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*Research article*

# Faber polynomial coefficients for meromorphic bi-subordinate functions of complex order

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**Abstract:** In this paper, we obtain the upper bounds for the $n-th$ $(n \geq 1)$ coefficients for meromorphic bi-subordinate functions of complex order by using Faber polynomial expansions. The results, which are presented in this paper, would generalize those in related works of several earlier authors.

**Keywords:** analytic functions; starlike functions; meromorphic functions; bi-univalent functions; subordination; Faber polynomial
**Mathematics Subject Classification:** 30C45, 30C80

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## 1. Introduction

Let $\Sigma$ be the class of meromorphic univalent functions in the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$ of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}. \tag{1.1}$$

Since $f \in \Sigma$ is univalent, it has an inverse $f^{-1}$, that satisfy

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad (M < |w| < \infty, \ M > 0).$$

A simple calculation shows that the function $g := f^{-1}$ is given by

$$\begin{aligned} g(w) &= w + B_0 + \sum_{n=1}^{\infty} \frac{B_n}{w^n} \qquad (1.2) \\ &= w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_1 + b_0^2 b_1 + b_1^2}{w^3} + \cdots . \end{aligned}$$

Analogous to the bi-univalent analytic functions, a function $f \in \Sigma$ is said to be meromorphic bi-univalent if $f^{-1} \in \Sigma$. We denote the family of all meromorphic bi-univalent functions by $\mathcal{M}_{\Sigma}$. Estimates on the coefficients of meromorphic univalent functions were widely investigated in theliterature, for example; Schiffer [18] obtained the estimate$|b_2| \leq 2/3$ for meromorphic univalent functions $f \in \Sigma$ with $b_0 = 0$ and Duren [6] proved that $|b_n| \leq 2/(n+1)$ for $f \in \Sigma$ with $b_k = 0$ for $1 \leq k \leq n/2$. For the coefficient of the inverse of meromorphic univalent functions Springer [20] proved that

$$|B_3| \leq 1 \quad \text{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-1)!}{n!(n-1)!}, \quad (n = 1, 2, \cdots).$$

In 1977, Kubota [13] has proved that the Springer's conjecture is true for $n = 3, 4, 5$ and subsequently Schober [19] obtained a sharp bounds for the coefficients $B_{2n-1}$, $1 \leq n \leq 7$. Recently, Kapoor and Mishra [12] found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order $\alpha$ in $\Delta$.

For a brief history and interesting examples of functions which are in (or are not in) the class $\mathcal{M}_{\Sigma}$, including various properties of such functions we refer the reader to the work of Hamidi et al. [9,10] and references therein. Bounds for the first few coefficients of various subclasses of bi-univalent functions were obtained by a variety of authors including [4, 8, 11, 15, 16, 22]. Not much was known about the bounds of the general coefficients $b_n; n \geq 1$ of subclasses of $\mathcal{M}_{\Sigma}$ up until the publication of the article [9, 10] by Hamidi, Halim and Jahangiri and followed by a number of related publications (see [5,14,21]). In this paper, we apply the Faber polynomial expansions to certain subclass of bi-univalent functions and obtain bounds for their $n-th; (n \geq 1)$ coefficients subject to a given gap series condition.

## 2. Coefficient estimates

An analytic function $f$ is subordinate to an analytic function $g$, written by $f \prec g$, provided that there is an analytic function $w$ defined on $\mathbb{D} =\{z \in \mathbb{C} : |z| < 1\}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$.

In the sequel, it is assumed that $\varphi$ is an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function is known to be typically real with the series expansion $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...$ where $B_1$, $B_2$ are real and $B_1 > 0$. We define the following comprehensive class of meromorphic functions:

**Definition 2.1.** *For* $0 \leq \lambda < 1$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$, *a function* $f \in \Sigma$ *given by (1.1) is said to be in the class* $\mathcal{M}_{\Sigma}(\lambda, \gamma; \varphi)$ *if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda)f(z)} - 1\right) \prec \varphi(z)$$

*and*

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + \lambda w^2 g''(w)}{\lambda w g'(w) + (1-\lambda)g(w)} - 1\right) \prec \varphi(w)$$

*where* $z, w \in \Delta$ *and the function* $g$ *is given by (1.2).*

A function $f \in \mathcal{M}_\Sigma(\lambda, \gamma; \varphi)$ is said the be generalized meromorphic bi-subordinate of complex order $\gamma$ and type $\lambda$.

By suitably specializing the parameters $\lambda$, $\gamma$ and the function $\varphi$, we state new subclass of meromorphic bi-univalent functions as illustrated in the following examples.

(1) $\mathcal{M}_\Sigma\,(0, 1; \varphi)) = \mathcal{M}_\Sigma(\varphi)$ is class of meromorphic Ma-Minda bi-starlike functions,

(2) $\mathcal{M}_\Sigma\,(0, 1; (1 + Az)/(1 + Bz)) = \mathcal{M}_\Sigma[A, B]$ $(-1 \le B < A \le 1)$ is class of meromorphic Janowski bi-starlike functions,

(3) $\mathcal{M}_\Sigma(0, (1-\beta)e^{-i\delta}\cos\delta; (1+z)/(1-z)) = \Sigma^*[\delta,\beta]$ $(|\delta| < \pi/2,\ 0 \le \beta < 1)$ is class of meromorphic bi- $\delta$-spirallike functions of order $\beta$,

(4) $\mathcal{M}_\Sigma(0, 1; (1+(1-2\beta)z)/(1-z)) = \Sigma^*(\beta)$ $(0 \le \beta < 1)$ is class of meromorphic bi-starlike functions of order $\beta$,

(5) $\mathcal{M}_\Sigma(0, 1; (1 + z)/(1 - z)) = \Sigma^*$ is class of meromorphic bi-starlike functions,

(6) $\mathcal{M}_\Sigma\left(0, 1; \left(\frac{1+z}{1-z}\right)^\beta\right) = \Sigma^*_\beta$ is class of meromorphic strongly bi-starlike functions of order $\beta$,

(7) $\mathcal{M}_\Sigma(0, \gamma; (1+z)/(1-z)) = \mathcal{S}^*[\gamma]$ is class of meromorphic bi-starlike functions of complex order.

In the following theorem, we use the Faber polynomials introduced by Faber [7] to obtain a bound for the general coefficients $|b_n|$ of the bi-univalent functions in $\mathcal{M}_\Sigma(\lambda, \gamma; \varphi)$ subject to a gap series condition.

**Theorem 2.1.** *Let $f \in \Sigma$ given by (1.1) in the class $\mathcal{M}_\Sigma(\lambda, \gamma; \varphi)$. If, $b_m$=0, $1 \le m \le n - 1$ for n being odd or if $b_m$=0, $0 \le m \le n - 1$ for n being even, then*

$$|b_n| \le \frac{|\gamma|\, B_1}{(n + 1)\, |1 - \lambda(n + 1|} \quad (n \ge 1).$$

*Proof.* If we write $\Lambda(f(z)) = \lambda z f'(z) + (1 - \lambda) f(z)$, then

$$\begin{aligned} f \ &\in\ \mathcal{M}_\Sigma(\lambda, \gamma; \varphi) \ \Leftrightarrow\ 1 + \frac{1}{\gamma}\left(\frac{z\Lambda'(f(z))}{\Lambda(f(z))} - 1\right) \prec \varphi(z) \\ g \ &=\ f^{-1} \in \mathcal{M}_\Sigma(\lambda, \gamma; \varphi) \ \Leftrightarrow\ 1 + \frac{1}{\gamma}\left(\frac{w\Lambda'(g(w))}{\Lambda(g(w))} - 1\right) \prec \varphi(w). \end{aligned}$$

Also, for the function $f(z) = z+b_0+\sum_{n=1}^\infty \frac{b_n}{z^n}$ we have $\Lambda(f(z)) = z+\sum_{n=0}^\infty a_n z^n$ where $a_n = (1-\lambda(n+1))b_n$.

Now, an application of Faber polynomial expansion to the power series $\mathcal{M}_\Sigma(\lambda, \gamma; \varphi)$ (e.g., see [2] or [3, equation (1.6)]) yields

$$1 + \frac{1}{\gamma}\left(\frac{z\Lambda'(f(z))}{\Lambda(f(z))} - 1\right) = 1 + \frac{1}{\gamma}\sum_{n=0}^{\infty} F_{n+1}\left(a_0, a_1, a_2, \cdots, a_n\right) \frac{1}{z^{n+1}} \tag{2.1}$$

where $F_{n+1}\left(a_0, a_1, a_2, \cdots, a_n\right)$ is a Faber polynomial of degree $n + 1$, i.e.,

$$F_n\left(a_0, a_1, a_2, \cdots, a_{n-1}\right) = \sum_{i_1+2i_2+\cdots+ni_n=n} A\left(i_1, i_2, \cdots, i_n\right)\left(a_0^{i_1} a_1^{i_2} \ldots a_{n-1}^{i_n}\right)$$

and

$$A\left(i_1, i_2, \cdots, i_n\right) := (-1)^{n+2i_1+\cdots+(n+1)i_n} \frac{(i_1 + i_2 + \cdots + i_n - 1)!n}{i_1! i_2! \ldots i_n!}.$$

The first few terms of $F_{n+1}(a_0, a_1, a_2, \cdots, a_n)$ are

$$\begin{aligned} F_1(a_0) &= -a_0, \\ F_2(a_0, a_1) &= a_0^2 - 2a_1, \\ F_3(a_0, a_1, a_2) &= -a_0^3 + 3a_0a_1 - 3a_2, \\ F_4(a_0, a_1, a_2, a_3) &= a_0^4 - 4a_0^2a_1 + 4a_0a_2 + 2a_1^2 - 4a_3, \\ F_5(a_0, a_1, a_2, a_3, a_4) &= -a_0^5 + 5a_0^3a_1 - 5a_0^2a_2 - 5\left(a_1^2 - a_3\right)a_0 + 5a_1a_2 - 5a_4. \end{aligned}$$

By the same token, the coefficients of the inverse map $g{=}f^{-1}$ may be expressed by

$$g(w) = f^{-1}(w) = w - b_0 - \sum_{n=1}^{\infty} \frac{1}{n} K_{n+1}^n \frac{1}{w^n} = w + \sum_{n=0}^{\infty} \beta_n \frac{1}{w^n}$$

where

$$\begin{aligned} K_{n+1}^n = \;& nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 + \frac{1}{2}n(n-1)(n-2)b_0^{n-3}(b_3 + b_1^2) \\ &+ \frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-4}(b_4 + 3b_1b_2) + \sum_{j\geq 5} b_0^{n-j}V_j \end{aligned}$$

and $V_j$ for $5 \leq j \leq n$ is a homogeneous polynomial in the variables $b_1, b_2, ..., b_n$.

The first few terms of $K_{n+1}^n$ are

$$\begin{aligned} K_2^1 &= b_1, \\ K_3^2 &= 2(b_0b_1 + b_2), \\ K_4^3 &= 3(b_0^2b_1 + 2b_0b_2 + b_3 + b_1^2), \\ K_5^4 &= 4\left(b_0^3b_1 + 3b_0^2b_2 + 3b_0(b_3 + b_1^2) + b_4 + 3b_1b_2\right). \end{aligned}$$

Obviously,

$$1 + \frac{1}{\gamma}\left(\frac{w\Lambda'(g(w))}{\Lambda(g(w))} - 1\right) = 1 + \frac{1}{\gamma}\sum_{n=0}^{\infty} F_{n+1}(A_0, A_1, A_2, \cdots, A_n)\frac{1}{w^{n+1}} \tag{2.2}$$

where $A_n = (1 - \lambda(n+1))\beta_n$. Since, the function $f$ in the class $\mathcal{M}_\Sigma(\lambda, \gamma; \varphi)$, by the definition of subordination, there exist two Schwarz functions $u(z) = \frac{c_1}{z} + \frac{c_2}{z^2} + ... + \frac{c_n}{z^n} + ...$, $|u(z)| < 1$, $z \in \Delta$ and $v(w) = \frac{d_1}{w} + \frac{d_2}{w^2} + ... + \frac{d_n}{w^n} + ...$, $|v(w)| < 1$, $w \in \Delta$, so that

$$1 + \frac{1}{\gamma}\left(\frac{z\Lambda'(f(z))}{\Lambda(f(z))} - 1\right) = \varphi(u(z)) = 1 + B_1 \sum_{n=1}^{\infty} K_n^{-1}(c_1, c_2, ..., c_n, B_1, B_2, ..., B_n)\frac{1}{z^n} \tag{2.3}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w\Lambda'(g(w))}{\Lambda(g(w))} - 1\right) = \varphi(v(w)) = 1 + B_1 \sum_{n=1}^{\infty} K_n^{-1}(d_1, d_2, ..., d_n, B_1, B_2, ..., B_n)\frac{1}{w^n}. \tag{2.4}$$

In general (e.g., see [1] and [2, equation (1.6)]), the coefficients $K_n^p := K_n^p(k_1, k_2, ..., k_n, B_1, B_2, ..., B_n)$ are given by

$$\begin{aligned} K_n^p &= \frac{p!(-1)^n}{(p-n)!n!}k_1^n\frac{B_n}{B_1} + \frac{p!(-1)^{n+1}}{(p-n+1)!(n-2)!}k_1^{n-2}k_2\frac{B_{n-1}}{B_1} \\ &+ \frac{p!(-1)^n}{(p-n+2)!(n-4)!}k_1^{n-3}k_3\frac{B_{n-2}}{B_1} \\ &+ \frac{p!}{(p-n+3)!(n-4)!}k_1^{n-4}\left[(-1)^{n+1}k_4\frac{B_{n-3}}{B_1} + (-1)^n\frac{p-n+3}{2}k_2^2\frac{B_{n-2}}{B_1}\right] \\ &+ \frac{p!}{(p-n+4)!(n-5)!}k_1^{n-5}\left[(-1)^n k_5\frac{B_{n-4}}{B_1} + (-1)^{n+1}(p-n+4)k_2k_3\frac{B_{n-3}}{B_1}\right] + \sum_{j\geq 6}k_1^{n-j}X_j \end{aligned}$$

where $X_j$ is a homogeneous polynomial of degree $j$ in the variables $k_2, k_3, ..., k_n$.

For the coefficients of the Schwarz functions $u(z)$ and $v(w)$, we have $|c_n| \leq 1$and $|d_n| \leq 1$ (e.g., see [17]).

Note that for $a_m$=0, $1 \leq m \leq n-1$ , we have

$$F_{n+1}(a_0, 0, 0, \cdots, 0, a_n) = (-1)^{n+1}a_0^{n+1} - (n+1)a_n.$$

Comparing the corresponding coefficients of (2.1) and (2.3) yields

$$\frac{1}{\gamma}F_{n+1}(a_0, a_1, a_2, \cdots, a_n) = B_1K_n^{-1}(c_1, c_2, ..., c_n, B_1, B_2, ..., B_n). \tag{2.5}$$

Then, under the assumption $a_m$=0, $1 \leq m \leq n-1$, we get

$$\frac{1}{\gamma}\left[(-1)^{n+1}a_0^{n+1} - (n+1)a_n\right] = \frac{1}{\gamma}\left[(-1)^{n+1}((1-\lambda(n+1))b_0)^{n+1} - (n+1)(1-\lambda(n+1))b_n\right] = B_1c_{n+1}. \tag{2.6}$$

Similarly, comparing the corresponding coefficients of (2.2) and (2.4) gives

$$\frac{1}{\gamma}F_{n+1}(A_0, A_1, A_2, \cdots, A_n) = B_1K_n^{-1}(d_1, d_2, ..., d_n, B_1, B_2, ..., B_n). \tag{2.7}$$

Note that, for $A_m$=0, $1 \leq m \leq n-1$, we have

$$\frac{1}{\gamma}\left[(-1)^{n+1}A_0^{n+1} - (n+1)A_n\right] = \frac{1}{\gamma}\left[(-1)^{n+1}((1-\lambda(n+1))\beta_0)^{n+1} - (n+1)(1-\lambda(n+1))\beta_n\right] = B_1d_{n+1}. \tag{2.8}$$

On the other hand, comparing the corresponding coefficients of the functions $f$ and $g = f^{-1}$, we obtain $\beta_0 = -b_0$ and $\beta_n = -b_n$ for $b_m$=0, $1 \leq m \leq n-1$.

Hence, when $n$ is odd, by using Eqs. (2.6), (2.8) and $\beta_0 = -b_0$ and $\beta_n = -b_n$, we obtain following system

$$\begin{aligned} \frac{1}{\gamma}\left[((1-\lambda(n+1))b_0)^{n+1} - (n+1)(1-\lambda(n+1))b_n\right] &= B_1c_{n+1}, \\ \frac{1}{\gamma}\left[((1-\lambda(n+1))b_0)^{n+1} + (n+1)(1-\lambda(n+1))b_n\right] &= B_1d_{n+1}. \end{aligned}$$

Subtracting two above equation, we have

$$\frac{2}{\gamma}\left[(n+1)(1-\lambda(n+1))b_n\right]=B_1\left(d_{n+1}-c_{n+1}\right).$$

Applying the $|c_n|\leq 1$and $|d_n|\leq 1$ yields

$$|b_n|\leq\frac{|\gamma|\,B_1}{(n+1)\,|1-\lambda(n+1|}.$$

Similarly, when $n$ is even, by using Eqs. (2.6), (2.8) with $b_m$=0, $0\leq m\leq n-1$, we obtain following system

$$\begin{aligned}\frac{1}{\gamma}\left[-(n+1)(1-\lambda(n+1))b_n\right] &= B_1c_{n+1},\\ \frac{1}{\gamma}\left[(n+1)(1-\lambda(n+1))b_n\right] &= B_1d_{n+1}.\end{aligned}$$

Hence

$$\frac{2}{\gamma}\left[(n+1)(1-\lambda(n+1))b_n\right]=B_1\left(d_{n+1}-c_{n+1}\right).$$

Applying the $|c_n|\leq 1$and $|d_n|\leq 1$ yields

$$|b_n|\leq\frac{|\gamma|\,B_1}{(n+1)\,|1-\lambda(n+1|}.$$

□

**Corollary 2.2.** *Let $f\in\Sigma$ given by (1.1) in the class $\mathcal{M}_\Sigma(\varphi)$. If, $b_m$=0, $1\leq m\leq n-1$ for n being odd or if $b_m$=0, $0\leq m\leq n-1$ for n being even, then*

$$|b_n|\leq\frac{B_1}{n+1}\quad(n\geq 1).$$

For functions in the class $\mathcal{M}_\Sigma(\lambda,\gamma;\varphi)$, the following initial coefficients estimation holds. To prove our next theorem, we shall need the following well-known lemma (see [17]).

**Lemma 2.1.** *( [17]) If $p\in\mathcal{P}$, the class of all functions with $\Re\left(p\left(z\right)\right)>0\ (z\in\mathbb{D})$, then $|p_n|\leq 2$ $(n\in\mathbb{N}=\{1,2,\cdots\})$, where $p\left(z\right)=1+\sum_{n=1}^{\infty}p_nz^n$.*

We know that $p(z)\in\mathcal{P}\ (z\in\mathbb{D})\Leftrightarrow p\left(\frac{1}{z}\right)\in\mathcal{P}\ (z\in\Delta)$. Define the functions $p$ and $q$ in $\mathcal{P}$ given by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\cdots$$

and

$$q(z)=\frac{1+v(z)}{1-v(z)}=1+\frac{q_1}{z}+\frac{q_2}{z^2}+\cdots,$$

where $u(z)=\frac{c_1}{z}+\frac{c_2}{z^2}+...+\frac{c_n}{z^n}+...$, $|u(z)|<1$, $z\in\Delta$ and $v(z)=\frac{d_1}{z}+\frac{d_2}{z^2}+...+\frac{d_n}{z^n}+...$, $|v(z)|<1$, $z\in\Delta$ are Schwarz functions (e.g., see [17]). It follows that

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1}{2}\frac{1}{z}+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)\frac{1}{z^2}+\cdots \tag{2.9}$$

and

$$v(z)=\frac{q(z)-1}{q(z)+1}=\frac{q_1}{2}\frac{1}{z}+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)\frac{1}{z^2}+\cdots . \tag{2.10}$$

**Theorem 2.3.** *Let $f$ given by (1.1) be in the class $\mathcal{M}_\Sigma(\lambda,\gamma;\varphi)$. Then*

$$|b_0|\leq \min\left\{\frac{\sqrt{|\gamma|\,(B_1+|B_2|)}}{1-\lambda},\frac{\sqrt{|\gamma|\,(|B_2-B_1|+B_1)}}{1-\lambda}\right\}$$

*and*

$$|b_1|\leq\frac{|\gamma|\,B_1}{2\,|1-2\lambda|}.$$

*Proof.* Let $f\in\mathcal{M}_\Sigma(\lambda,\gamma;\varphi)$. Then, there are analytic functions $u,v:\Delta\to\mathbb{C}$, with $u(\infty)=v(\infty)=0$, satisfying

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right) = \varphi(u(z)) \tag{2.11}$$
$$\text{and } 1+\frac{1}{\gamma}\left(\frac{wg'(w)+\lambda w^2g''(w)}{\lambda wg'(w)+(1-\lambda)g(w)}-1\right) = \varphi(v(w)),\ \left(\ g:=f^{-1}\right).$$

Since

$$\begin{aligned}&1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)\\ =\ & 1-\frac{(1-\lambda)b_0}{\gamma}\ \frac{1}{z}+\frac{(1-\lambda)^2b_0^2-2(1-2\lambda)b_1}{\gamma}\frac{1}{z^2}+...\end{aligned}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left(\frac{wg'(w)+\lambda w^2g''(w)}{\lambda wg'(w)+(1-\lambda)g(w)}-1\right)\\ =\ & 1+\frac{(1-\lambda)b_0}{\gamma}\ \frac{1}{w}+\frac{(1-\lambda)^2b_0^2+2(1-2\lambda)b_1}{\gamma}\frac{1}{w^2}+...\end{aligned}$$

then (2.3) and (2.4) yield

$$-\frac{(1-\lambda)b_0}{\gamma}=B_1c_1, \tag{2.12}$$

$$\frac{(1-\lambda)^2b_0^2-2(1-2\lambda)b_1}{\gamma}=B_1c_2+B_2c_1^2, \tag{2.13}$$

$$\frac{(1-\lambda)b_0}{\gamma}=B_1d_1 \tag{2.14}$$

and

$$\frac{(1-\lambda)^2 b_0^2 + 2(1-2\lambda)b_1}{\gamma} = B_1 d_2 + B_2 d_1^2. \tag{2.15}$$

Now, considering (2.12) and (2.14), we get

$$c_1 = -d_1. \tag{2.16}$$

Also, from (2.13) and (2.15), we find that

$$b_0^2 = \frac{\gamma\left[B_1(c_2+d_2)+2B_2c_1^2\right]}{2(1-\lambda)^2}$$

which, in view of the inequalities $|c_n| \le 1$and $|d_n| \le 1$ yield

$$|b_0|^2 \le \frac{|\gamma|\,(B_1+|B_2|)}{(1-\lambda)^2}.$$

Since $B_1 > 0$, the last inequality gives the desired first estimate on $|b_0|$ given in the theorem. On the other hand, comparing the coefficients of (2.9) and (2.10) with (2.11), we have

$$-\frac{(1-\lambda)b_0}{\gamma} = B_1\frac{p_1}{2}, \tag{2.17}$$

$$\frac{(1-\lambda)^2 b_0^2 - 2(1-2\lambda)b_1}{\gamma} = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2 \tag{2.18}$$

$$\frac{(1-\lambda)b_0}{\gamma} = B_1\frac{q_1}{2} \tag{2.19}$$

and

$$\frac{(1-\lambda)^2 b_0^2 + 2(1-2\lambda)b_1}{\gamma} = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2. \tag{2.20}$$

From (2.17) and (2.19), we get $p_1 = -q_1$. Considering the sums of (2.18) and (2.20) with $p_1 = -q_1$, we have

$$b_0^2 = \frac{\gamma}{4(1-\lambda)^2}\left[p_1^2(B_2-B_1) + B_1(p_2+q_2)\right].$$

Applying Lemma 2.1 for the coefficients $p_1$, $p_2$ and $q_2$, we obtain

$$|b_0| \le \frac{\sqrt{|\gamma|\,(|B_2-B_1|+B_1)}}{1-\lambda}$$

that gives the second estimate on $|b_0|$ given in the theorem.

Next, in order to find the bound on $|b_1|$, by further computations from (2.13), (2.15) and (2.16) lead to

$$\frac{4(1-2\lambda)b_1}{\gamma} = B_1(d_2-c_2).$$

Applying the inequalities $|c_n| \le 1$and $|d_n| \le 1$, we readily get

$$|b_1| \le \frac{|\gamma|\,B_1}{2\,|1-2\lambda|}$$

which is the bound on $|b_1|$. □

**Corollary 2.4.** *Let f given by (1.1) be in the class $\mathcal{M}_\Sigma(\varphi)$. Then,*

$$|b_0| \le \min\left\{ \sqrt{B_1 + |B_2|},\ \sqrt{|B_2 - B_1| + B_1} \right\}$$

*and*

$$|b_1| \le \frac{B_1}{2}.$$

**Remark 2.5.** *Taking $\varphi(z) = (1 + (1 - 2\beta)z)/(1 - z)$ in Corollary 2.2 and 2.4, we obtain results of [10].*

## Conflict of interest

The authors declare no conflict of interest.

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