equation by ∇u and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^3} \nabla(u\cdot\nabla u)\nabla u\,dx - \int_{\mathbb{R}^3} \nabla(\nabla\cdot(\nabla d\odot\nabla d))\nabla u\,dx \\
&= -\int_{\mathbb{R}^3} \partial_k u_j\partial_j u_i\partial_k u_i\,dx - \int_{\mathbb{R}^3} \partial_i\partial_j\partial_l d_k\partial_j d_k\partial_l u_i\,dx \\
&\quad - \int_{\mathbb{R}^3} \partial_i\partial_j d_k\partial_j\partial_l d_k\partial_l u_i\,dx - \int_{\mathbb{R}^3} \partial_i\partial_l d_k\partial_j\partial_j d_k\partial_l u_i\,dx \\
&\quad - \int_{\mathbb{R}^3} \partial_i d_k\partial_j\partial_j\partial_l d_k\partial_l u_i\,dx,
\end{aligned}
\tag{3.4}
$$

here and in what follows we adopt the Einstein convention summation over repeated indices.

Then, applying ∇^2 on the second equation of (1.7), multiplying the resulting equation by $\nabla^2 d$ and using integration by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla^2 d(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla d)\nabla^2 d\,dx - \int_{\mathbb{R}^3} \nabla^2 f(d)\nabla^2 d\,dx \\
&= -\int_{\mathbb{R}^3} \partial_j\partial_l u_i\partial_i d_k\partial_j\partial_l d_k\,dx - \int_{\mathbb{R}^3} \partial_l u_i\partial_i\partial_j d_k\partial_j\partial_l d_k\,dx \\
&\quad - \int_{\mathbb{R}^3} \partial_j u_i\partial_i\partial_l d_k\partial_j\partial_l d_k\,dx - \int_{\mathbb{R}^3} u_i\partial_i\partial_j\partial_l d_k\partial_j\partial_l d_k\,dx \\
&\quad - \int_{\mathbb{R}^3} \partial_j\partial_l f(d)\cdot\partial_j\partial_l d\,dx.
\end{aligned}
\tag{3.5}
$$

Combining (3.4) and (3.5) together, integrating by parts and using divergence free condition, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2) + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \partial_k u_j\partial_j u_i\partial_k u_i\,dx - \int_{\mathbb{R}^3} \partial_i\partial_l d_k\partial_j\partial_j d_k\partial_l u_i\,dx \\
&\quad - \int_{\mathbb{R}^3} \partial_j u_i\partial_i\partial_l d_k\partial_j\partial_l d_k\,dx - \int_{\mathbb{R}^3} \partial_j\partial_l f(d)\cdot\partial_j\partial_l d\,dx \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{3.6}
$$

Then the terms I_1, I_2, I_3, I_4 on the right-hand side of Eq. (3.6) can be estimated as

$$
\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3} \partial_k u_j\partial_j u_i\partial_k u_i\,dx = \int_{\mathbb{R}^3} \partial_k u_j u_i\partial_j\partial_k u_i\,dx \\
&\le \|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2},
\end{aligned}
\tag{3.7}
$$

where, we have used divergence free property and integration by parts.

$$
\begin{aligned}
I_2 + I_3 &= -\int_{\mathbb{R}^3} \partial_i\partial_l d_k\partial_j\partial_j d_k\partial_l u_i\,dx - \int_{\mathbb{R}^3} \partial_j u_i\partial_i\partial_l d_k\partial_j\partial_l d_k\,dx \\
&= \int_{\mathbb{R}^3} \left(\partial_i\partial_l\partial_l d_k\partial_j\partial_j d_k + \partial_i\partial_l d_k\partial_j\partial_j\partial_l d_k\right) u_i\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^3} u_i \left(\partial_i \partial_j \partial_l d_k \partial_j \partial_l d_k + \partial_i \partial_l d_k \partial_j \partial_j \partial_l d_k \right) dx \\
&\leq \|u\|_{L^\infty} \|\nabla^2 d\|_{L^2} \|\nabla^3 d\|_{L^2}.
\end{aligned} \tag{3.8}
$$

The term I_4, can be estimated as

$$
\begin{aligned}
I_4 &= -\int_{\mathbb{R}^3} \partial_j \partial_l f(d) \cdot \partial_j \partial_l d \, dx = \int_{\mathbb{R}^3} \partial_l \left(|d|^2 d_k - d_k \right) \partial_j \partial_j \partial_l d_k dx \\
&\leq \int_{\mathbb{R}^3} \left(|d|^2 |\nabla d| |\nabla^3 d| + |\nabla d| |\nabla^3 d| \right) dx \\
&\leq \|\nabla d\|_{L^2} \|\nabla^3 d\|_{L^2} \\
&\leq \delta \|\nabla^3 d\|_{L^2}^2 + C,
\end{aligned} \tag{3.9}
$$

where we have used the fact $|d| \leq 1$ and (3.3).

Taking (3.6)–(3.9) into consideration, which together with Lemma 2.4, we have

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2 \right) + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 \\
&\leqslant C\|u\|_\infty \left(\|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + \|\nabla^2 d\|_{L^2} \|\nabla^3 d\|_{L^2} \right) + C \\
&\leqslant C \left(1 + \|u\|_{V_\Theta} (\|u\|_{H^2} + e)^{2(1-\alpha)} \right) \left(\|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + \|\nabla^2 d\|_{L^2} \|\nabla^3 d\|_{L^2} \right) + C \\
&\leqslant C \left(1 + \|u\|_{V_\Theta} \right) \|u\|_{H^2}^{2(1-\alpha)} \left(\|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + \|\nabla^2 d\|_{L^2} \|\nabla^3 d\|_{L^2} \right) + C \\
&\leqslant C \left(1 + \|u\|_{V_\Theta} \right) \|u\|_{H^2}^{2(1-\alpha)} \left(\|\nabla^2 u\|_{L^2}^{\frac{3}{2}} + \|\nabla^3 d\|_{L^2}^{\frac{3}{2}} \right) + C \\
&\leqslant \delta \left(\|u\|_{H^2}^{\frac{8(1-\alpha)}{7-4\alpha}} \|\nabla^2 u\|_{L^2}^{\frac{6}{7-4\alpha}} + \|u\|_{H^2}^{\frac{8(1-\alpha)}{7-4\alpha}} \|\nabla^3 d\|_{L^2}^{\frac{6}{7-4\alpha}} \right) + C \left(1 + \|u\|_{V_\Theta} \right)^{\frac{4}{4\alpha-3}} \\
&\leqslant \delta \left(\|u\|_{H^2}^2 + \|d\|_{H^3}^2 \right) + C \left(1 + \|u\|_{V_\Theta} \right)^{\frac{4}{4\alpha-3}},
\end{aligned} \tag{3.10}
$$

for any $\delta > 0$, $\frac{3}{4} < \alpha < 1$, where in the fourth inequality we have used the following Gagliardo-Nirenberg inequality

$$
\|\nabla u\|_{L^2} \leq C \|u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}}.
$$

Then integrating inequality (3.10) over time $(0, T)$ and by condition (2.11), we obtain

$$
\sup_{t \in [0,T]} \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2 + \int_0^T \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 dt \leqslant C. \tag{3.11}
$$

This completes the proof of theorem 2.1. □

3.2. Proof of Theorem 2.2.

Proof. From the proof of Theorem 2.1 and Lemma 2.3, $I_i (i = 1, 2, 3, 4)$ can be estimated as

$$
\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3} \partial_k u_j \partial_j u_i \partial_k u_i \, dx \leq \|\nabla u\|_{L^3}^3 \\
&\leq C \|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}} \|\nabla u\|_{\dot{H}^{\frac{r}{2}}}^2
\end{aligned}
$$

$$\leq C\|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}}\|\nabla u\|_{L^2}^{2-r}\|\nabla^2 u\|_{L^2}^r$$
$$\leq \delta\|\nabla^2 u\|_2^2 + C\|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}}\|\nabla u\|_{L^2}^2, \tag{3.12}$$

and

$$\begin{aligned} I_2 + I_3 &= -\int_{\mathbb{R}^3}\partial_i\partial_l d_k\partial_j\partial_j d_k\partial_l u_i\,dx - \int_{\mathbb{R}^3}\partial_j u_i\partial_i\partial_l d_k\partial_j\partial_l d_k\,dx \\ &\leq \|\nabla u\|_{L^2}\|\nabla^2 d\|_{L^4}^2 \\ &\leq C\|\nabla u\|_{L^2}\|d\|_{L^\infty}\|\nabla^3 d\|_{L^2} \\ &\leq \delta\|\nabla^3 d\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2, \end{aligned} \tag{3.13}$$

where we have used the Gagliardo-Nirenberg inequality $\|\nabla^2 d\|_{L^4}^2 \leq C\|d\|_{L^\infty}\|\nabla^3 d\|_{L^2}$.

Similarly,

$$I_4 = -\int_{\mathbb{R}^3}\Delta f(d)\cdot\Delta d\,dx \leq \delta\|\nabla^3 d\|_{L^2}^2 + C. \tag{3.14}$$

Plugging (3.12)–(3.14) into (3.6) we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2) + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 \\ &\leq C\left(\|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}} + 1\right)\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\right), \end{aligned} \tag{3.15}$$

from which, by using Gronwall's inequality and condition (2.13) we have

$$\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2 + \int_0^T\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 dt \leqslant C. \tag{3.16}$$

This completes the proof of Theorem 2.2. □

4. Blow-up Criterion for Ericksen-Leslie system

In this section, we give some blow-up criterions for Ericksen-Leslie model. The local existence of the classical solution to the system (1.1) satisfying (see [40] for example)

$$u \in C\left(0, T^*; H^3(\mathbb{R}^3, \mathbb{R}^3)\right) \cap C^1\left(0, T^*; H^2(\mathbb{R}^3, \mathbb{R}^3)\right),$$
$$d \in C\left(0, T^*; H^4(\mathbb{R}^3, \mathbb{S}^2)\right) \cap C^1\left(0, T^*; H^3(\mathbb{R}^3, \mathbb{S}^2)\right).$$

Let T^* is the maximal time of the existence for the local solution, then by standard continuation argument and under the assumption of (4.1), global in time solution is obtained.

Theorem 4.1. *Let the initial data $(u_0, d_0) \in H^3(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and the pair (u, d) be the smooth solution to the nematic liquid crystal flows (1.1) on time $[0, T^*)$ for some $0 < T^* < +\infty$. If there exists a positive constant M such that*

$$\int_0^{T^*}\left(\|\nabla u(t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}} + \|\nabla^2 d(t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}}\right)dt \leq M < +\infty, \quad 0 < r < 2, \tag{4.1}$$

then (u, d) is smooth up to time $t = T^*$.

Remark 4.1. *Under the divergence free condition and* $\|\nabla^2 d(t)\|_{\dot{B}^{-r}_{\infty,\infty}} \le C\|\nabla d(t)\|_{\dot{B}^{1-r}_{\infty,\infty}}$, *from (4.1), we have the following blow-up criterion*

$$\int_0^{T^*} \left(\|\omega(t)\|^{\frac{2}{2-r}}_{\dot{B}^{-r}_{\infty,\infty}} + \|\nabla d(t)\|^{\frac{2}{2-r}}_{\dot{B}^{1-r}_{\infty,\infty}} \right) dt \le M < +\infty, \quad 0 < r < 2, \tag{4.2}$$

where $\omega = \nabla \times u$, *this improves the result (1.5) in [42].*

Combining the proof of Theorem 2.1 and Theorem 4.1, we have the following corollary:

Corollary 4.1. *Assume that the initial data* $(u_0, d_0) \in H^3(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$. *Let* (u, d) *be a local smooth solution of the system (1.1). Suppose that*

$$\int_0^{T^*} \left(\|u(t)\|^{\frac{4}{4\alpha-3}}_{V_\Theta} + \|\nabla d(t)\|^{\frac{2}{2-r}}_{\dot{B}^{1-r}_{\infty,\infty}} \right) dt \le M < +\infty, \tag{4.3}$$

where $\frac{3}{4} < \alpha < 1$, $0 < r < 2$, *then the solution* (u, d) *can be extended past time* T^*.

4.1. Proof of Theorem 4.1.

Proof. (L^2 estimate). First, from Eq. (3.1) we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2} = -\int_{\mathbb{R}^3} (\Delta d \cdot \nabla d) \cdot u \, dx. \tag{4.4}$$

Next, multiplying both sides of the second equation of (1.1) by $-\Delta d$ and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla d\|^2_{L^2} + \|\Delta d\|^2_{L^2} &= \int_{\mathbb{R}^3} (u \cdot \nabla) d \cdot \Delta d \, dx - \int_{\mathbb{R}^3} |\nabla d|^2 d \cdot \Delta d \, dx \\ &= \int_{\mathbb{R}^3} (u \cdot \nabla) d \cdot \Delta d \, dx - \int_{\mathbb{R}^3} |d \cdot \Delta d|^2 \, dx \\ &\le \int_{\mathbb{R}^3} (u \cdot \nabla) d \cdot \Delta d \, dx + \int_{\mathbb{R}^3} |\Delta d|^2 \, dx,\end{aligned} \tag{4.5}$$

where we have used the facts $|d| = 1$ and $|\nabla d|^2 = -d \cdot \Delta d$.

Then adding (4.4) and (4.5) together, we obtain

$$\|u(t)\|^2_{L^2} + \|\nabla d\|_{L^2} + \int_0^T \|\nabla u(t)\|^2_{L^2} dt \le C. \tag{4.6}$$

(Lower-order estimate). Applying ∇ on first equation of (1.1) and multiplying the resulting equation by ∇u. Using divergence free property and integration by parts we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2} + \|\nabla^2 u(t)\|^2_{L^2} = -\int_{\mathbb{R}^3} \nabla(u \cdot \nabla u) \nabla u \, dx - \int_{\mathbb{R}^3} \nabla(\nabla \cdot (\nabla d \odot \nabla d)) \nabla u \, dx. \tag{4.7}$$

Applying ∇^2 on second equation of (1.1), multiplying the resulting equation with $\nabla^2 d$, integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla^2 d(t)\|^2_{L^2} + \|\nabla^3 d(t)\|^2_{L^2} = -\int_{\mathbb{R}^3} \nabla^2(u \cdot \nabla d) \nabla^2 d \, dx - \int_{\mathbb{R}^3} \nabla^2(|\nabla d|^2 d) \nabla^2 d \, dx. \tag{4.8}$$

Adding (4.7) and (4.8) together we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2) + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \nabla(u\cdot\nabla u)\nabla u\,dx - \int_{\mathbb{R}^3} \nabla(\nabla\cdot(\nabla d\odot\nabla d))\nabla u\,dx \\
&\quad - \int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla d)\nabla^2 d\,dx - \int_{\mathbb{R}^3} \nabla^2(|\nabla d|^2 d)\nabla^2 d\,dx \\
&= J_1 + J_2 + J_3 + J_4,
\end{aligned}
\tag{4.9}
$$

where

$$
J_1 = -\int_{\mathbb{R}^3} \nabla(u\cdot\nabla u)\nabla u\,dx \le \delta\|\nabla^2 u\|_2^2 + C\|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}}\|\nabla u\|_{L^2}^2, \tag{4.10}
$$

and

$$
\begin{aligned}
J_2 &= -\int_{\mathbb{R}^3} \nabla(\nabla\cdot(\nabla d\odot\nabla d))\nabla u\,dx \\
&\le C\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3}\|\nabla^2 u\|_{L^2} \\
&\le C\|d\|_{L^\infty}^{\frac{1}{2}}\|\nabla^2 d\|_{L^3}^{\frac{3}{2}}\|\nabla^2 u\|_{L^2} \\
&\le \delta\|\nabla^2 u\|_{L^2}^2 + C(\delta)\|\nabla^2 d\|_{L^3}^3 \\
&\le \delta\|\nabla^2 u\|_{L^2}^2 + \delta\|\nabla^3 d\|_2^2 + C(\delta)\|\nabla^2 d\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}}\|\nabla^2 d\|_{L^2}^2,
\end{aligned}
\tag{4.11}
$$

where in the second inequality we have used the following Gagliardo-Nirenberg inequality

$$
\|\nabla d\|_{L^6} \le C\|d\|_{L^\infty}^{\frac{1}{2}}\|\nabla^2 d\|_{L^3}^{\frac{1}{2}}.
$$

The term J_3, is estimated as

$$
\begin{aligned}
J_3 &= -\int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla d)\nabla^2 d\,dx \\
&\le \|\nabla^2 u\|_{L^2}\|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3} + \|\nabla u\|_{L^3}\|\nabla^2 d\|_{L^3}^2 \\
&\le \|d\|_{L^\infty}^{\frac{1}{2}}\|\nabla^2 d\|_{L^3}^{\frac{3}{2}}\|\nabla^2 u\|_{L^2} \\
&\qquad + \|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{1}{3}}\|\nabla u\|_{L^2}^{\frac{2-r}{3}}\|\nabla^2 u\|_{L^2}^{\frac{r}{3}}\|\nabla^2 d\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{3}}\|\nabla^2 d\|_{L^2}^{\frac{2(2-r)}{3}}\|\nabla^3 d\|_{L^2}^{\frac{2r}{3}} \\
&\le \delta\|\nabla^2 u\|_{L^2}^2 + C(\delta)\|\nabla^2 d\|_{L^3}^3 + \delta\left(\|\nabla u\|_{L^2}^{\frac{2}{3}}\|\nabla^3 d\|_{L^2}^{\frac{4}{3}}\right) \\
&\qquad + C\|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{3(2-r)}}\|\nabla u\|_{L^2}^{\frac{2}{3}}\|\nabla^2 d\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{4}{3(2-r)}}\|\nabla^2 d\|_{L^2}^{\frac{4}{3}} \\
&\le \delta\left(\|\nabla u\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2\right) + C(\delta)\left(\|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}} + \|\nabla^2 d\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}}\right)\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\right),
\end{aligned}
\tag{4.12}
$$

and

$$
J_4 = -\int_{\mathbb{R}^3} \nabla^2(|\nabla d|^2 d)\nabla^2 d\,dx
$$

$$
\begin{aligned}
&\leq C\left(\|\nabla^2 d\|_{L^3}^3 + \|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3}\|\nabla^3 d\|_{L^2} + \|\nabla d\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2\right)\\
&\leq \delta\|\nabla^3 d\|_{L^2}^2 + C(\delta)\|\nabla^2 d\|_{L^3}^3\\
&\leq \delta\|\nabla^3 d\|_{L^2}^2 + \delta\|\nabla^3 d\|_2^2 + C(\delta)\|\nabla^2 d\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}}\|\nabla^2 d\|_{L^2}^2, \qquad (4.13)
\end{aligned}
$$

Plugging (4.10)–(4.13) into (4.9) we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2) + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2\\
&\leq C\left(\|\nabla u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{(2-r)}} + \|\nabla^2 d\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{(2-r)}}\right)\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\right), \qquad (4.14)
\end{aligned}
$$

by using Gronwall's inequality and condition (4.1) we have

$$
\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2 + \int_0^T \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 dt \leqslant C. \qquad (4.15)
$$

(Higher-order estimate). For the completeness of our proof, we obtain the higher-order estimates, which can be found in [29]. Applying ∇^3 on the first equation of (1.1), taking inner product with $\nabla^3 u$ and integration over domain $\mathbb{R}^3$. After integrating by parts we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^4 u(t)\|_{L^2}^2\\
&= -\int_{\mathbb{R}^3}\nabla^3(u\cdot\nabla u)\cdot\nabla^3 u\,dx - \int_{\mathbb{R}^3}\nabla^3(\Delta d\cdot\nabla d)\cdot\nabla^3 d\,dx\\
&:= K_1 + K_2. \qquad (4.16)
\end{aligned}
$$

Now, we estimate $K_i (i = 1, 2)$ one by one, by using commutator estimate we have

$$
\begin{aligned}
K_1 &= -\int_{\mathbb{R}^3}\nabla^3(u\cdot\nabla u)\cdot\nabla^3 u\,dx = -\int_{\mathbb{R}^3}[\nabla^3, u\cdot\nabla]u\cdot\nabla^3 u\,dx\\
&\leqslant \|[\nabla^3, u\cdot\nabla]u\|_{L^{\frac{3}{2}}}\|\nabla^3 u\|_{L^3}\\
&\leqslant C\left(\|\nabla u\|_{L^3}\|\nabla^3 u\|_{L^3} + \|\nabla^3 u\|_{L^3}\|\nabla u\|_{L^3}\right)\|\nabla^3 u\|_{L^3}\\
&\leqslant C\|\nabla u\|_{L^3}\|\nabla^3 u\|_{L^3}^2\\
&\leqslant C\|\nabla u\|_{L^2}^{\frac{5}{6}}\|\nabla^4 u\|_{L^2}^{\frac{1}{6}}\|\nabla u\|_{L^2}^{\frac{1}{3}}\|\nabla^4 u\|_{L^2}^{\frac{5}{3}}\\
&\leqslant C\|\nabla u\|_{L^2}^{\frac{7}{6}}\|\nabla^4 u\|_{L^2}^{\frac{11}{6}}\\
&\leqslant \delta\|\nabla^4 u\|_{L^2}^2 + C(\delta)\|\nabla u\|_{L^2}^{14}, \qquad (4.17)
\end{aligned}
$$

and

$$
\begin{aligned}
K_2 &= -\int_{\mathbb{R}^3}\nabla^3(\Delta d\cdot\nabla d)\cdot\nabla^3 d\,dx\\
&\leq \delta\|\nabla^4 u\|_{L^2}^2 + C(\delta)\int_{\mathbb{R}^3}|\nabla^2(\Delta d\cdot\nabla d)|^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \delta\|\nabla^4 u\|_{L^2}^2 + C(\delta)\int_{\mathbb{R}^3}\left(|\nabla^4 d|^2|\nabla d|^2 + |\nabla^2 d|^2|\nabla^3 d|^2\right)dx\\
&\leq \delta\|\nabla^4 u\|_{L^2}^2 + C(\delta)\left(\|\nabla^4 d\|_{L^3}^2\|\nabla d\|_{L^6}^2 + \|\nabla^2 d\|_{L^4}^2\|\nabla^3 d\|_{L^4}^2\right)\\
&\leq \delta\|\nabla^4 u\|_{L^2}^2 + C(\delta)\left(\|\Delta d\|_{L^2}^{\frac{7}{3}}\|\nabla^5 d\|_{L^2}^{\frac{5}{3}} + \|\Delta d\|_{L^2}^{\frac{19}{6}}\|\nabla^5 d\|_{L^2}^{\frac{5}{6}}\right)\\
&\leq \delta\|\nabla^4 u\|_{L^2}^2 + \delta\|\nabla^5 d\|_{L^2}^2 + C(\delta)\left(\|\Delta d\|_{L^2}^{14} + \|\Delta d\|_{L^2}^{\frac{38}{7}}\right),
\end{aligned}
\tag{4.18}
$$

where we have used Young's inequality and the following Gagliardo-Nirenberg inequalities

$$
\|\nabla^2 d\|_{L^4} \leq C\|\Delta d\|_{L^2}^{\frac{3}{4}}\|\nabla^5 d\|_{L^2}^{\frac{1}{4}},
$$

$$
\|\nabla^3 d\|_{L^4} \leq C\|\Delta d\|_{L^2}^{\frac{5}{6}}\|\nabla^5 d\|_{L^2}^{\frac{1}{6}},
$$

$$
\|\nabla^4 d\|_{L^3} \leq C\|\Delta d\|_{L^2}^{\frac{1}{6}}\|\nabla^5 d\|_{L^2}^{\frac{5}{6}}.
$$

Next, applying ∇^4 on the second equation of (1.1) and taking inner product with $\nabla^4 d$ we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^4 d(t)\|_{L^2}^2 + \|\nabla^5 d(t)\|_{L^2}^2\\
&= -\int_{\mathbb{R}^3}\nabla^4(u\cdot\nabla d)\cdot\nabla^4 d\,dx + \int_{\mathbb{R}^3}\nabla^4(d|\nabla d|^2)\cdot\nabla^4 d\,dx\\
&:= K_3 + K_4,
\end{aligned}
\tag{4.19}
$$

the right-hand side of above equality can be estimate as

$$
\begin{aligned}
K_3 &= \int_{\mathbb{R}^3}[\nabla^4(u\cdot\nabla d) - (u\cdot\nabla)\nabla^4 d]\cdot\nabla^4 d\,dx\\
&\leq C\|[\nabla^4(u\cdot\nabla d) - (u\cdot\nabla)\nabla^4 d]\|_{L^{\frac{3}{2}}}\|\nabla^4 d\|_{L^3}\\
&\leq C\|\nabla d\|_{L^6}\|\nabla^4 u\|_{L^2}\|\nabla^4 d\|_{L^3} + C\|\nabla u\|_{L^2}\|\nabla^4 d\|_{L^2}\|\nabla^4 d\|_{L^3}\\
&\leq \delta\|\nabla^4 u\|_{L^2}^2 + C(\delta)\left(\|\Delta d\|_{L^2}^2\|\nabla^4 d\|_{L^3}^2 + \|\nabla u\|_{L^2}\|\nabla^5 d\|_{L^2}\|\nabla^4 d\|_{L^3}\right)\\
&\leq \delta\|\nabla^4 u\|_{L^2}^2 + C(\delta)\left(\|\Delta d\|_{L^2}^2\|\Delta d\|_{L^2}^{\frac{1}{3}}\|\nabla^5 d\|_{L^2}^{\frac{5}{3}} + \|\nabla u\|_{L^2}\|\Delta d\|_{L^2}^{\frac{1}{6}}\|\nabla^5 d\|_{L^2}^{\frac{11}{6}}\right)\\
&\leq \delta\|\nabla^4 u\|_{L^2}^2 + \delta\|\nabla^5 d\|_{L^2}^2 + C(\delta)\left((\|\Delta d\|_{L^2}^{14} + \|\nabla u\|_{L^2}^{24} + \|\Delta d\|_{L^2}^4\right),
\end{aligned}
\tag{4.20}
$$

where we have used the divergence free condition and Lemma 2.1. The second term is estimated as

$$
\begin{aligned}
K_4 &= \int_{\mathbb{R}^3}\nabla^4(|\nabla d|^2 d)\cdot\nabla^4 d\,dx = -\int_{\mathbb{R}^3}\nabla^3(|\nabla d|^2 d)\cdot\nabla^5 d\,dx\\
&= -\int_{\mathbb{R}^3}\left(\nabla^3(|\nabla d|^2)d\cdot\nabla^5 d + 3\nabla^2(|\nabla d|^2)\nabla d\cdot\nabla^5 d + 3\nabla(|\nabla d|^2)\nabla^2 d\cdot\nabla^5 d\right)dx\\
&\quad -\int_{\mathbb{R}^3}|\nabla d|^2\nabla^3 d\cdot\nabla^5 d\,dx\\
&\leq C\|\nabla^5 d\|_{L^2}\left(\|\nabla d\|_{L^6}\|\nabla^4 d\|_{L^3} + \|\nabla^2 d\|_{L^4}\|\nabla^3 d\|_{L^4} + \|\nabla d\|_{L^6}^2\|\nabla^3 d\|_{L^6} + \|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^6}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|\nabla^5 d\|_{L^2}\left(\|\Delta d\|_{L^2}^{\frac{7}{6}}\|\nabla^5 d\|_{L^2}^{\frac{5}{6}} + \|\Delta d\|_{L^2}^{\frac{7}{3}}\|\nabla^5 d\|_{L^2}^{\frac{2}{3}}\right)\\
&\leq \frac{1}{4}\|\nabla^5 d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^{14}\\
&\leq \delta\|\nabla^5 u\|_{L^2}^2 + C(\delta)\|\nabla^2 d\|_{L^2}^{14}.
\end{aligned}
\tag{4.21}
$$

By combining (4.17)–(4.21) with (4.16) and integration over time $[0, T]$, we have

$$
\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^4 d(t)\|_{L^2}^2 + \int_0^T \|\nabla^4 u(t)\|_{L^2}^2 + \|\nabla^5 d(t)\|_{L^2}^2 dt \leq C. \tag{4.22}
$$

This completes the proof of Theorem 4.1. □

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Conflict of interest

All authors declare no conflicts of interest.

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